Oriented cycle game

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joint work with Dennis Clemens
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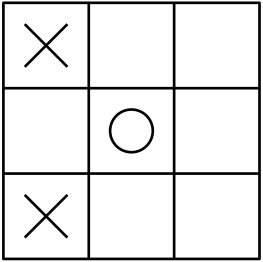
Combinatorial games

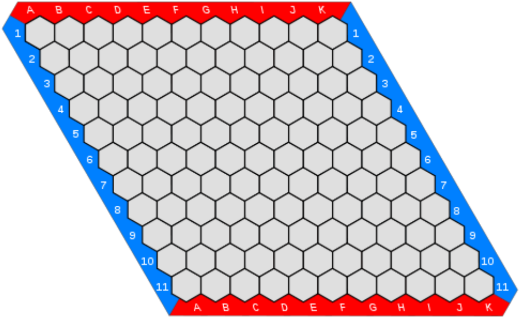
- Tic-tac-toe (Noughts & Crosses)



- Generalizations: $[n] \times [n]$, $[n]^d$

Combinatorial games

- Hex



Combinatorial games

- (usually) finite boards
- 2 players
- perfect information
  $\rightarrow$ "trivial" in classical Game Theory
- "Combinatorial chaos":
  - $[n]^d$ (fixed n, large d) is a First-player win
  - Hex is a First-player win
  - proofs for both are existential
  - no explicit strategy known, even for 11×11 board in Hex

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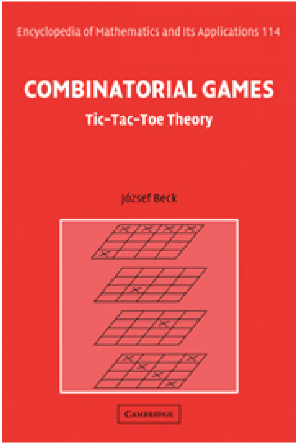
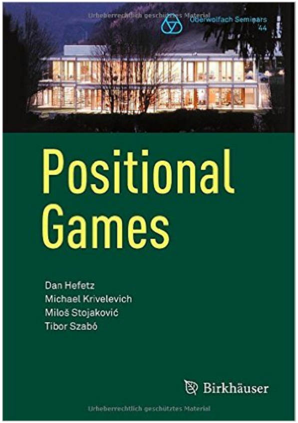
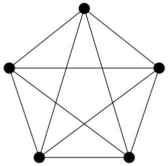




Orientation games

- 2 players: OMaker (First) and OBreaker (Second)
- alternately direct edges of K_n



→ tournament on n vertices

- OMaker wins if tounament has some predefined property $\mathcal{P}$.
- Otherwise, OBreaker wins.

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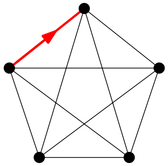

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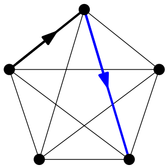

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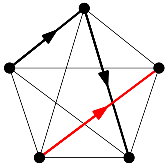

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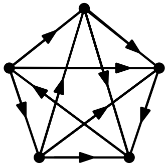

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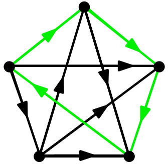

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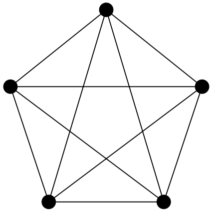
Oriented cycle game

- Final tournament contains a directed cycle
  - $\leftrightarrow$ Final tournament contains a cyclic triangle
- OBreaker tries to achieve transitive tournament



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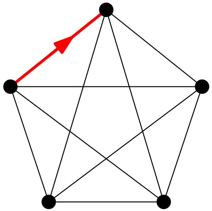

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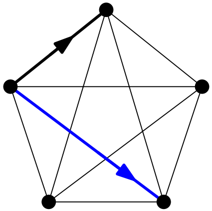

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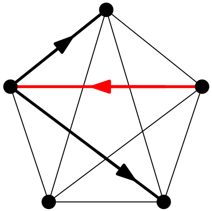

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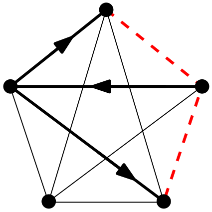

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Oriented cycle game

→ More power for OBreaker : direct more edges per round

Definition

(Monotone) $(1; b)$ oriented cycle game:

→ OBreaker directs up to b edges per round

Strict $(1; b)$ oriented cycle game:

→ OBreaker directs exactly b edges per round

Question (Bollobás & Szabó 1998)

What is the largest bias b such that OMaker has a winning strategy in the (strict) $(1; b)$ oriented cycle game?

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Threshold bias for the oriented cycle game

$$t(n) := \max\{b : \text{ OMaker wins the } (1; b) \text{ oriented cycle game}\}$$
$$t^+(n) := \max\{b : \text{ OMaker wins the strict } (1; b) \text{ oriented cycle game}\}$$
$$\rightarrow t(n) \leqslant t^+(n)$$

Theorem (Bollobás & Szabó, 1998)

OMaker wins the $(1; b)$ oriented cycle game for $b \leqslant \lfloor (2 - \sqrt{3})n \rfloor$.

Theorem (Ben-Eliezer, Krivelevich & Sudakov, 2012)

OMaker wins the $(1; b)$ oriented cycle game for $b \leqslant n/2 - 2$.

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OMaker's strategy for $b = n/2 - 2$

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Lemma

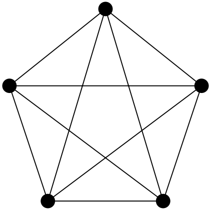
OMaker has a strategy such that for every $t \leqslant n - 1$, the digraph H_t obtained after t rounds contains a directed path of length t.

Strategy:

- Create a Hamilton path in at most $n - 1$ rounds.
- Then direct one "backward edge".

Threshold bias for the oriented cycle game

- $\rightarrow$ $n/2 - 2 \leqslant t(n) \leqslant t^{+}(n) \leqslant n - 3$
- $\rightarrow$ OBreaker strategy for $b \geqslant n - 2$ for the monotone rules:



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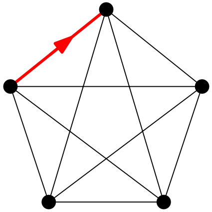

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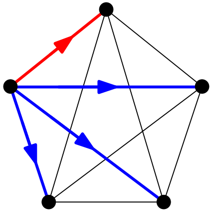

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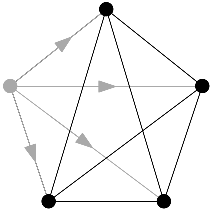

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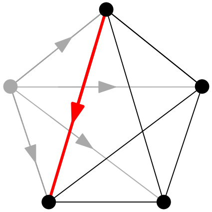

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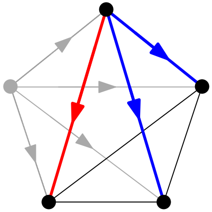

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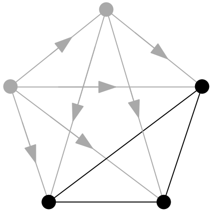

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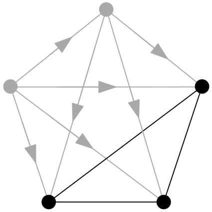

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OBreaker can do better

→ Strategy for OBreaker to build the transitive tournament when $b \geqslant 5n/6 + 2$.

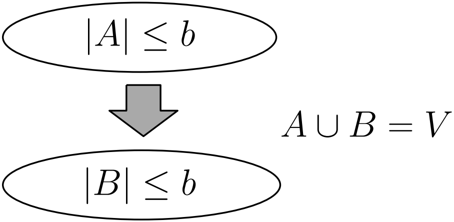
Theorem (Clemens, L 2015+)

- Monotone: $t(n) \leqslant 5n/6 + 1$.
- Strict: $t^{+}(n) \leqslant 19n/20$.

Idea for OBreaker's strategy (monotone rules)

→ strategy for OBreaker to build the transitive tournament

→ Goal: build a *UDB* (uniformly directed biclique):



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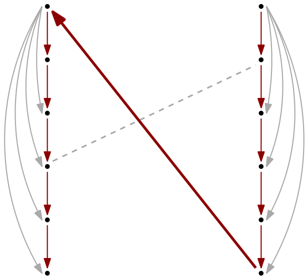


Idea for OBreaker's strategy (monotone rules)

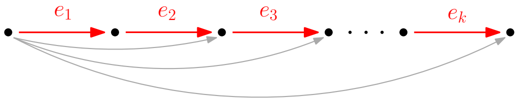
→ $D[V \setminus A]$ and $D[V \setminus B]$ always forms an α-structure

Definition

A digraph D is called an α-structure of rank k if there exist arcs $e_1, \dots, e_k \in D$ such that the map

$$\alpha((i,j)) := (e_i^+, e_j^-)$$

is a surjection $\alpha : \{(i,j) : 1 \leqslant i \leqslant j \leqslant k\} \to D$.



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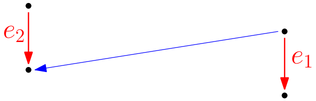
Advantages if, say, $D[V \setminus A]$ forms an α-structure of rank k:

- $D[V \setminus A]$ is acyclic and OMaker cannot close a cycle in $V \setminus A$.
- It takes OBreaker at most k edges to direct to incorporate OMaker's new directed edge (if in $V \setminus A$) into that α-structure.
- Incorporating a new arc increases the rank by 1.

α-structure – example



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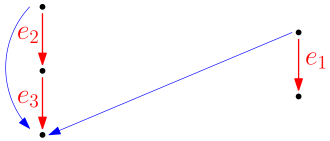

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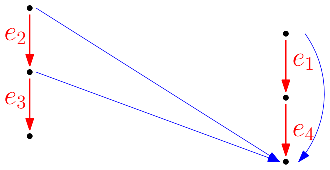

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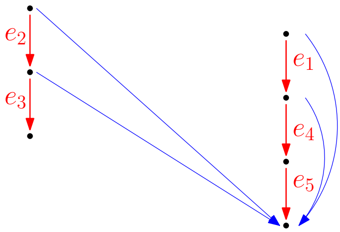

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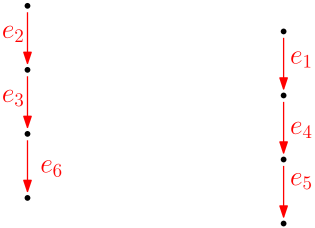

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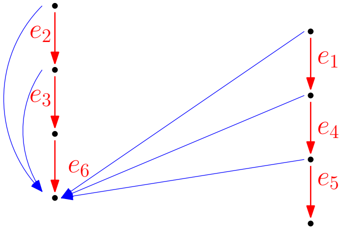

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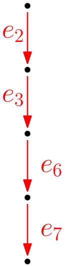

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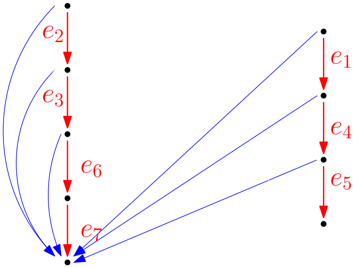

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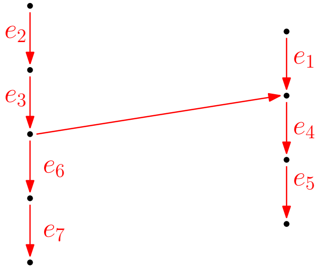

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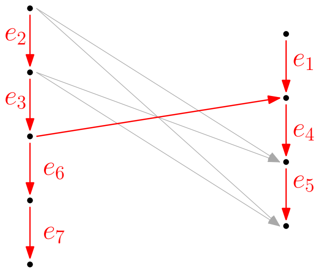

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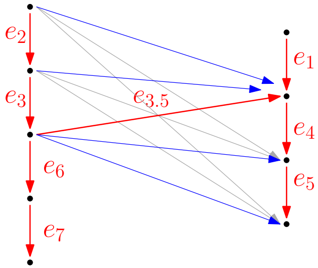

Idea for OBreaker's strategy (monotone rules)

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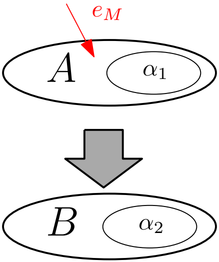
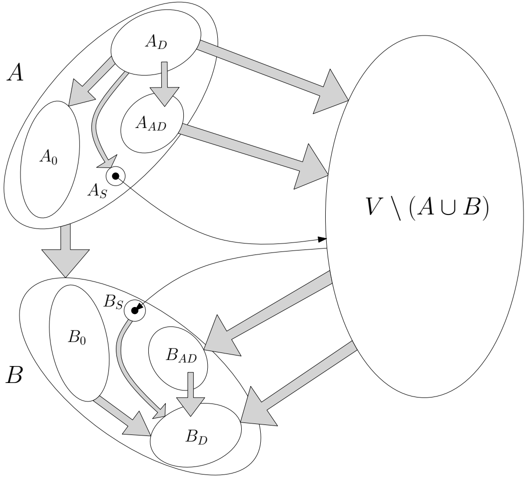


$\rightarrow$ need $\mathrm{rank}(\alpha_1)$ edges to incorporate e_M into α_1

$\rightarrow$ $|B|$ edges to "add e_M to A"

Idea for OBreaker's strategy (strict rules)



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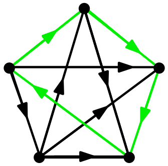

$\to$ tournament on n vertices

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Related properties

- T_n contains a Hamilton cycle
  - $\rightarrow$ $t(n, \mathcal{P}_{Ham}) = \frac{n}{\ln n}(1 + o(1))$
    (Ben-Eliezer, Krivelevich, Sudakov 2012)
- T_n contains a copy of some fixed digraph H on k vertices
  - $\rightarrow$ $t(n, \mathcal{P}_H) = O\left(n^{k/r}\right)$, where r is the minimal number of arcs that have to be deleted in H to make H acyclic (BKS 2012)
- T_n contains some fixed tournament T_k on k vertices
  - $\rightarrow$ How large can k be such that OMaker has a winning strategy in the $(1;1)$ orientation game, independent of the chosen tournament T_k?
  - $\rightarrow$ $2 \log n \leqslant k_t \leqslant 4 \log n$ (Gebauer, Clemens, L 2015+)

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  (Ben-Eliezer, Krivelevich, Sudakov 2012)

- T_n contains a copy of some fixed digraph H on k vertices

  $\rightarrow$ $t(n, \mathcal{P}_H) = O\left(n^{k/r}\right)$, where r is the minimal number of arcs that have to be deleted in H to make H acyclic (BKS 2012)

- T_n contains some fixed tournament T_k on k vertices

  $\rightarrow$ How large can k be such that OMaker has a winning strategy in the $(1;1)$ orientation game, independent of the chosen tournament T_k?
  $\rightarrow$ $2 \log n \leqslant k_t \leqslant 4 \log n$ (Gebauer, Clemens, L 2015+)

Thank you for your attention!