Complete Tripartite Graphs and their Competition Numbers

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Competition Graphs

Definition 1

Let $D = (V, A)$ be a digraph. The competition graph of D is the simple graph $G = (V, E)$ where

$$\{u, v\} \in E \text{ if and only if } N^+(u) \cap N^+(v) \neq \emptyset.$$

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Which graphs are competition graphs?

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$$\theta_e(G) = \min\{|\mathcal{S}| : \mathcal{S} \text{ is an edge clique cover of } G\}.$$

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Definition 2

The competition number of G is

$$k(G) = \min\{k : \; G \cup I_k \text{ is the competition graph of an acyclic digraph}\}$$

A digraph $D = (V, A)$ is acyclic if and only if there is an ordering $v_1, v_2, \ldots, v_n$ of the vertices in V such that if $(v_i, v_j) \in A$, then $i < j$.

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G is the competition graph of an acyclic digraph if and only if

- there is an ordering $v_1, \ldots, v_n$ and
- there is an edge clique cover $\{S_1, \ldots, S_n\}$

such that $S_i \subseteq \{v_1, \ldots, v_{i-1}\}$ for each i.

Theorem 1

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Theorem 3

For positive integers x, y and z where $2 \leq x \leq y \leq z$,

$$k(K_{x,y,z}) = \begin{cases} yz - 2y - z + 4, & \text{if } x = y \\ yz - z - y - x + 3, & \text{if } x \neq y \end{cases}$$

Theorem 4

If $n \geq 5$ is odd, then $n^2 - 4n + 7 \leq k(K_n^4) \leq n^2 - 4n + 8$.

Theorem 5

If n is prime and $m \leq n$, then $k(K_n^m) \leq n^2 - 2n + 3$.

Theorem 6

$k(K_{n,n,n}) = n^2 - 3n + 4$

Proof.

Let L be the latin square of order n such that $(a, b, c) \in L$ if and only if $c \equiv a + b - 1 \mod n$.

Consider the cliques $\Delta(1, 1, 1)$, $\Delta(2, n, 1)$, $\Delta(1, n, n)$, $\Delta(n, 1, n)$, $\Delta(n, 2, 1)$, $\Delta(1, 2, 2)$, and

$$\Delta(n-1, 2, n),\ \Delta(2, n-1, n),\ \Delta(1, n-1, n-1),$$
$$\Delta(n-2, 2, n-1),\ \Delta(2, n-2, n-1),\ \Delta(1, n-2, n-2), \ldots$$

Consider $K_{x,y,z}$ $(x \leq y \leq z)$.

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Definition 3

An r-multi latin square of order n is an $n \times n$ array of nr symbols such that

- *each symbol appears once in each row and column and*
- *each cell contains r symbols.*

$K_{2,4,6}$

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1,2	4,5	3,7	6,8
5,6	7,8	1,2	3,4
7,8	2,3	4,6	1,5
3,4	1,6	5,8	2,7

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1,2	4,5	3	6
3,4	1,6	5	2

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1,2	4,5	3	6
3,4	1,6	5	2

$$\mathcal{F} = \{\Delta(1,1,1), \Delta(1,1,2), \Delta(1,2,4), \Delta(1,2,5), \Delta(1,3,3), \Delta(1,4,6), \\ \Delta(2,1,3), \Delta(2,1,4), \Delta(2,2,1), \Delta(2,2,6), \Delta(2,3,5), \Delta(2,4,2),$$

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3,4	1,6	5,8	2,7

1,2	4,5	3	6
3,4	1,6	5	2

$$\mathcal{F} = \{\Delta(1,1,1), \Delta(1,1,2), \Delta(1,2,4), \Delta(1,2,5), \Delta(1,3,3), \Delta(1,4,6), \Delta(2,1,3), \Delta(2,1,4), \Delta(2,2,1), \Delta(2,2,6), \Delta(2,3,5), \Delta(2,4,2),$$

$$\Delta(1,5), \Delta(1,6), \Delta(3,1), \Delta(3,2), \Delta(4,3), \Delta(4,4), \Delta(2,2), \Delta(2,3), \Delta(3,4), \Delta(3,6), \Delta(4,1), \Delta(4,5)\}$$

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Let $R' = \{r'_i : 1 \leq i \leq x\}$ be a set of rows and let $S' = \{s'_i : 1 \leq i \leq z\}$ be a set of z symbols.

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Let $R' = \{r_i' : 1 \leq i \leq x\}$ be a set of rows and let $S' = \{s_i' : 1 \leq i \leq z\}$ be a set of z symbols.

Set $L(R', C, S') = \{(r_i', c_j, s_k') : (r_i', c_j, s_k') \in L, r_i' \in R', s_k' \in S'\}$.

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Lemma 1

The family

$$\mathcal{F} = \{\Delta(i,j,k) : (r'_i, c_j, s'_k) \in L(R', C, S')\} \cup \{\Delta(j,k) : (r_i, c_j, s_k) \in L(R \setminus R', C, S')\}$$

is an edge clique cover of $K_{x,y,z}$. Moreover, $\theta(K_{x,y,z}) = yz$.

Theorem 7

For positive integers x, y and z where $2 \leq x \leq y \leq z$,

$$k(K_{x,y,z}) = \begin{cases} yz - 2y - z + 4, & \text{if } x = y \\ yz - z - y - x + 3, & \text{if } x \neq y \end{cases}$$

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Let $R' = \{r_1, \ldots, r_{x-1}, r_y\}$ and let $S' = \{s_1, \ldots, s_z\}$.

Example: $x = 3$, $y = 5$, $z = 13$

1,6,11	2,7,12	3,8,13	4,9,14	5,10,15
2,7,12	3,8,13	4,9,14	5,10,15	1,6,11
3,8,13	4,9,14	5,10,15	1,6,11	2,7,12
4,9,14	5,10,15	1,6,11	2,7,12	3,8,13
5,10,15	1,6,11	2,7,12	3,8,13	4,9,14

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4,9,14	5,10,15	1,6,11	2,7,12	3,8,13
5,10,15	1,6,11	2,7,12	3,8,13	4,9,14

1,6,11	2,7,12	3,8,13	4,9	5,10
2,7,12	3,8,13	4,9	5,10	1,6,11
5,10	1,6,11	2,7,12	3,8,13	4,9

Case 1: $x = y$

$$\Delta_1 = \{u_1, v_1, w_1\},\ \Delta_2 = \{u_2, v_y, w_1\},\ \Delta_3 = \{u_1, v_y, w_y\},$$

$$\Delta_4 = \{u_y, v_1, w_y\},\ \Delta_5 = \{u_y, v_2, w_1\},\ \Delta_6 = \{u_1, v_2, w_2\}$$

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$0 \leq s \leq y - 4$:

$$\Delta_{3s+7} = \{u_{y-s-1}, v_2, w_{y-s}\},$$
$$\Delta_{3s+8} = \{u_2, v_{y-s-1}, w_{y-s}\}, \text{ and}$$
$$\Delta_{3s+9} = \{u_1, v_{y-s-1}, w_{y-s-1}\}$$

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$0 \leq s \leq z - y - 1$:

$$\Delta_{3y-2+s} = \{u, v, w_{y+s+1}\}$$

Case 2: $x < y$

$$\Delta_1 = \{u_x, v_1, w_y\}, \quad \Delta_2 = \{v_2, w_y\}, \quad \Delta_3 = \{u_x, v_2, w_1\},$$
$$\Delta_4 = \{u_1, v_1, w_1\}, \quad \Delta_5 = \{u_1, v_2, w_2\}, \quad \Delta_6 = \{u_2, v_1, w_2\},$$
$$\Delta_7 = \{u_2, v_y, w_1\}$$

Case 2: $x < y$

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$$\Delta_7 = \{u_2, v_y, w_1\}$$

$0 \leq s \leq y - x - 1$:

$$\Delta_{2s+8} = \{u_x, v_{y-s}, w_{y-s-1}\}, \text{ and}$$
$$\Delta_{2s+9} = \{u_1, v_{y-s-1}, w_{y-s-1}\}$$

$0 \le s \le x - 4$:

$$\Delta_{3s+2(y-x)+8} = \{u_{x-s-1}, v_2, w_{x-s}\},$$
$$\Delta_{3s+2(y-x)+9} = \{u_2, v_{x-s-1}, w_{x-s}\}, \text{ and}$$
$$\Delta_{3s+2(y-x)+10} = \{u_1, v_{x-s-1}, w_{x-s-1}\}$$

$0 \le s \le z - y - 1$:

$$\Delta_{2y+x-1+s} = \{u, v, w_{y+s+1}\}$$