Spanning Substructures in Randomly Perturbed Graphs and Hypergraphs

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Joint work with Michael Krivelevich and Benny Sudakov

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Hamiltonicity

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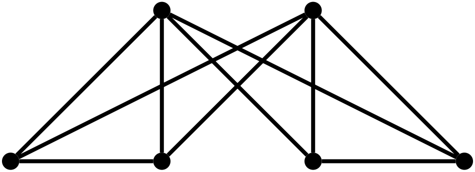

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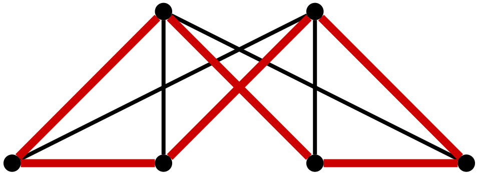

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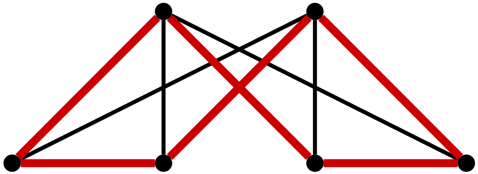

Theorem. Checking whether a graph is Hamiltonian is **NP-complete**

Dirac's Theorem

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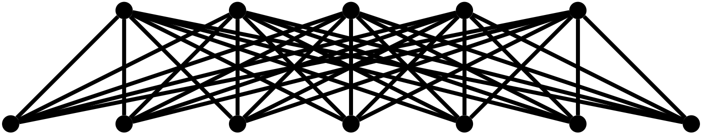

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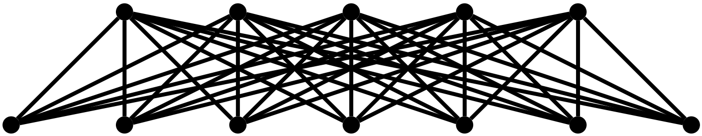

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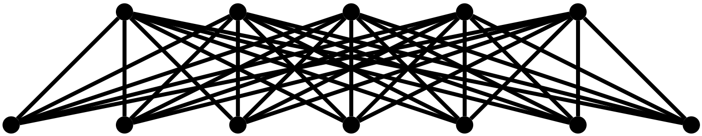

- "Dense" will always refer to high minimum degree
- Theorem also holds for directed graphs (Ghouila-Houri 1960)

Random Graphs

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Definition. We say that some property P holds for $\mathbb{G}(n, m(n))$ **almost surely** if

$$\lim_{n \to \infty} \mathbb{P}(P \text{ holds for } \mathbb{G}(n, m(n))) = 1.$$

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Answer. Consider **randomly perturbed** graphs.

Randomly perturbed graphs: a model

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- There are lots of other models of random perturbation, which are for most purposes equivalent.
- This model naturally extends $\mathbb{G}(n, m)$: let G be the n-vertex graph with no edges.

Motivation: smoothed analysis

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This was introduced by Spielman and Teng, and was effective for explaining why the simplex algorithm is efficient in practice.

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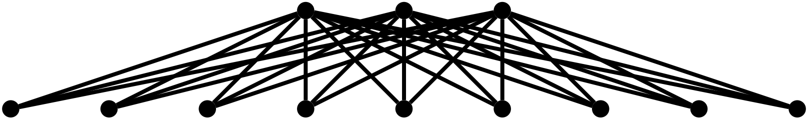

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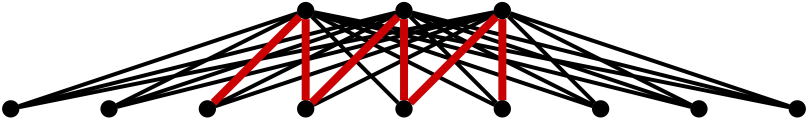

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Corollary (Krivelevich, K., Sudakov). A dense graph plus linearly many random edges is almost surely Hamiltonian.

Expansion and pancyclicity

Theorem (Krivelevich, K., Sudakov). "Dense **digraphs** with good expansion properties are Hamiltonian"

Let D be a **digraph** on n vertices with minimum degree at least $\mathbf{4k}$.

Suppose for every pair of disjoint sets $A, B \subseteq V(D)$ with $|A| = |B| \geq k$, there is an edge from A to B.

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Then D is **pancyclic** (has cycles of every possible length).

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Embedding Hamilton cycles: Rotation-Extension

- If we cannot greedily extend a path,
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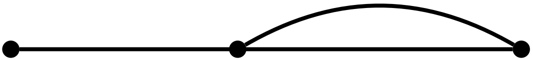

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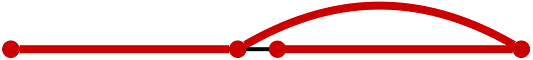

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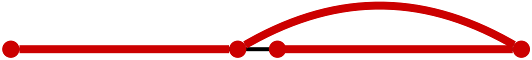

- Continue rotating and extending until we reach a Hamilton path, then close into a Hamilton cycle with a similar "rotation"

Generalization

We generalize in two directions:

- More general kinds of spanning subgraphs than Hamilton cycles
- hypergraphs

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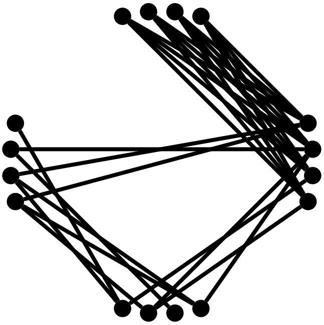
The rotation-extension idea fails in both these cases. We need to take a more "global" approach.

Szemerédi's regularity lemma

Lemma (Szemerédi). We can split almost all the vertices of any graph into a **constant** number of "clusters" in such a way that the edges between every pair of clusters are random-like.



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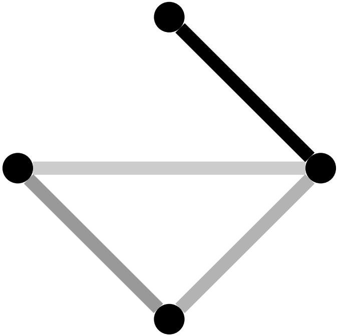

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Bounded-degree spanning trees

Theorem (Komlós, Sárközy, Szemerédi 1995). For any ε, Δ and large enough n:
Let T be an n-vertex tree with maximum degree at most Δ;
Let G be an n-vertex graph with minimum degree at least $\left(\frac{1}{2} + \varepsilon\right)n$.
Then G contains a copy of T.

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Theorem (Montgomery). For any Δ:
Let T be an n-vertex tree with maximum degree at most Δ.
If $m \geq \Delta n(\log n)^5$ for large c then $\mathbb{G}(n,m)$ contains T almost surely.

Spanning trees in randomly perturbed graphs

Theorem (Krivelevich, K., Sudakov). Let G be an n-vertex graph with minimum degree at least αn;
Let T be an n-vertex tree with maximum degree at most Δ.
If $m \geq cn$ for large $c = c(\alpha, \Delta)$ then $\mathbb{G}(G, m)$ contains T almost surely.

Proof ingredients and ideas

Theorem (Alon, Krivelevich, Sudakov). We can almost surely find bounded-degree **almost-spanning** trees (trees of size $(1-\varepsilon)n$) in $\mathbb{G}(n, cn)$, for large c.

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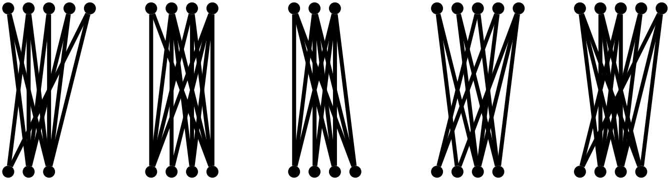
Theorem (Alon, Krivelevich, Sudakov). "We can almost surely find bounded-degree **almost-spanning** trees in $\mathbb{G}(n, O(n))$".

Lemma (Krivelevich, K., Sudakov). "We can partition the vertices of a dense ($\delta \geq \alpha n$) graph into $O(1)$ pairs of clusters of comparable sizes, in such a way that the edges between pairs are **super-regular**".

$G \supseteq$ 

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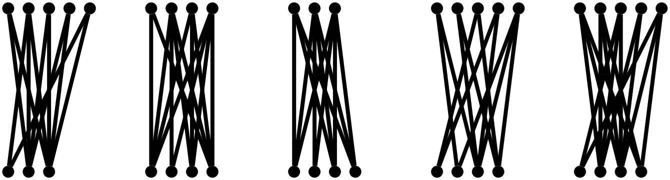
$G \supseteq$ 

Blow-up Lemma (Komlós, Sárközy, Szemerédi). "It's easy to embed bounded-degree graphs into super-regular pairs".

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Theorem (Alon, Krivelevich, Sudakov). "We can almost surely find bounded-degree **almost-spanning** trees in $\mathbb{G}(n, O(n))$".

Lemma (Krivelevich, K., Sudakov). "We can decompose a dense graph into $O(1)$ **super-regular pairs** of comparable sizes".

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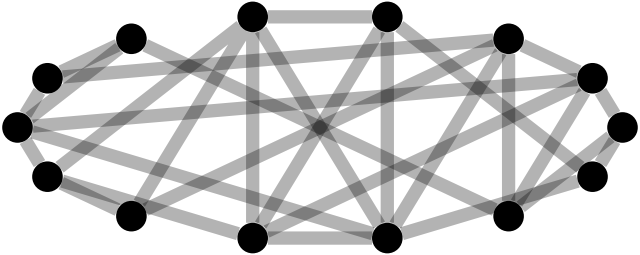
Proof Sketch. We have a dense graph G and random edges $R \in \mathbb{G}(n, O(n))$. We want to find a spanning tree T in $G \cup R$.

- Decompose G into super-regular pairs.
- Embed "most" of T, mainly using R, in a way that is compatible with the decomposition of G.
- Finish the embedding using the super-regular pairs in G.

Decomposing into super-regular pairs (for experts)



If G has minimum degree $\left(\frac{1}{2}+\varepsilon\right)n$ then we can obtain a decomposition into super-regular pairs by finding a perfect matching of the cluster graph obtained by Szemerédi's regularity lemma.

This idea was used by Komlós, Sárközy and Szemerédi to prove a Dirac-type theorem for spanning trees.

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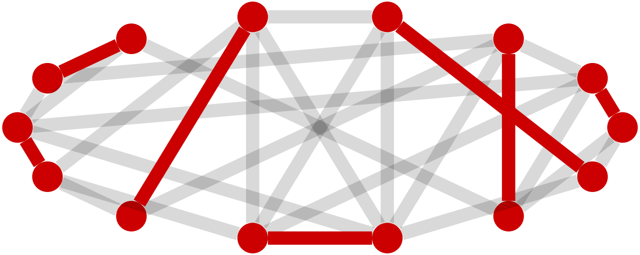

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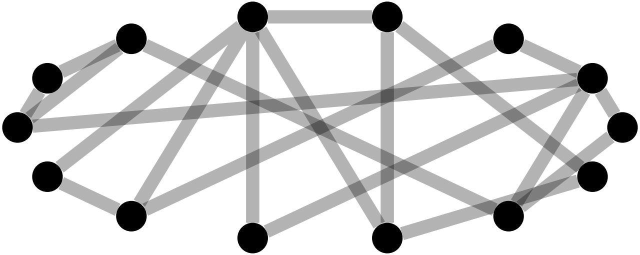

If G has minimum degree αn for small α, we can instead find a cover of the cluster graph by small stars (with up to $1/\alpha$ leaves), then "merge" those stars into pairs.

The clusters will not be the same size, but the variation in their sizes will depend only on α.

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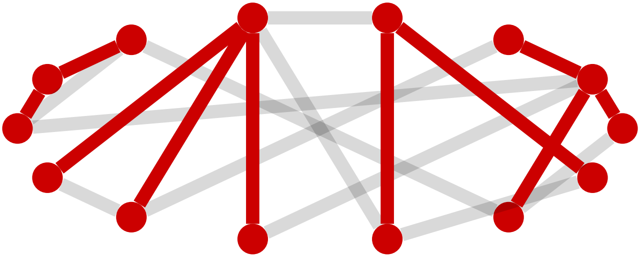

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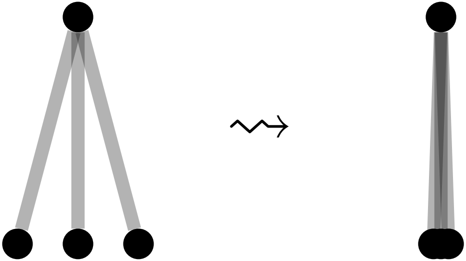

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A structural dichotomy for trees

Definition. A **bare path** is a path in a graph where every vertex has degree 2.

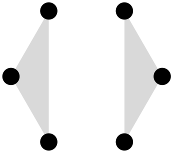
Theorem (Krivelevich 2010). Let T be a tree on n vertices with at most ℓ leaves. Then T contains a collection of about $n/k - 2\ell$ vertex-disjoint bare paths of length k.

In particular, all spanning trees either have $\Omega(n)$ leaves, or they are almost entirely composed of bare paths.

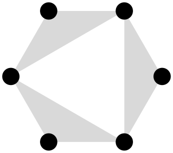
If we want to embed most of T, a convenient choice is to embed T without some leaves, or T without some bare paths

Cycles and density in k-uniform hypergraphs

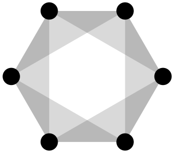
Definition. In a **loose** cycle, consecutive edges intersect in one vertex. In a **tight** cycle, they intersect in $k-1$ vertices.



matching



loose cycle



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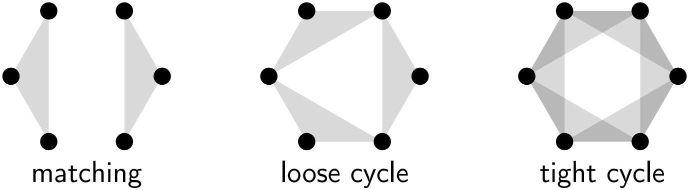

Definition. The **degree** of a set of vertices is the number of edges that includes that set.

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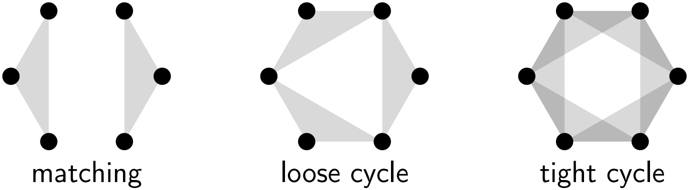

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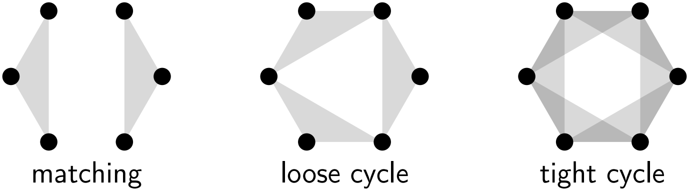

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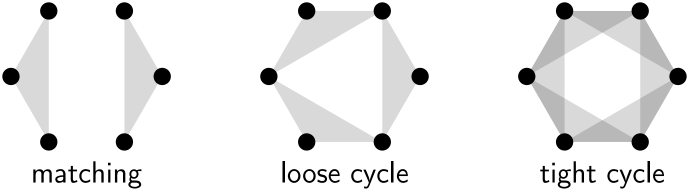

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- Usually consider $(k-1)$-degree

Randomly perturbed dense hypergraphs

Theorem (Krivelevich, K., Sudakov). Consider a k-uniform hypergraph with minimum $(k-1)$-degree at least αn, and add cn random edges (for large $c = c(\alpha)$). Then

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(b) We almost surely get a perfect matching

Proof sketch of hypergraph theorems

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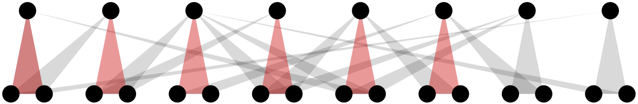
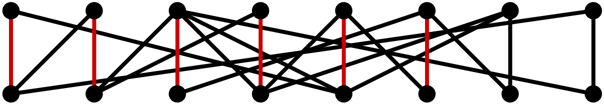
- Greedily find almost all of a perfect matching or Hamilton cycle using only the linearly many random edges
- Use this partial structure to define a map from hypergraphs to bipartite graphs



Proof sketch of hypergraph theorems

- Greedily find almost all of a perfect matching or Hamilton cycle using only the linearly many random edges
- Use this partial structure to define a map from hypergraphs to bipartite graphs



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- To prove the key lemma, prove that every subset "expands"

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Goal. G is a dense bipartite graph on $A \cup B$;
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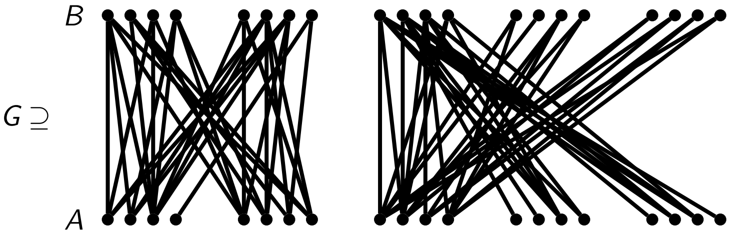
Szemerédi's regularity lemma

Idea. Use a bipartite version of szemeredi's regularity lemma to "classify" the subsets.



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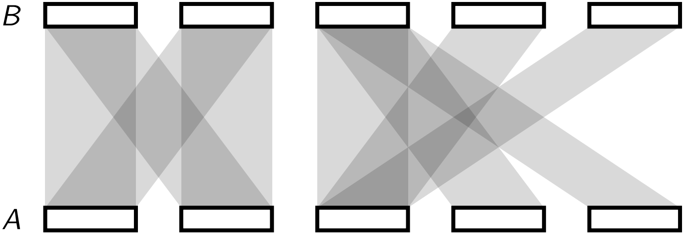

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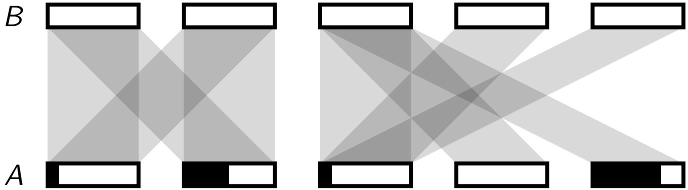

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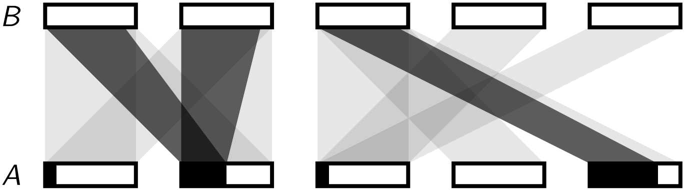

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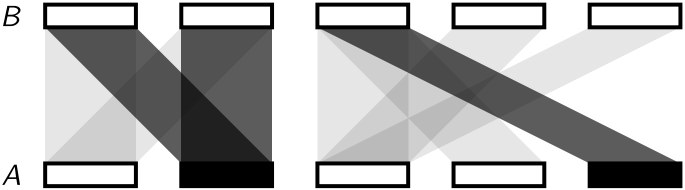

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  - Maybe one can "derandomize" the spanning trees theorem, as with Hamilton cycles.
- We can ask for different types of hypergraph Hamilton cycles, in particular **tight** cycles.
  And we can inpose weaker hypergraph density conditions (minimum 1-degree?)