On the Erdős-Hajnal conjecture for trees

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joint work with Marcin Pilipczuk,
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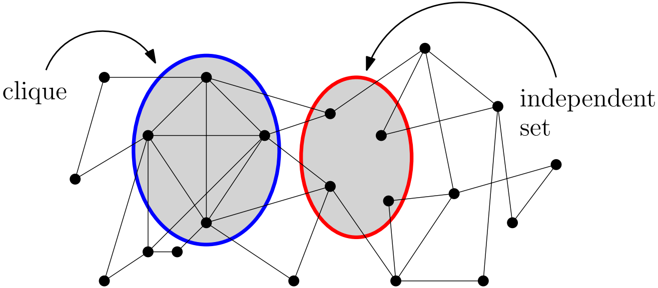
Discrete Maths Seminar 2017

Introduction

Graph G, n vertices

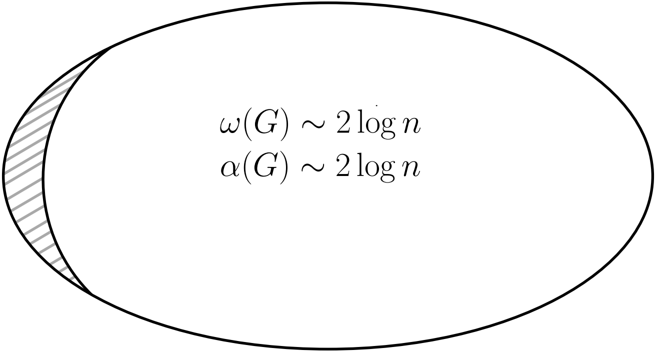


$$\omega(G) = \max\{|S| \ : \ G[S] \text{ is a clique}\}$$

$$\alpha(G) = \max\{|S| \ : \ G[S] \text{ is an independent set}\}$$

Typical graphs

all graphs on n vertices:

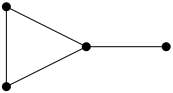


But also: almost all graphs contain all "small" subgraphs.

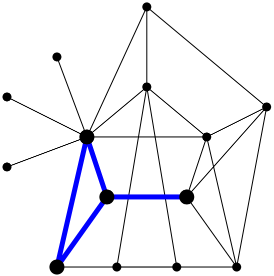
"Containing small subgraphs"

→ G contains H as an induced subgraph

H



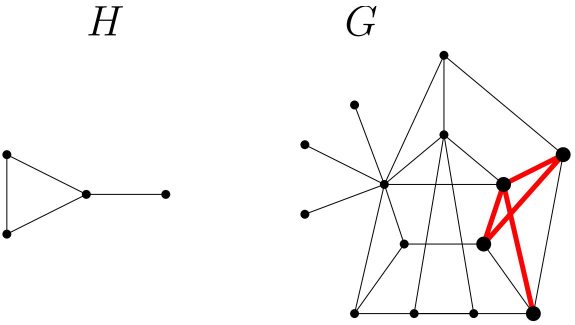
G



Induced copy of H

"Containing small subgraphs"

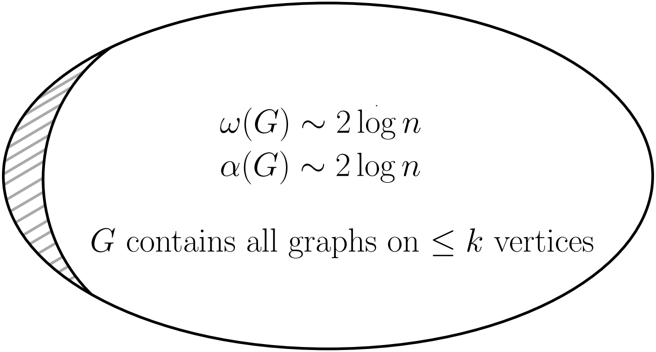
→ G contains H as an induced subgraph



Not an induced copy of H

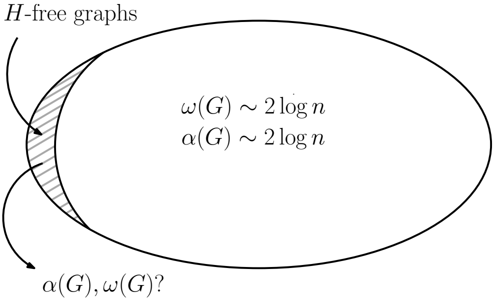
Typical graphs

Fix k. Let n be large. All graphs on n vertices:



H-free graphs

Fix graph H.



- G is H-free if it does not contain H as an induced subgraph
- $\hom(G) = \max\{\alpha(G), \omega(G)\}$

The Erdős-Hajnal conjecture

$\mathrm{hom}(G) = \max\{\alpha(G), \omega(G)\}$

Theorem (Erdős & Hajnal, 1989)

For every graph H there exists a constant $c = c(H)$ such that every H-free graph G on n vertices satisfies

$$\mathrm{hom}(G) \geqslant e^{c(H)\sqrt{\log n}}.$$

Conjecture (Erdős & Hajnal, 1977)

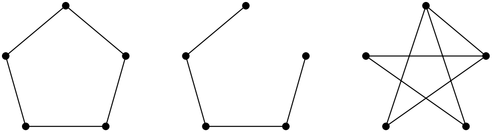
For every graph H there exists a constant $c = c(H)$ such that every H-free graph G on n vertices satisfies

$$\mathrm{hom}(G) \geqslant e^{c(H)\log n} = n^{c(H)}.$$

The Erdős-Hajnal conjecture

is known to be true if

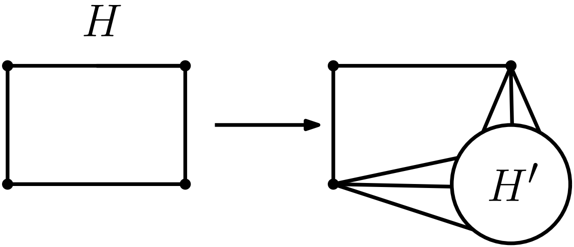
- $H = K_k$ (for every $k \geqslant 2$)
- $v(H) \leqslant 4$
- $v(H) = 5$ and H is not one of those:



- H is obtained through the "Substitution method"

The substitution method

- Alon, Pach, Solymosi (2001)
- H, H' graphs that satisfy the EH conjecture



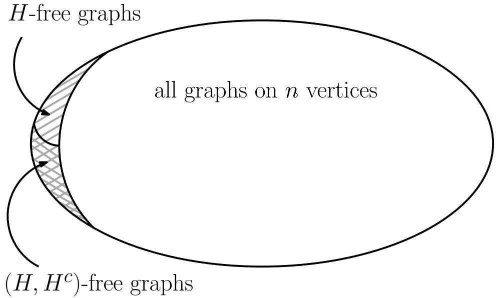
Weakening the conjecture

- forbid both H and H^{c} (the complement) as induced subgraphs

Symmetric EH conjecture (Gyarfas 1997, Chudnovsky 2014)

For every graph H there exists a constant $c = c(H)$ such that every (H, H^{c})-free graph on n vertices satisfies

$$\hom(G) \geqslant n^{c(H)}.$$



Weakening the conjecture

- forbid both H and H^{c} (the complement) as induced subgraphs

Symmetric EH conjecture (Gyarfas 1997, Chudnovsky 2014)

For every graph H there exists a constant $c = c(H)$ such that every (H, H^{c})-free graph on n vertices satisfies

$$\hom(G) \geqslant n^{c(H)}.$$

The symmetric EH conjecture is known to be true for H if

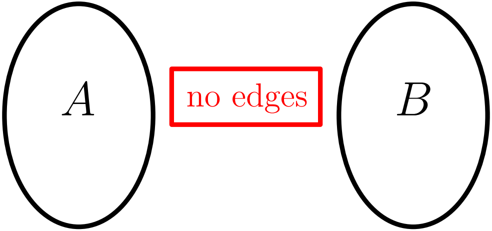
- the EH conjecture is true for H;
- $H = P_k$ (any $k \geqslant 1$; Bousquet, Lagoutte, Thomassé 2015)
- $H = H_k$ (any $k \geqslant 1$; Choromanski, Falik, L, Patel, Pilizcuk 2015+)
- Still open: C_5

Proving something stronger

Strong Sparse EH-property

A graph H has the strong sparse EH-property if there exists $\varepsilon > 0$ such that every H-free graph G on $n \geq 2$ vertices

- either has $\Delta(G) \geqslant \varepsilon n$, or
- there are two disjoint sets $A, B \subseteq V(G)$ such that $E(A, B) = \emptyset$ and $|A|, |B| \geqslant \varepsilon n$.



Proving something stronger

Sparse Strong EH-property

A graph H has the sparse strong EH-property if there exists $\varepsilon > 0$ such that every H-free graph G on $n \geq 2$ vertices

- either has $\Delta(G) \geqslant \varepsilon n$, or
- there are two disjoint sets $A, B \subseteq V(G)$ such that $E(A, B) = \emptyset$ and $|A|, |B| \geqslant \varepsilon n.$

- Sparse strong EH-property $\Longrightarrow$ symmetric EH conjecture.
- H has sparse strong EH-property $\Longrightarrow$ H is acyclic.
- $H = P_k$ has the sparse strong EH-property (Bousquet, Lagoutte, Thomassé 2015)
- $H = H_k$ has the sparse strong EH-property (Choromanski, Falik, L, Patel, Pilizcuk 2015+)

Symmetric EH for trees

Conjecture

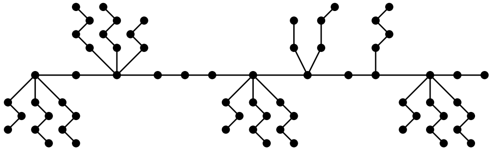
A graph H has the sparse strong EH-property $\iff$ H is a forest.

Symmetric EH for trees

Conjecture

A graph H has the sparse strong EH-property $\iff$ H is a forest.

- A caterpillar subdivision is a tree in which all vertices of degree $\geqslant 3$ lie on a common path.



Theorem (L, Pilipzcuk, Seymour, Spirkl 2017+)

Every caterpillar subdivision has the sparse strong EH-property.