Elusive problems in extremal graph theory

Andrzej Grzesik (Krakow)
Dan Kráľ (Warwick)
László Miklós Lovász (MIT/Stanford)

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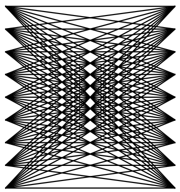
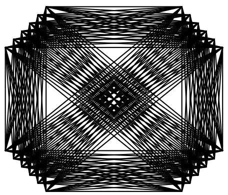
Turán Problems

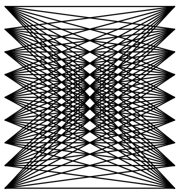
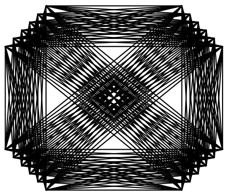
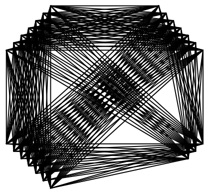
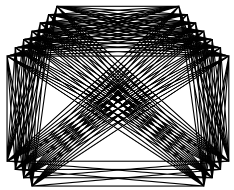
- Maximum edge-density of H-free graph
- Mantel's Theorem (1907): $\frac{1}{2}$ for $H = K_3$ $(K_{\frac{n}{2},\frac{n}{2}})$
- Turán's Theorem (1941): $\frac{\ell-2}{\ell-1}$ for $H = K_\ell$ $(K_{\frac{n}{\ell-1},\dots,\frac{n}{\ell-1}})$
- Erdős-Stone Theorem (1946): $\frac{\chi(H)-2}{\chi(H)-1}$
- extremal examples unique up to $o(n^2)$ edges





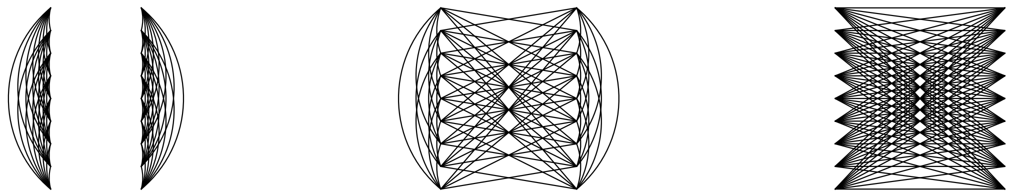
Edge vs. Triangle Problem

- Minimum density of K_3 for a specific edge-density
- determined by Razborov (2008), $K_{\alpha n, \ldots, \alpha n, (1-k\alpha)n}$
- extensions by Nikiforov (2011) and Reiher (2016) for K_ℓ
- Pikhurko and Razborov (2017) gave extremal examples
  generally not unique, can be made unique by $\overline{K_n} = 0$

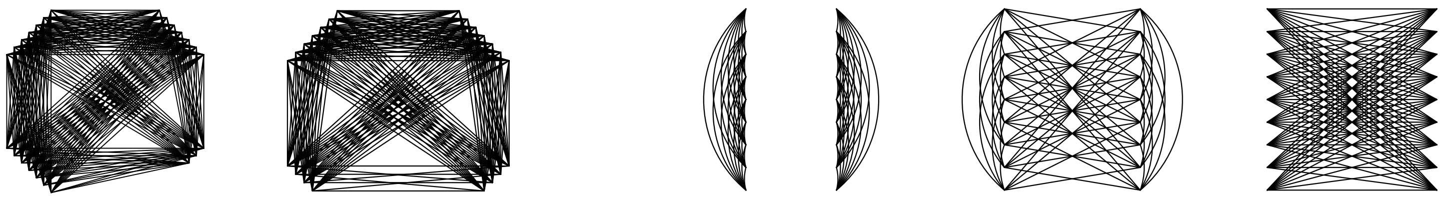
Another example

- Minimum sum of densities of K_3 and $\overline{K_3}$
- Goodman's Bound (1959): $K_3 + \overline{K_3} \geq \frac{1}{4}$
  every $n/2$-regular graph is a minimizer
- minimizer can be made unique $K_3 = 0$, or $\overline{K_3} = 0$, or $C_4 = 1/16$ (Erdős-Rényi random graph $G_{n,1/2}$)



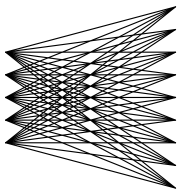
This Talk

- Conjecture (Lovász 2008, Lovász and Szegedy 2011)
  Every finite feasible set $H_i = d_i$, $i = 1, \ldots, k$,
  can be extended to a finite feasible set
  with an asymptotically unique structure.
- Every extremal problem has a finitely forcible optimum.
- Theorem (Grzesik, K., Lovász Jr.): FALSE

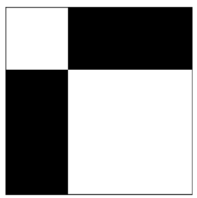


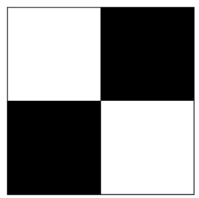
Limits of dense graphs

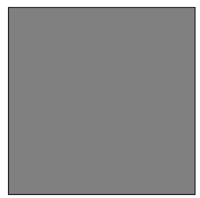
- $d(H,G)$ = probability $|H|$-vertex subgraph of G is H
- a sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H,G_n)$ converges for every H
- examples: K_n, $K_{\alpha n,n}$, blow ups $G[K_n]$ Erdős-Rényi random graphs $G_{n,p}$, planar graphs
- graphon $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)$

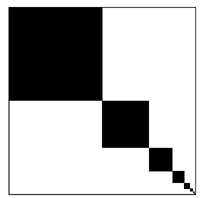


$$\begin{matrix} 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1&1&1\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0\\ 1&1&1&1&0&0&0&0&0&0&0&0 \end{matrix}$$









Finitely forcible graph limits

- a graphon W is finitely forcible if there exist $H_1, \ldots, H_k$ and $d_1, \ldots, d_k$ such that W is the only graphon with the expected density of H_i equal to d_i
- $\Leftrightarrow$ the only graphon minimizing $\sum \alpha_j d(H'_j, W)$
density calculation: $\sum (H_i - d_i)^2 = \sum \alpha_j H'_j$
- Lovász, Sós (2008): Step graphons are finitely forcible.

Statement of Problem

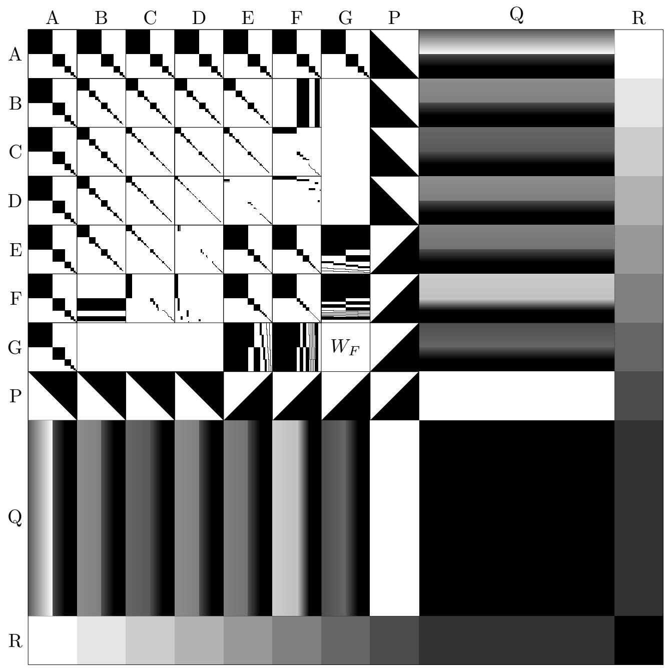
- Conjecture (Lovász 2008, Lovász and Szegedy 2011):
  Every extremal problem $\min \sum \alpha_j d(H_j, W)$
  has a finitely forcible optimal solution.

- extremal graph theory problem $\to$
  finitely forcible optimal solution $\to$
  "simple structure" gives new bounds on old problems

- Conjectures (Lovász and Szegedy):
  The space $T(W)$ of a finitely forcible W is compact.
  The space $T(W)$ has finite dimension.

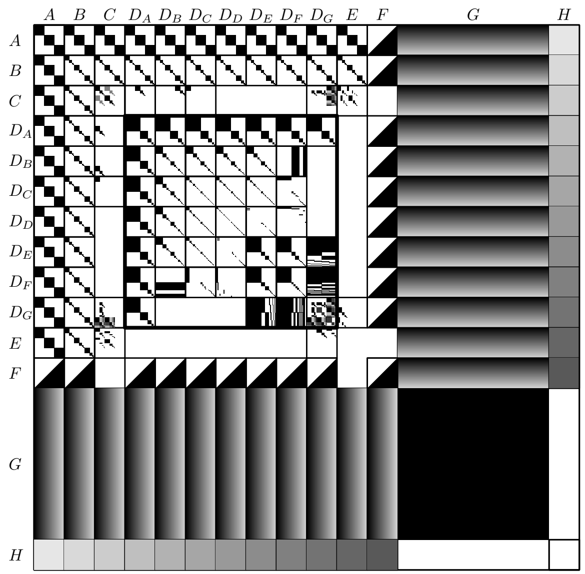
Finitely forcible graph limits

- Theorem (Cooper, K., Martins):
  Every graphon is a subgraphon of a finitely forcible graphon.



Main result

- Theorem (Grzesik, K., Lovász Jr.)
  $\exists$ graphon family $\mathcal{W}$, graphs H_i, reals d_i, $i = 1, \ldots, m$
  $W \in \mathcal{W} \Leftrightarrow d(H_i, W) = d_i$ for $i = 1, \ldots, m$
  no graphon in $\mathcal{W}$ is finitely forcible



Some details of the proof

- graphons $W_P(\vec{z})$, $\vec{z} \in [0,1]^{\mathbb{N}}$
  $\vec{z}$ satisfies polynomial inequalities in P (e.g. $z_1 + z_2^2 \leq 1$)
- construct $J_i \subseteq [0,1]$, inequalities P and inj. maps f_i
  $f_i(x_1, \ldots, x_i) = (z_1, \ldots, z_{\frac{(i+1)(i+2)}{2}})$
  $(x_1) \to (z_1, z_2)$, $(x_1, x_2) \to (z_1, z_2, z_3, z_4, z_5)$, etc.
  $d(H_i, W_P(\vec{z})) = g_i(x_1, \ldots, x_i)$ if $x_k \in J_k$, $k \in \mathbb{N}$
  each J_i has positive measure
- $\Rightarrow$ no graphon in $W_P(\vec{z})$ is finitely forcible

Some details of the proof

- graphons $W_P(\vec{z})$, $\vec{z} \in [0,1]^{\mathbb{N}}$
  $\vec{z}$ satisfies polynomial inequalities in P (e.g. $z_1 + z_2^2 \leq 1$)
- independent of P: there exist graphs $H_1, \dots, H_k$
  there exist polynomials $q_1, \dots, q_\ell$ in $d(H_i, W)$
- for every P: there exist reals $\alpha_1, \dots, \alpha_\ell$
  $W_P(\vec{z})$ are precisely graphons satisfying $q_i = \alpha_i$
- analysis of the dependance of $d(H_i, W_P(\vec{z}))$ on P
  approximation of maps f_i by polynomial inequalities

Possible extensions

- techniques universal to prove more general results
  equalize other functions than subgraph densities
- Theorem (Grzesik, K., Lovász Jr.)
  $\exists$ graphon family $\mathcal{W}$, graphs H_i, reals d_i, $i = 1, \ldots, m$
  $W \in \mathcal{W} \Leftrightarrow d(H_i, W) = d_i$ for $i = 1, \ldots, m$
  no graphon in $\mathcal{W}$ is finitely forcible
  all graphons in $\mathcal{W}$ have the same entropy
- extremal problems with no typical structure

Thank you for your attention!