Graphs with three eigenvalues

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Monash University, March 2, 2015

Outline

Definitions

- Let $\Gamma = (V, E)$ be a graph.
- The distance $d(x, y)$ between two vertices x and y is the length of a shortest path connecting them.
- The maximum distance between two vertices in Γ is the diameter $D = D(\Gamma)$.
- The valency k_x of x is the number of vertices adjacent to it.
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- A graph is regular with valency k if each vertex has k neighbours.
- The adjacency matrix A of Γ is the matrix whose rows and columns are indexed by the vertices of Γ and the (x, y)-entry is 1 whenever x and y are adjacent and 0 otherwise.
- The eigenvalues of the graph Γ are the eigenvalues of A.

Strongly regular graphs

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- The Petersen graph is a strongly regular graph with parameters $(10, 3, 0, 1)$.
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- The line graph of a complete bipartite graph $K_{t,t}$, $L(K_{t,t})$, is a SRG $(t^2, 2(t-1), t-2, 2)$.
- There are many more examples, coming from all parts in combinatorics.

Strongly regular graphs 2

A connected strongly regular graph has at most diameter two, and has at most three distinct eigenvalues. We can characterize the strongly regular graphs by this property.

Theorem

A connected regular graph Γ has at most three eigenvalues if and only if it is strongly regular.

Small number of distinct eigenvalues

Now we will discuss graphs with a small number of distinct eigenvalues. If Γ is a connected graph with t distinct eigenvalues then the diameter of Γ is bounded by $t - 1$. So a connected graph with at most two distinct eigenvalues is just a complete graph and hence is regular.

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Also the cone over the Petersen graph (i.e. you add a new vertex and join the new vertex with all the other vertices) is a non-regular graph with exactly three distinct eigenvalues.

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1. Introduction
   - Definitions
   - History
2. Theory
   - Basic Theory
3. Our results
   - Bound
   - Complement
   - Neumaier's result
4. Many valencies
   - Many valencies

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- In 1998 Muzychuk and Klin gave more examples of such graphs.
- In 1998 E. van Dam gave the basic theory for such graphs, and also give some new examples. Also he classified the graphs with exactly three distinct eigenvalues having smallest eigenvalue at least -2.

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- As $B := (A - \theta_1 I)(A - \theta_2 I)$ has rank 1 and is positive semi-definite we have $B = \mathbf{x}\mathbf{x}^T$ for some eigenvector $\mathbf{x}$ of A corresponding to eigenvalue θ_0.

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- By Perron-Frobenius theorem again wlog all entries of $\mathbf{x}$ are positive.

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- This gives $k_u = -\theta_1\theta_2 + x_u^2$ for u a vertex,
- $\lambda_{uv} = \theta_1 + \theta_2 + x_u x_v$, for $u \sim v$,
- $\mu_{xy} = x_u x_v$ for u and v non-adjacent.

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From now on we will assume $\theta_1 > 0$ and hence θ_0 is an integer.

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Let Γ be connected graph with n vertices with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$, with respective multiplicities $m_0 = 1, m_1, m_2$. Assume that $m_1 = m_2$, then there exists a $b = 1 \pmod 4$ and $b \leq n$, such that $\theta_1 = (-1+\sqrt{b})/2$, $\theta_2 = (-1-\sqrt{b})/2$, $\theta_0 = (n-1)/2$. (In particular n is odd). Moreover Γ is regular if and only if $n = b$.

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Note that the cone over the Petersen graph is an example with $m_1 = m_2$. Also note that if not all eigenvalues are integral then $m_1 = m_2$. De Caen et al. gave an example with non-integral eigenvalues with $b = 41$ and $n = 43$.

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Also note that if not all eigenvalues are integral then $m_1 = m_2$. De Caen et al. gave an example with non-integral eigenvalues with $b = 41$ and $n = 43$.
A consequence of our next result is that we can bound n by a function in b.

Outline

A bound on the number of vertices

Lemma

Let Γ be a non-regular connected graph with exactly three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$. Let $u \sim v$ with $k_u < k_v$. Then $k_u \geq \lambda_{uv} + 1$. This gives $x_v - 1 \leq x_u(x_u - x_v) \leq -\theta_1\theta_2 + \theta_1 + \theta_2$, and hence $x_v \leq -(\theta_1 + 1)(\theta_2 + 1)$.

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This implies:

Proposition

Let Γ be a non-regular connected graph on n vertices with three distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$ with respective multiplicities $1, m_1, m_2$. Let Δ be the maximal valency in Γ and let $\ell := \min\{1 - (\theta_1 + 1)(\theta_2 + 1), -\theta_1\theta_2 + 1\}$. Then the following hold:

1. $\Delta \leq (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1\theta_2)$;
2. If $\Delta \neq n - 1$, then $\Delta \leq \ell^2 - \theta_1\theta_2$;
3. $n \leq \max\{(\ell^2 - \theta_1\theta_2 - 1)^2 + 1, (1 - (\theta_1 + 1)(\theta_2 + 1))^2 - \theta_1\theta_2 + 1\}$.

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Complement

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Theorem

Let Γ be a non-regular connected graph with three distinct eigenvalues on n vertices. If the complement of Γ is disconnected, then either Γ has a vertex with degree $n-1$, or Γ is complete bipartite.

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Neumaier's Theorem

Neumaier (1979) showed the following result.

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Let m be a positive integer. Let Γ be a connected and coconnected (i.e the complement is connected) strongly regular graph with minimal eigenvalue $-m$. Then either the number of vertices is bounded by a function in m, or Γ belongs to one of two infinite (one parameter) families of strongly regular graphs (and we know how to construct all of them if the number of vertices is large enough)

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How can we generalize this result to graphs with three distinct eigenvalues?

Neumaier's Theorem 2

Question 1:

Let m be a positive integer.

- Are there only finitely many non-regular connected graphs with distinct eigenvalues $\theta_0 > \theta_1 > \theta_2$ such that $0 < \theta_1 \leq m$?
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- On this moment we are working on the second part of this question for graphs with exactly two different valencies.
- Note that the answer for the question (i) is negative if you allow four distinct eigenvalues as the friendship graphs show.

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- We also constructed some new graphs with three distinct eigenvalues and showed some non-existence results.

Now we look at the following result.

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Sketch of proof

- Let Γ has valencies $k_1 > k_2$. Let $x_i = \sqrt{k_i + \theta_1\theta_2}, (i = 1, 2)$.
- Let $V_i = \{u \mid k_u = k_i\}$, $(i = 1, 2)$.
- (Van Dam) Then the partition $\{V_1, V_2\}$ is an equitable partition of Γ.
- One can show that $x_1x_2 = -(\theta_1 + 1)\theta_2$ or $x_1x_2 = -(\theta_2 + 1)\theta_1$.
- The case $x_1x_2 = -(\theta_2 + 1)\theta_1$ gives $x_2^2 \geq -\theta_2$.
- The case $x_1x_2 = -(\theta_1 + 1)\theta_2$ is more difficult. In this use that there must be a pair of adjacent vertices u, v such that $k_u = k_v = k_2$ and $\lambda_{uv} \geq 0$. Then one gets also a lower bound on k_2 of order θ_2.
- Using those lower bounds on k_2 one gets that the multiplicity of θ_2 is bounded above by a function in θ_2.
- Bell and Rowlinson showed that if an eigenvalue multiplicity is $n - m$, then either the corresponding eigenvalue equals 0 or -1 or $n \leq (m + 1)m/2$. This shows the theorem.

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Challenge:

Construct more connected non-regular graphs with three distinct eigenvalues.

Outline

Graphs with a few eigenvalues and many valencies

(This is joint work with E. van Dam and Mr. Xia)
First we construct graphs with five eigenvalues:

- Let $m = 2t + 1$ be a positive odd integer.
- Let $r_i = 2^i$ and $s_i = 2^{m-i}$ for $i = 0, 1, \ldots, t$.
- Take Γ be the disjoint union of $K_{r_0,s_0}, K_{r_1,s_1}, \ldots, K_{r_t,s_t}$.
- Then Γ has three eigenvalues and $2t + 2$ valencies.

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- Let Δ be the complement of Γ. Then Δ has still $2t + 2$ valencies, is connected and has exactly 5 eigenvalues, of which 3 have multiplicity 1.

Graphs with a few eigenvalues and many valencies, 2

Now examples with four eigenvalues with many valencies.

- Consider the Paley graph $P(p)$ where p is a prime such that $p = 1$ mod 4.
- (vertex set $\{0, 1, 2 \ldots, p-1\}$, $x \sim y$ if $x - y$ is a square (not zero))

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- Now consider H the disjoint union of $P(p_t)$ and an isolated vertex.
- Let G be a graph on t vertices, there exists a set of vertices S of $P(p_t)$ such the induced subgraph on S of $P(p_t)$ is G.
- Now switch the graph H with respect S, to obtain $\hat{H}$.

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- Now switch the graph H with respect S, to obtain $\hat{H}$.
- Now $\hat{H}$ has in general 4 distinct eigenvalues, of which two are simple, and the spectrum only depends on the number of edges of G.

Thank you for your attention.