Graphs with three eigenvalues

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Outline

1 Introduction
   - Definitions
     - History

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   - Basic Theory

3 Our results
   - Bound
   - Complement
   - Neumaier’s result

4 Many valencies
   - Many valencies
Definitions

- Let $\Gamma = (V, E)$ be a graph.
- The distance $d(x, y)$ between two vertices $x$ and $y$ is the length of a shortest path connecting them.
- The maximum distance between two vertices in $\Gamma$ is the diameter $D = D(\Gamma)$.
- The valency $k_x$ of $x$ is the number of vertices adjacent to it.
- A graph is regular with valency $k$ if each vertex has $k$ neighbours.
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The adjacency matrix $A$ of $\Gamma$ is the matrix whose rows and columns are indexed by the vertices of $\Gamma$ and the $(x, y)$-entry is 1 whenever $x$ and $y$ are adjacent and 0 otherwise.

The eigenvalues of the graph $\Gamma$ are the eigenvalues of $A$. 

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Strongly regular graphs

A strongly regular graph (SRG) with parameters \((n, k, \lambda, \mu)\) is a \(k\)-regular graph on \(n\) vertices such that

- each pair of adjacent vertices have \(\lambda\) common neighbours;
- each pair of distinct non-adjacent vertices have \(\mu\) common neighbours.
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**Examples**

- The Petersen graph is a strongly regular graph with parameters \( (10, 3, 0, 1) \).
- The line graph of a complete graph on \( t \) vertices \( L(K_t) \) is a SRG \( (t(t-1)/2, 2(t-2), t-2, 4) \).
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- The line graph of a complete bipartite graph \(K_{t,t} \), \(L(K_{t,t})\), is a SRG \((t^2, 2(t-1), t-2, 2)\).
- There are many more examples, coming from all parts in combinatorics.
A connected strongly regular graph has at most diameter two, and has at most three distinct eigenvalues. We can characterize the strongly regular graphs by this property.

**Theorem**

A connected regular graph $\Gamma$ has at most three eigenvalues if and only if it is strongly regular.
Now we will discuss graphs with a small number of distinct eigenvalues. If $\Gamma$ is a connected graph with $t$ distinct eigenvalues then the diameter of $\Gamma$ is bounded by $t - 1$. So a connected graph with at most two distinct eigenvalues is just a complete graph and hence is regular.
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In 1998 E. van Dam gave the basic theory for such graphs, and also give some new examples. Also he classified the graphs with exactly three distinct eigenvalues having smallest eigenvalue at least $-2$. 
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- Let $A$ be the adjacency matrix of $\Gamma$.
- As $B := (A - \theta_1 I)(A - \theta_2 I)$ has rank 1 and is positive semi-definite we have $B = xx^T$ for some eigenvector $x$ of $A$ corresponding to eigenvalue $\theta_0$. 


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- By Perron-Frobenius theorem again wlog all entries of $x$ are positive.
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This gives $k_u = -\theta_1 \theta_2 + x_u^2$ for $u$ a vertex,

$\lambda_{uv} = \theta_1 + \theta_2 + x_u x_v$, for $u \sim v$,

$\mu_{xy} = x_u x_v$ for $u$ and $v$ non-adjacent.
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$\theta_1 \geq 0$ with equality if and only if $\Gamma$ is complete bipartite.

$\theta_2 \leq -\sqrt{2}$ with equality if and only if $\Gamma$ is the path of length 2.

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**Theorem (Van Dam)**

Let $\Gamma$ be connected graph with $n$ vertices with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$, with respective multiplicities $m_0 = 1, m_1, m_2$. Assume that $m_1 = m_2$, then there exists a $b = 1 \pmod{4}$ and $b \leq n$, such that $\theta_1 = (-1 + \sqrt{b})/2$, $\theta_2 = (-1 - \sqrt{b})/2$, $\theta_0 = (n - 1)/2$. (In particular $n$ is odd). Moreover $\Gamma$ is regular if and only if $n = b$. 
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Note that the cone over the Petersen graph is an example with \(m_1 = m_2\).

Also note that if not all eigenvalues are integral then \(m_1 = m_2\). De Caen et al. gave an example with non-integral eigenvalues with \(b = 41\) and \(n = 43\).
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A consequence of our next result is that we can bound \(n\) by a function in \(b\).
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Lemma

Let Γ be a non-regular connected graph with exactly three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$. Let $u \sim v$ with $k_u < k_v$. Then $k_u \geq \lambda_{uv} + 1$. This gives $x_v - 1 \leq x_u(x_u - x_v) \leq -\theta_1\theta_2 + \theta_1 + \theta_2$, and hence $x_v \leq -(\theta_1 + 1)(\theta_2 + 1)$. 
A bound on the number of vertices

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This implies:

**Proposition**

Let $\Gamma$ be a non-regular connected graph on $n$ vertices with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$ with respective multiplicities $1, m_1, m_2$. Let $\Delta$ be the maximal valency in $\Gamma$ and let $\ell := \min\{1 - (\theta_1 + 1)(\theta_2 + 1), -\theta_1 \theta_2 + 1\}$. Then the following hold:

1. $\Delta \leq (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1 \theta_2$;
2. If $\Delta \neq n - 1$, then $\Delta \leq \ell^2 - \theta_1 \theta_2$;
3. $n \leq \max\{(\ell^2 - \theta_1 \theta_2 - 1)^2 + 1, (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1 \theta_2 + 1\}$. 
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The complement of a strongly regular graph is strongly regular. The complement of a non-regular graph with three distinct eigenvalues has usually more than three distinct eigenvalues. But it is quite easy to see that the number of distinct eigenvalues cannot be more than five.
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**Theorem**

Let $\Gamma$ be a non-regular connected graph with three distinct eigenvalues on $n$ vertices. If the complement of $\Gamma$ is disconnected, then either $\Gamma$ has a vertex with degree $n-1$, or $\Gamma$ is complete bipartite.
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Neumaier’s Theorem

Neumaier (1979) showed the following result.

**Neumaier’s Theorem**

Let $m$ be a positive integer. Let $\Gamma$ be a connected and coconnected (i.e. the complement is connected) strongly regular graph with minimal eigenvalue $-m$. Then either the number of vertices is bounded by a function in $m$, or $\Gamma$ belongs to one of two infinite (one parameter) families of strongly regular graphs (and we know how to construct all of them if the number of vertices is large enough).
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How can we generalize this result to graphs with three distinct eigenvalues?
Neumaier’s Theorem 2

Question 1:

Let $m$ be a positive integer.

- Are there only finitely many non-regular connected graphs with distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$ such that $0 < \theta_1 \leq m$?

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- Note that the answer for the question (i) is negative if you allow four distinct eigenvalues as the friendship graphs show.
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We also constructed some new graphs with three distinct eigenvalues and showed some non-existence results.
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**Theorem**

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- The case $x_1 x_2 = -(\theta_2 + 1)\theta_1$ gives $x_2^2 \geq -\theta_2$. 

Now we look at the following result.

Theorem

For given positive integer $\alpha$, there are finitely many connected graphs with eigenvalues $\theta_0 > \theta_1 > \theta_2$ with exactly two valencies and $\theta_1 = \alpha$.

Sketch of proof

- Let $\Gamma$ has valencies $k_1 > k_2$. Let $x_i = \sqrt{k_i + \theta_1 \theta_2}$, $(i = 1, 2)$.
- Let $V_i = \{u \mid k_u = k_i\}$, $(i = 1, 2)$.
- (Van Dam) Then the partition $\{V_1, V_2\}$ is an equitable partition of $\Gamma$.
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- The case $x_1 x_2 = - (\theta_2 + 1) \theta_1$ is more difficult. In this use that there must be a pair of adjacent vertices $u, v$ such that $k_u = k_v = k_2$ and $\lambda_{uv} \geq 0$. Then one gets also a lower bound on $k_2$ of order $\theta_2$. 

Bell and Rowlinson showed that if an eigenvalue multiplicity is $n - m$, then either the corresponding eigenvalue equals 0 or $-1$ or $n \leq (m + 1)m/2$. This shows the theorem.
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This was shown for cones, i.e. graphs with a vertex of valency \( n - 1 \). There are examples of connected graphs with four distinct eigenvalues with the number of different valencies as large as you want. (I will discuss it below, in more detail.) The connectedness part in Question 2 is essential as all complete bipartite graphs with the same number of edges have the same distinct eigenvalues.
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**Challenge:**

Construct more connected non-regular graphs with three distinct eigenvalues.
Outline

1 Introduction
   • Definitions
   • History

2 Theory
   • Basic Theory

3 Our results
   • Bound
   • Complement
   • Neumaier’s result

4 Many valencies
   • Many valencies
Graphs with a few eigenvalues and many valencies

(This is joint work with E. van Dam and Mr. Xia)

First we construct graphs with five eigenvalues:

- Let \( m = 2t + 1 \) be a positive odd integer.
- Let \( r_i = 2^i \) and \( s_i = 2^{m-i} \) for \( i = 0, 1, \ldots, t \).
- Take \( \Gamma \) be the disjoint union of \( K_{r_0,s_0}, K_{r_1,s_1}, \ldots, K_{r_t,s_t} \).
- Then \( \Gamma \) has three eigenvalues and \( 2t + 2 \) valencies.
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- Then $\Gamma$ has three eigenvalues and $2t + 2$ valencies.
- Let $\Delta$ be the complement of $\Gamma$. Then $\Delta$ has still $2t + 2$ valencies, is connected and has exactly 5 eigenvalues, of which 3 have multiplicity 1.
Graphs with a few eigenvalues and many valencies, 2

Now examples with four eigenvalues with many valencies.

- Consider the Paley graph $P(p)$ where $p$ is a prime such that $p = 1 \mod 4$.
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- Now consider $H$ the disjoint union of $P(p_t)$ and an isolated vertex.
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- Now switch the graph $H$ with respect $S$, to obtain $\hat{H}$.
- Now $\hat{H}$ has in general 4 distinct eigenvalues, of which two are simple, and the spectrum only depends on the number of edges of $G$. 
Thank you for your attention.