An Introduction to Tries

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Monash University

21.09.2015

Given: Words, e.g. in binary code

$$\Xi_1 = 11010\ldots, \quad \Xi_2 = 00011\ldots, \quad \Xi_3 = 01101\ldots,$$
$$\Xi_4 = 00000\ldots, \quad \Xi_5 = 11111\ldots, \quad \Xi_6 = 11100\ldots$$

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$\rightarrow$ Use tree-like data structures such as a Trie (Information re*trie*val)

Constructing a Trie

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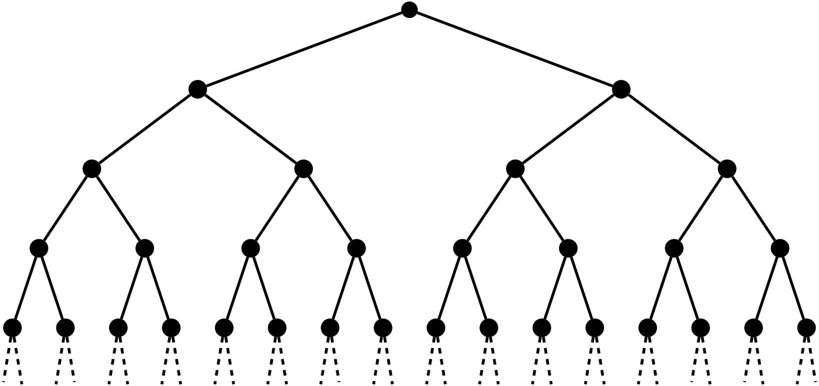

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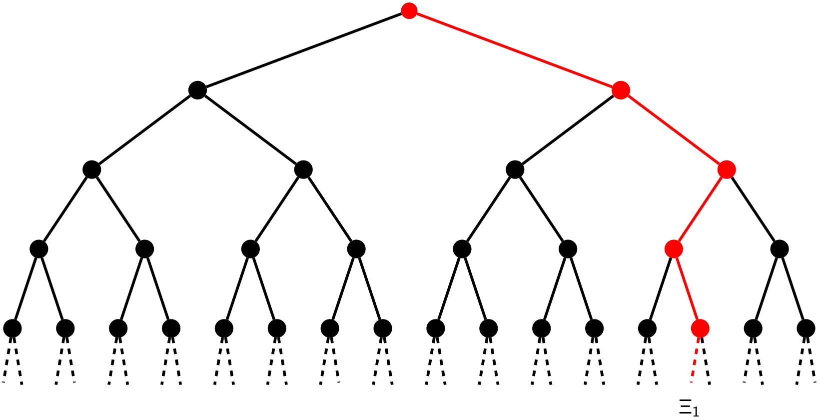

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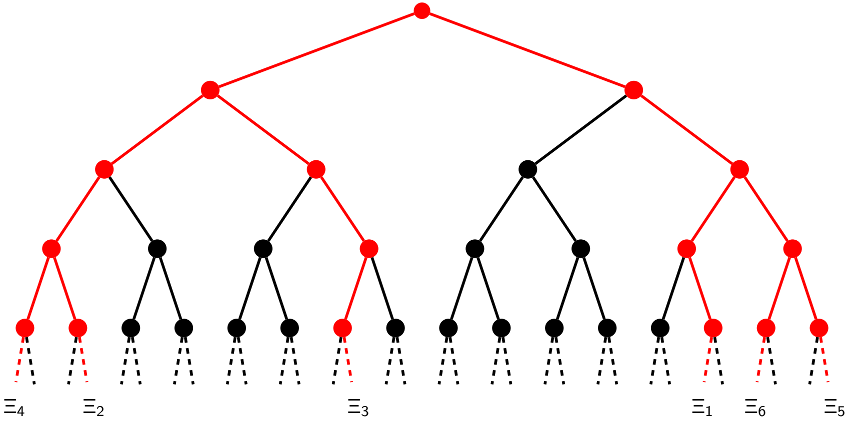

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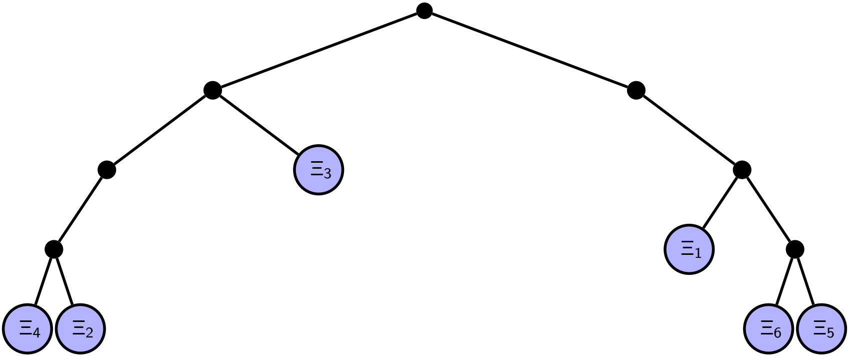
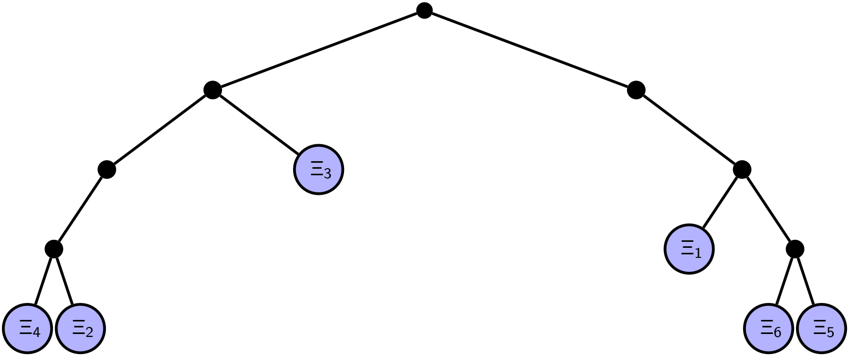


Searching

Search for $\Xi_1 = 11010\ldots$



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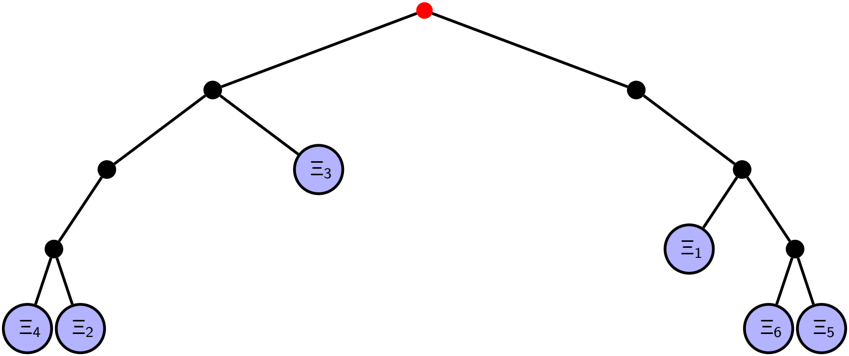

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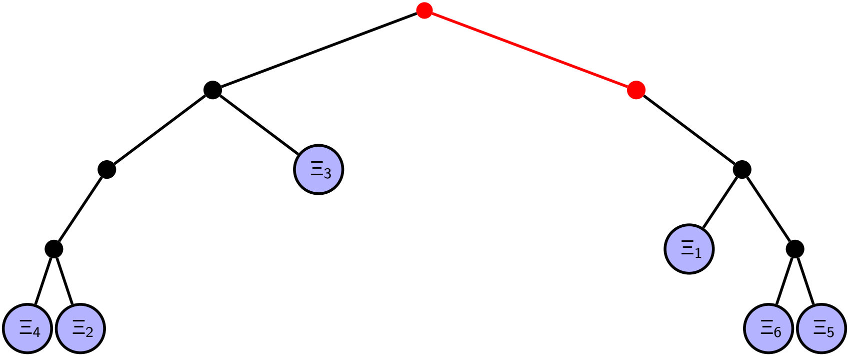

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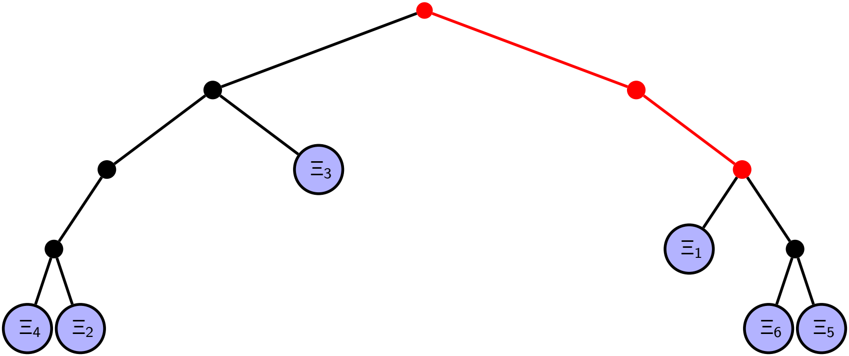

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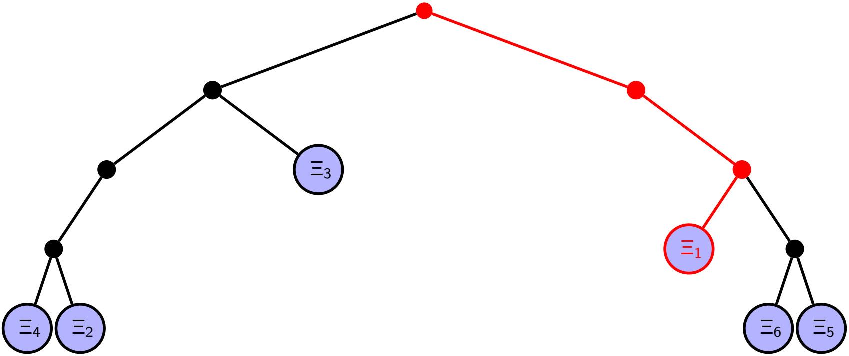

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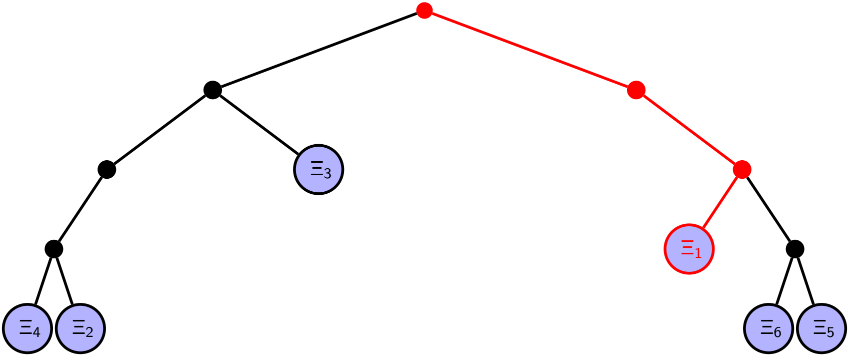

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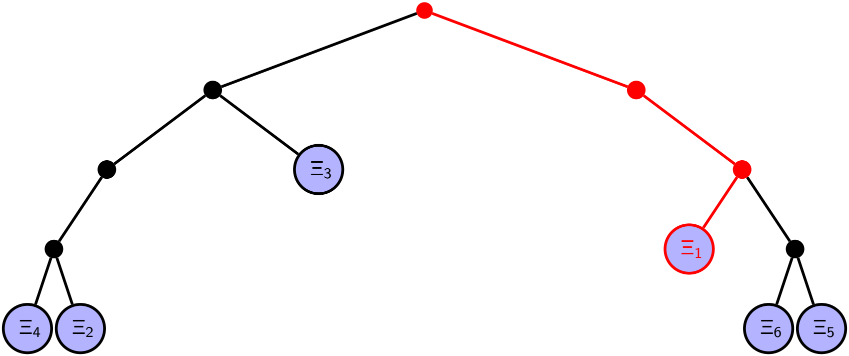

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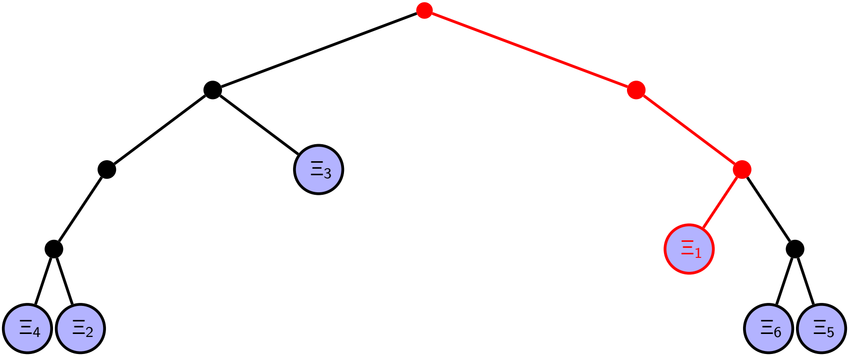

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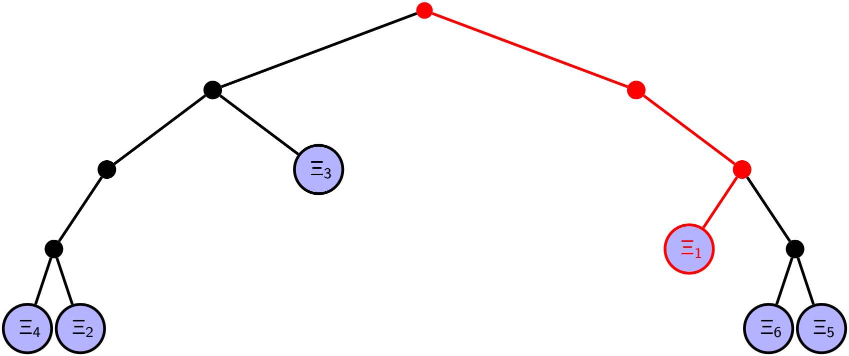

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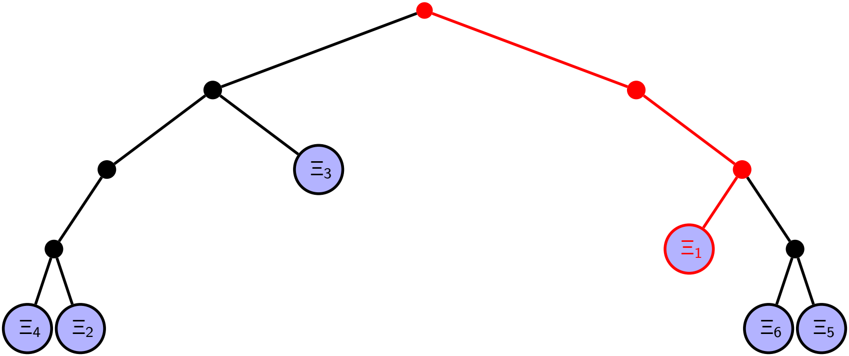

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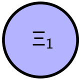
More general models allow $\xi_1, \xi_2, \ldots$ to be dependent
(e.g. evolving as a Markov chain)

A recursive construction of the Trie



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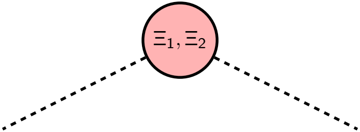

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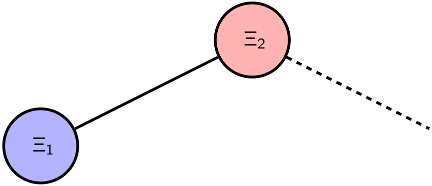

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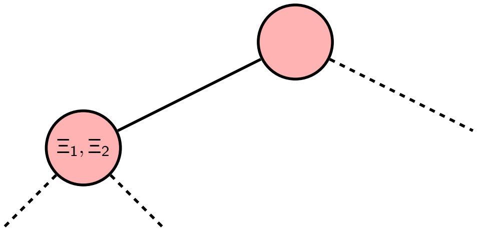

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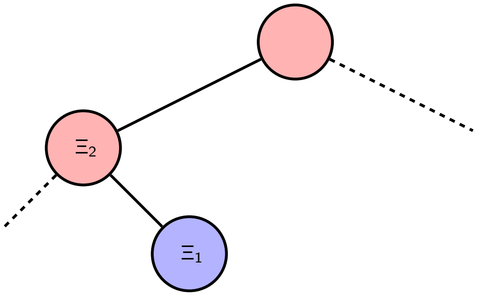

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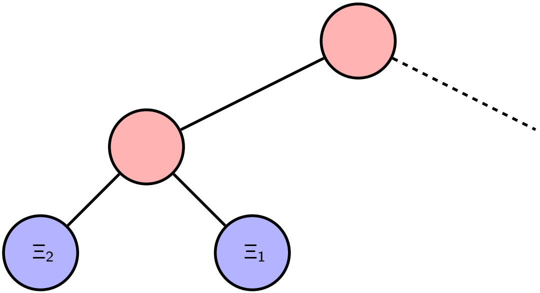

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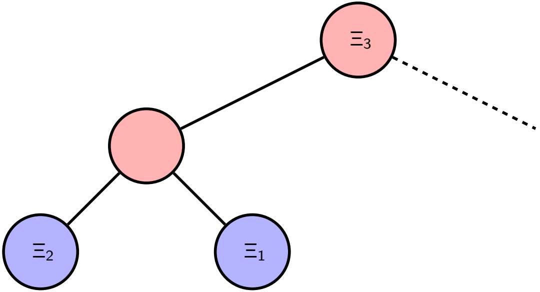

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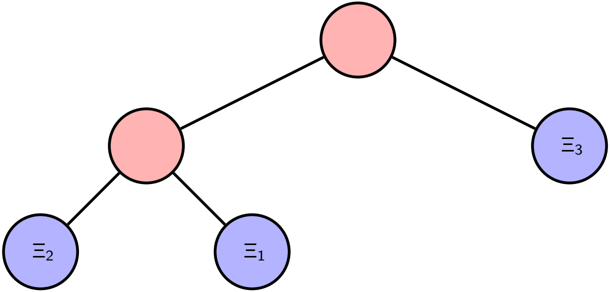

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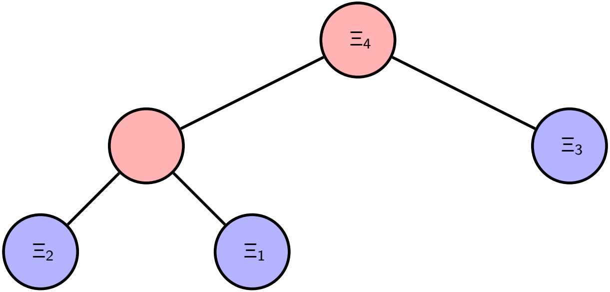

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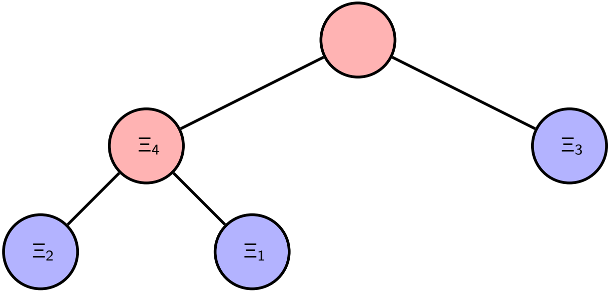

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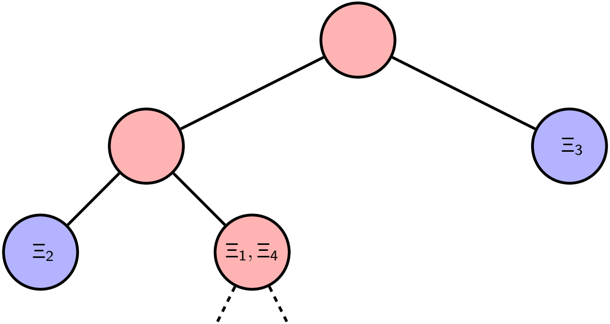

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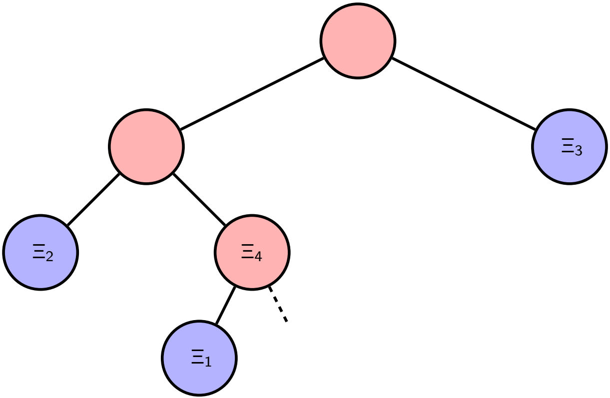

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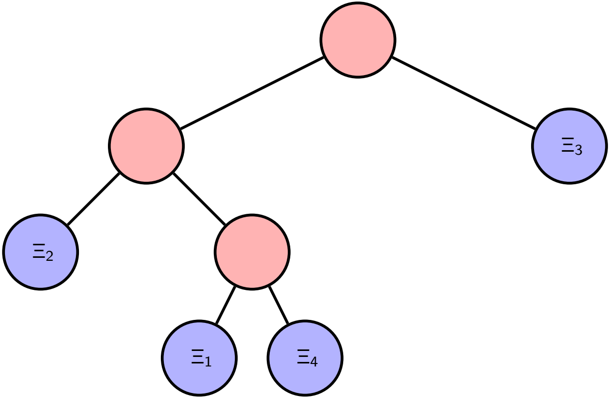

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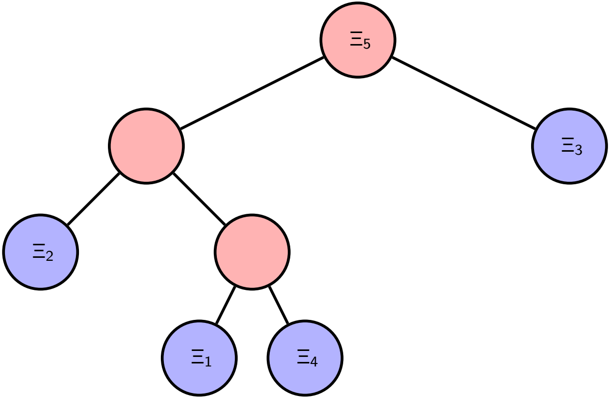

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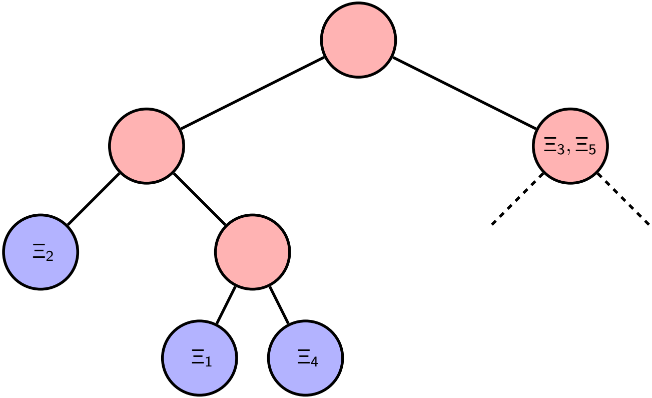

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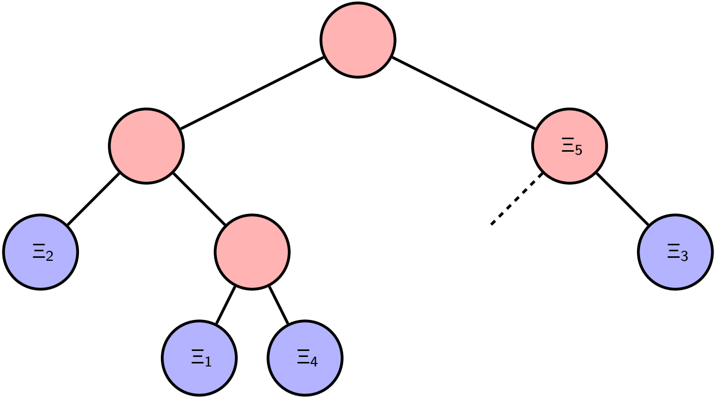

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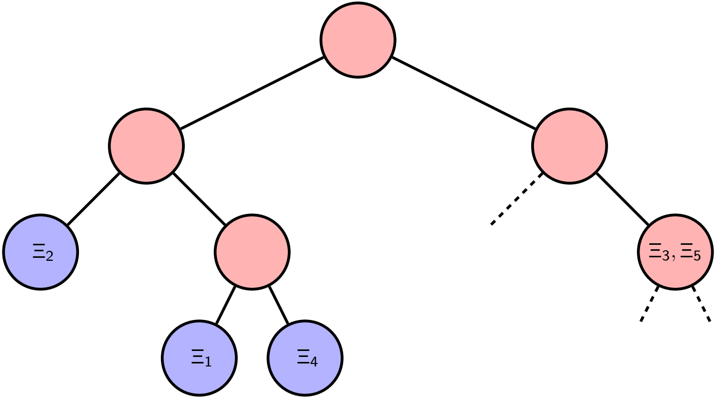

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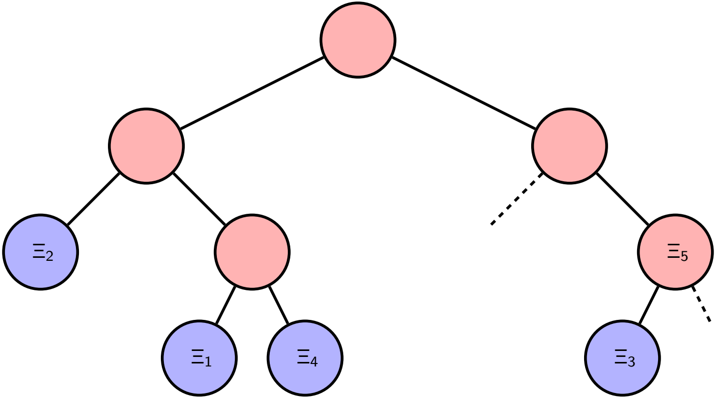

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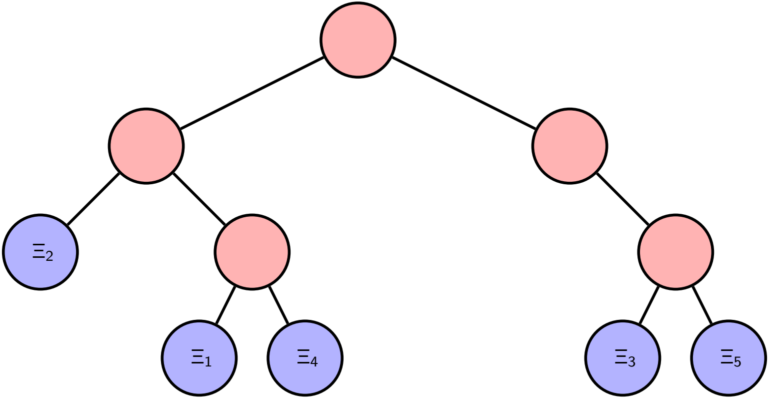

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$$\mathbb{P}(D_n \le \alpha \log_2(n)) = \left(1 - n^{-\alpha}\right)^{n-1} \overset{n\to\infty}{\longrightarrow} \begin{cases} 1, & \text{if } \alpha > 1, \\ 0, & \text{if } \alpha < 1. \end{cases}$$

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- **Thm (Szpankowski '86):** $\mathrm{Var}(D_n) \sim \Phi(\log_2(n))$ with periodic function Φ

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Consequence: $\mathbb{P}\left(H_n > \alpha \log_2(n)\right) \to 0$ for $\alpha > 2$

Results on H_n

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$$\frac{H_n}{2\log_2(n)} \xrightarrow{\mathbb{P}} 1$$

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$$\mathbb{E}[H_n] \sim 2\log_2(n) \qquad (n \to \infty)$$

(Flajolet, Steyaert '82 $\rightarrow$ periodic second order term)

Summary: Typical depth: $\log_2(n)$, height: $2\log_2(n)$.

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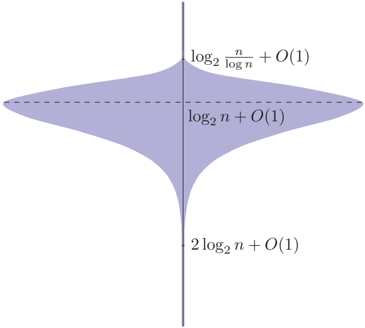
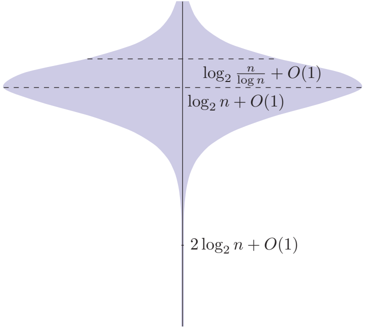
Profile (Park, Hwang, Nicodème, Szpankowski):



(External nodes/Leaves)



(Internal nodes)

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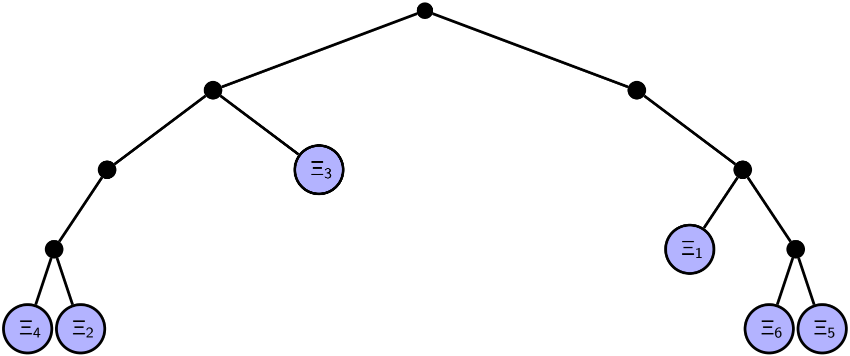

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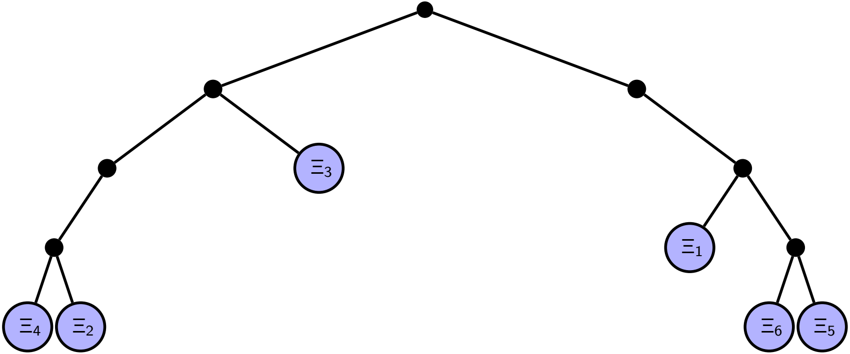
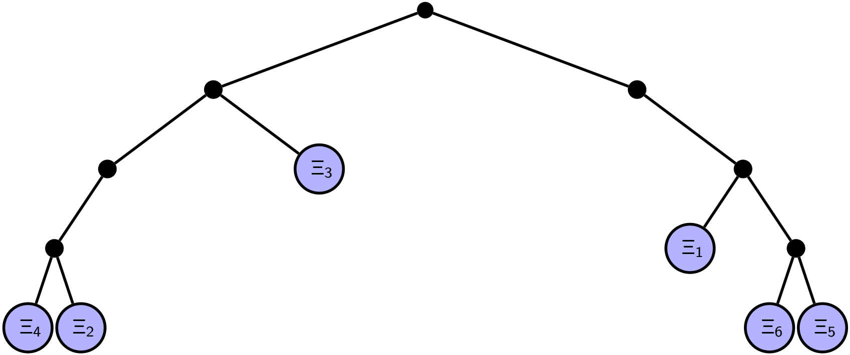


Example: $L_6 = 2 + 3 + 4 \cdot 4 = 21$

A Recursion for L_n



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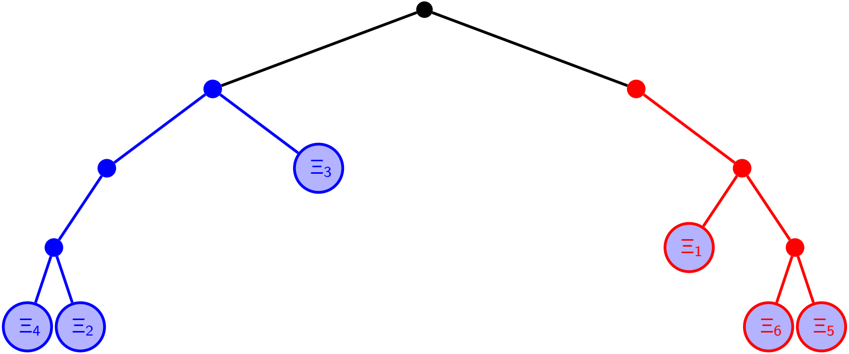

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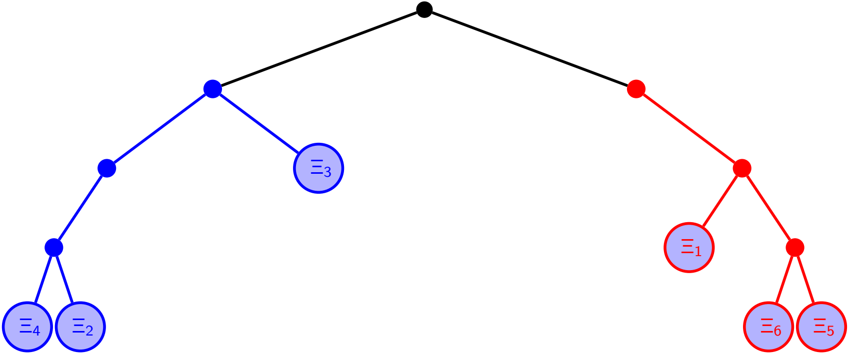

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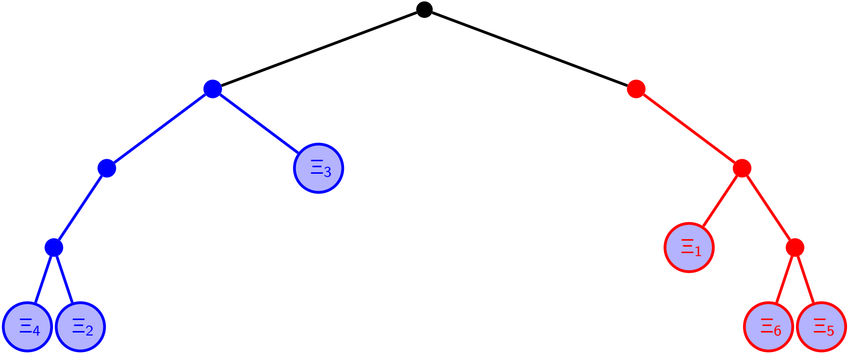

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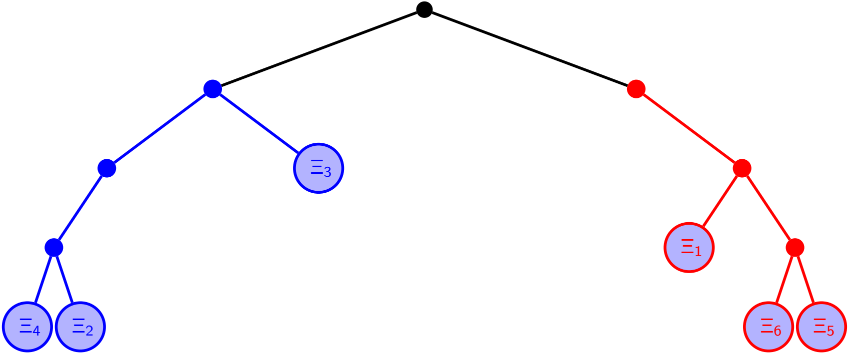

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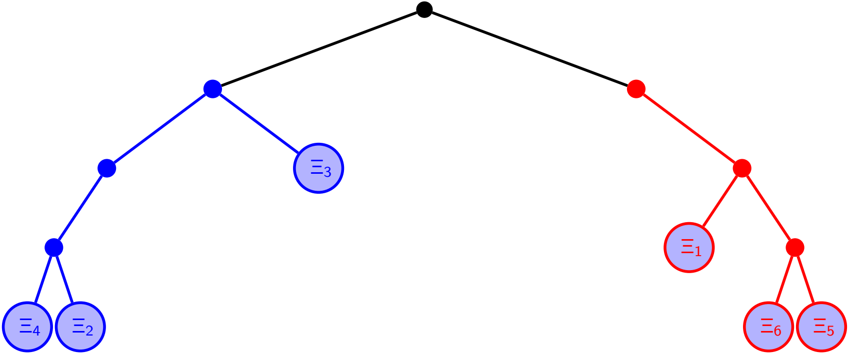

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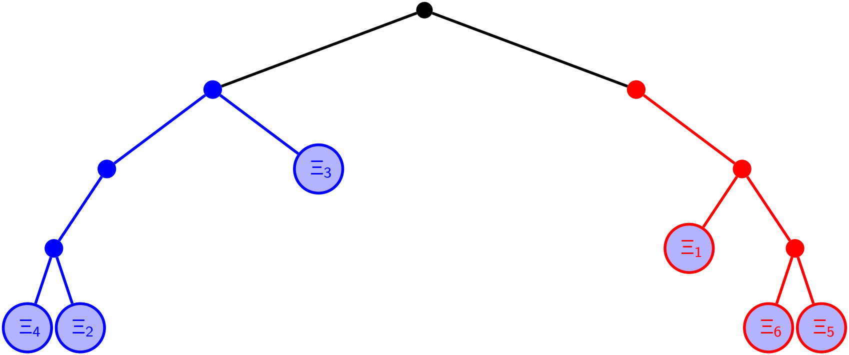

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- Input model not very realistic, what about more general input models?

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Even more general: Dynamical Sources Model by Vallée

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Depth for Markov Sources:

$$\mathbb{E}[D_n] \sim \frac{1}{\mathcal{H}} \log(n)$$

Results for the Markov Source Model

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