Mixing Times for the Random Cluster Model

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Markov Chains

A Markov chain on state space Ω is a sequence of random elements $X_1, X_2, \ldots$ of Ω such that $\mathbb{P}(X_{t+1} = y | X_t = x) = p_{xy}$.

- The day of the week
- Rolling a dice
- Random walk on a $\mathbb{Z}_n$ (cycle)
- Random walk on S_{52} (card shuffling)

Limiting Distributions

- The day of the week: π does not exist.
- Rolling a dice: $\pi =$ uniform distribution
- Random walk on a $\mathbb{Z}_n$ (cycle): $\pi =$ uniform distribution (only when n is odd)
- Random walk on S_{52} (card shuffling): $\pi =$ uniform distribution

Markov chain Monte Carlo

- Sample from difficult distributions.
- $d(\mu, \nu) = \sup_{A \subseteq \Omega} |\mu(A) - \nu(A)|$
- $\tau_\epsilon = \inf\{t : \forall x \in \Omega, d(\delta_x P^t, \pi) \leq \epsilon\}$

Mixing Times

- The day of the week: no stationary distribution
- Rolling a n-sided die: $\tau_\epsilon = 1$
- Random walk on a $\mathbb{Z}_n$: $\tau_\epsilon \geq n$
- Random walk on S_n: $\tau_\epsilon =??$

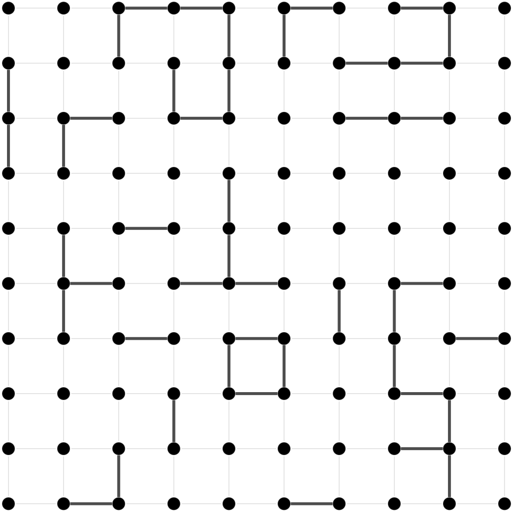
Random Cluster Model (Fortuin and Kastelyn, 1969)

Let $G = (V, E)$ be a graph.

$$\pi^{RC}(A) \propto q^{k(A)} \prod_{ij \in A} p \prod_{ij \notin A} (1-p), \quad A \in 2^E$$

$p \in [0, 1]$ and $q > 0$ are parameters.

Random Cluster Model (Fortuin and Kastelyn, 1969)



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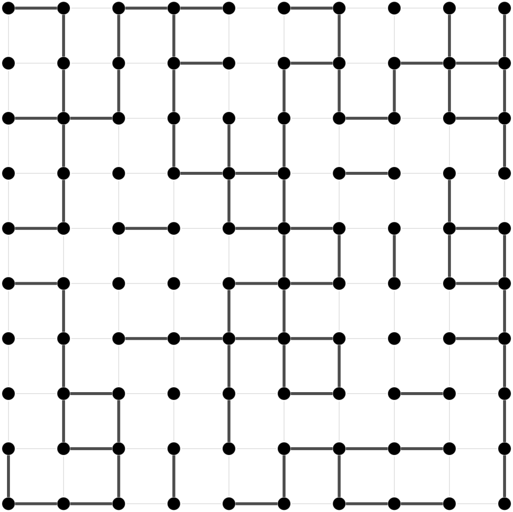

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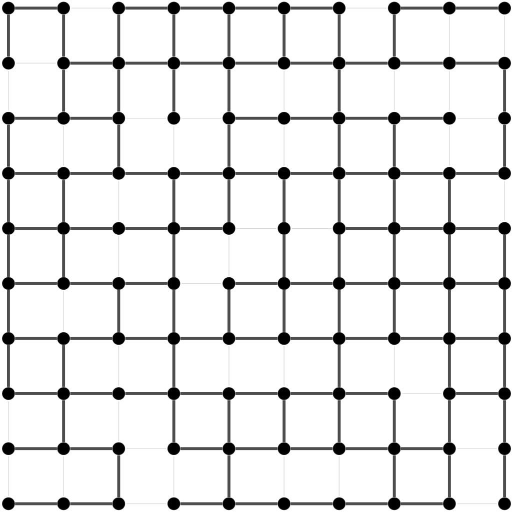

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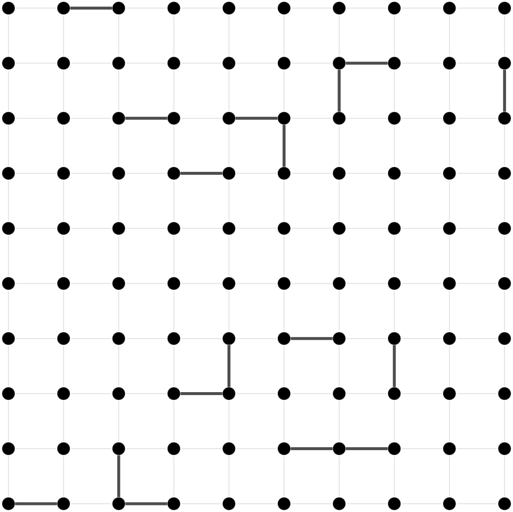

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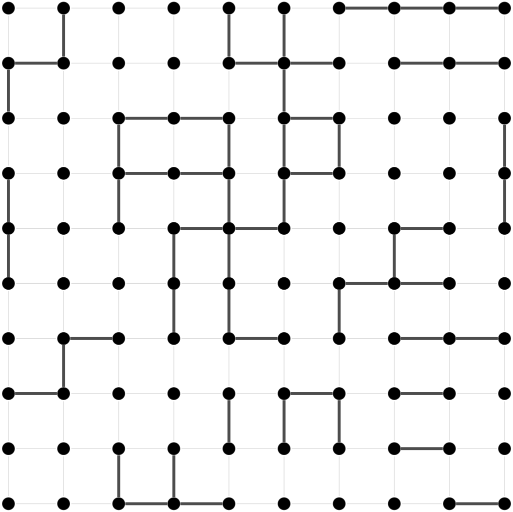

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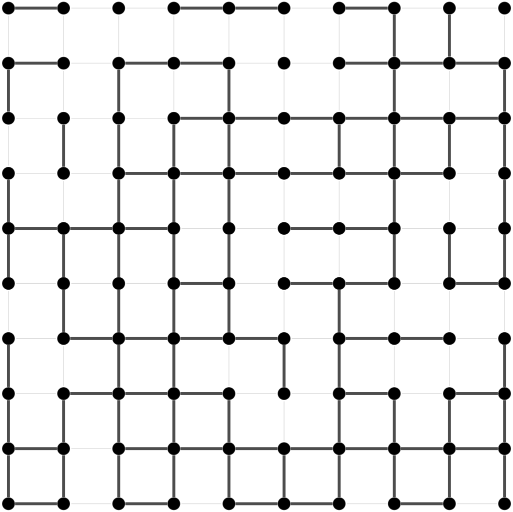


$$\pi^{RC} \propto q^{k(A)} \prod_{ij \in A} p \prod_{ij \notin A} (1 - p), \quad A \in 2^E$$

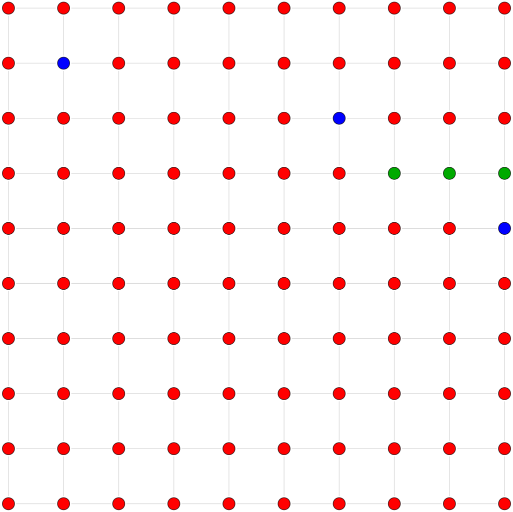
Potts Model (1951)

$$\pi^P(\sigma) \propto \exp[-\mathcal{H}_T(\sigma)], \quad \sigma \in \{1,2,\ldots,q\}^V$$

$$\mathcal{H}_T(\sigma) = \sum_{\substack{ij\in E\\ \sigma_i\neq\sigma_j}} \frac{1}{T}, \quad \sigma \in \{1,2,\ldots,q\}^V$$

$$\pi^P(\sigma) \propto \prod_{\substack{ij\in E\\ \sigma_i\neq\sigma_j}} (1-p), \quad \sigma \in \{1,2,\ldots,q\}^V$$

Potts Model (1951)



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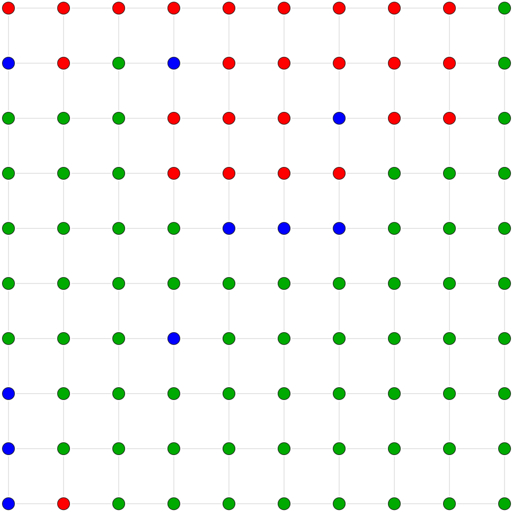

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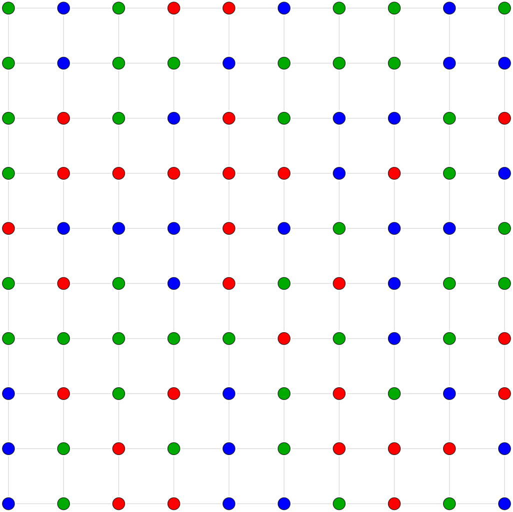

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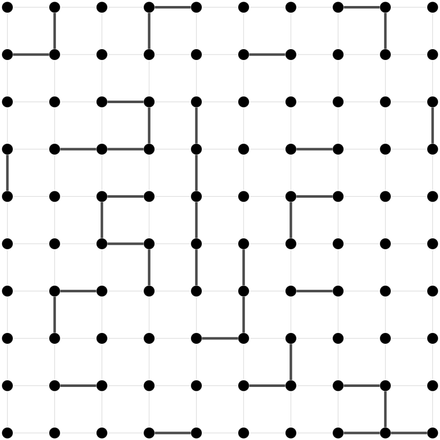
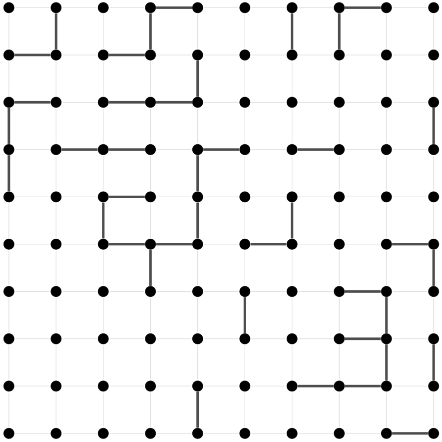
Edwards-Sokal Coupling (1989)

Let $X \sim \pi^{\mathrm{RC}}$.



Edwards-Sokal Coupling (1989)

To each cluster of (V, X_t), assign a colour in $\{1, \ldots, q\}$, uniformly and randomly...



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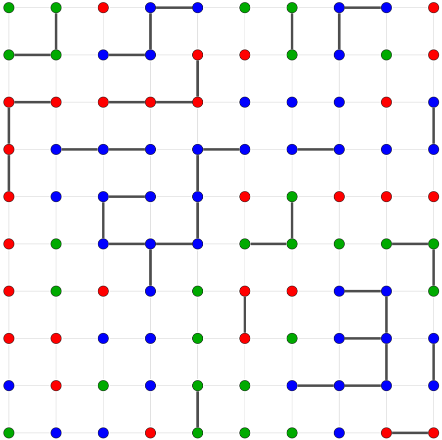
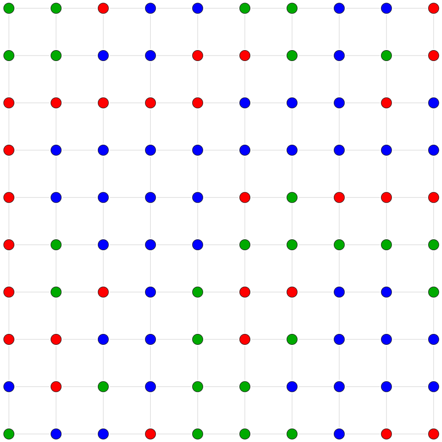


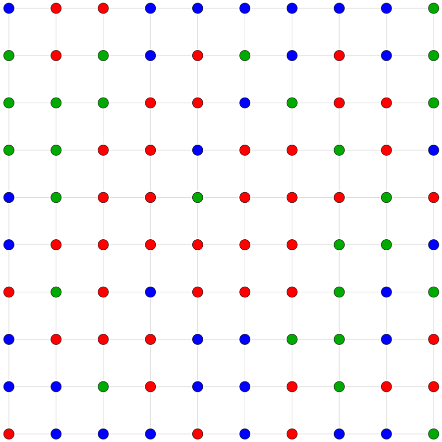
Edwards-Sokal Coupling (1989)

The resulting $\sigma \in \{1, 2, \ldots, q\}^V$ has distribution π^P.



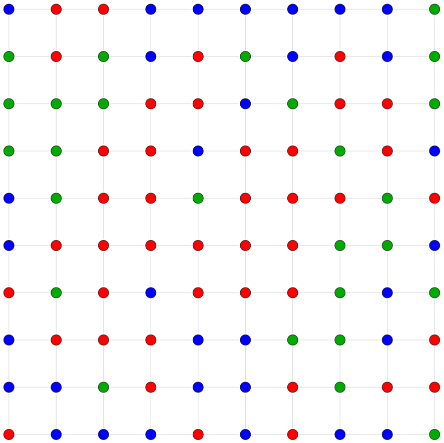
Edwards-Sokal Coupling (1989)

On the other hand, suppose $\sigma \sim \pi^P$.



Edwards-Sokal Coupling (1989)

Draw all edges with same coloured endpoints...



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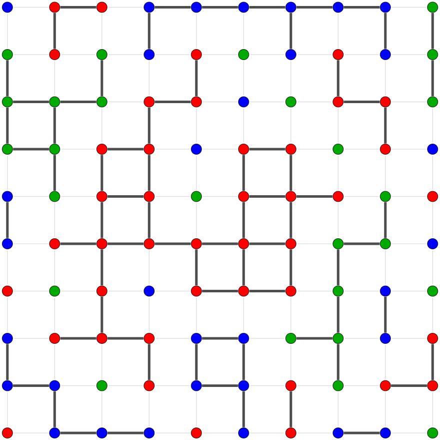
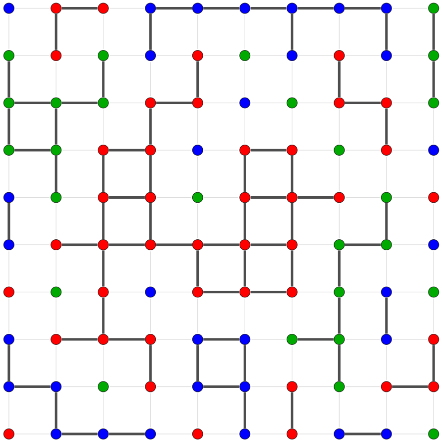


Edwards-Sokal Coupling (1989)

Keep each edge with probability p.



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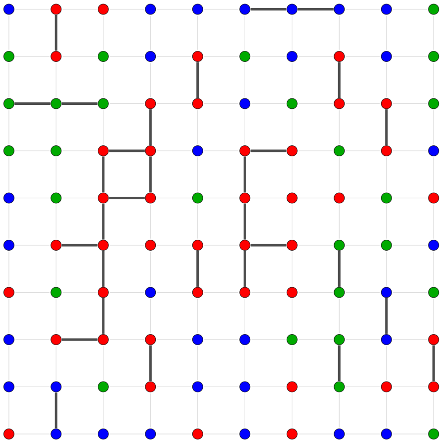
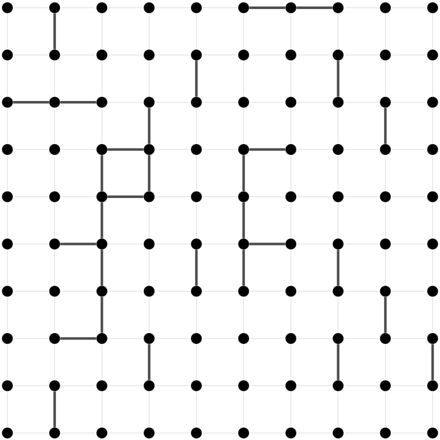


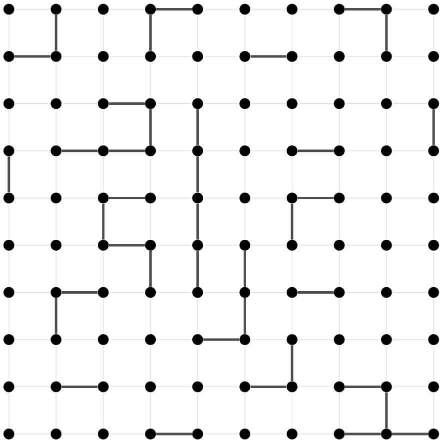
Edwards-Sokal Coupling (1989)

The result is an element of 2^E distributed like π^{RC}.



Swendsen-Wang Chain (1987)

Current state is $X_t \subseteq E$.



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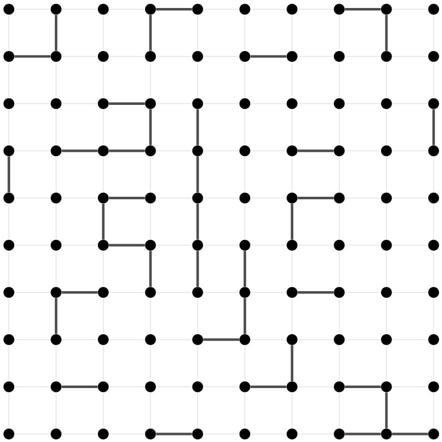
To each cluster of (V, X_t), assign a colour in $\{1, \ldots, q\}$, uniformly and randomly.



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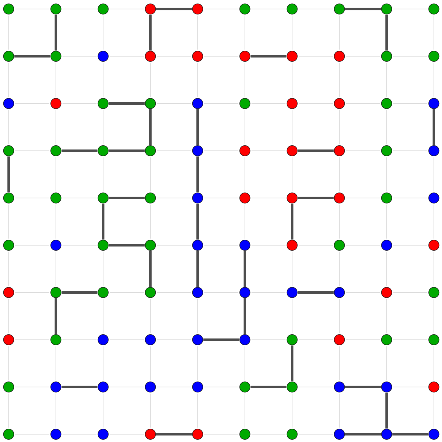
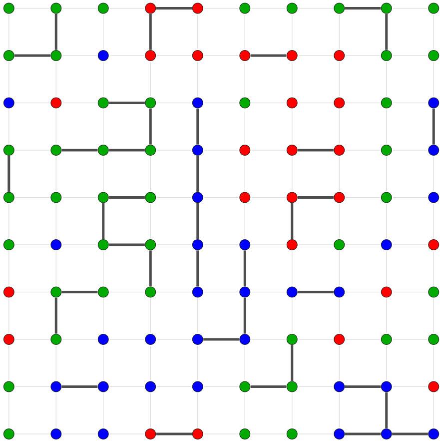


Swendsen-Wang Chain (1987)

Add in edges with the same colour.



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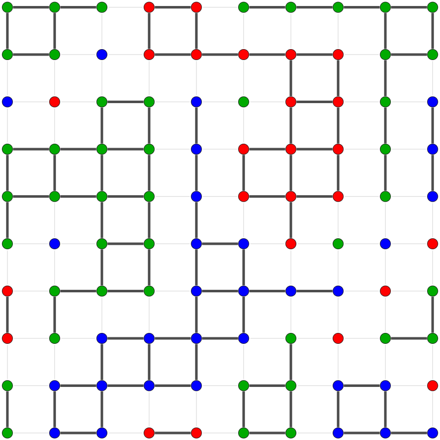

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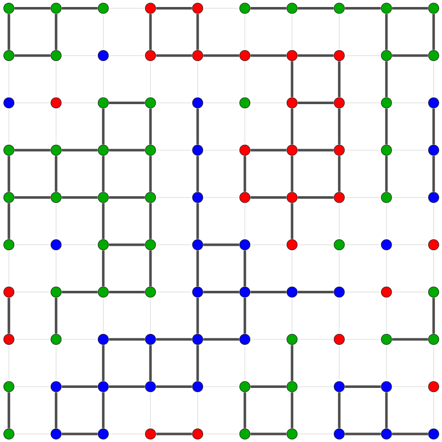

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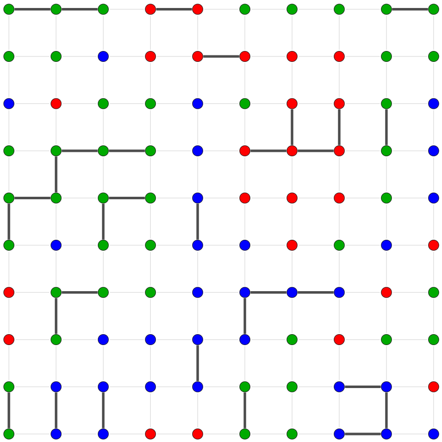
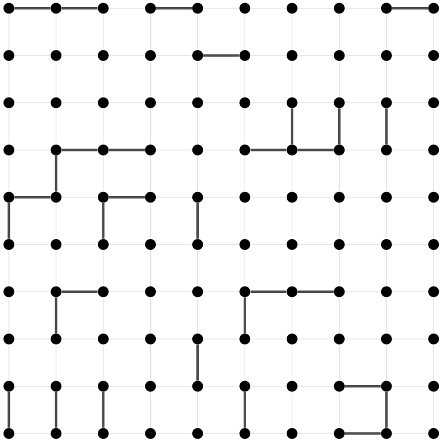


Swendsen-Wang Chain (1987)

New state X_{t+1}:

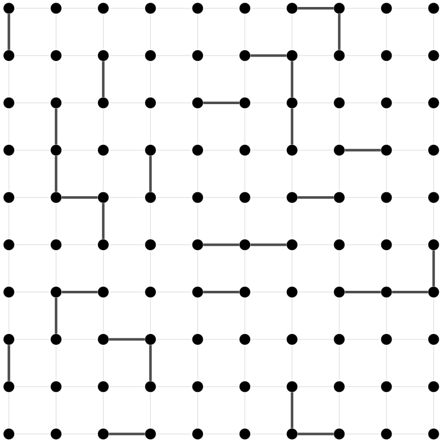


- What if q is non-integer?
- Set $q \in \{1, 2, \dots\}$ to be the integer part and $\delta > 0$ to be the non-integer part.

$$\pi^{RC}(A) \propto (q+\delta)^{k(A)} \prod_{ij \in A} p \prod_{ij \notin A} (1-p), \quad A \in 2^E$$

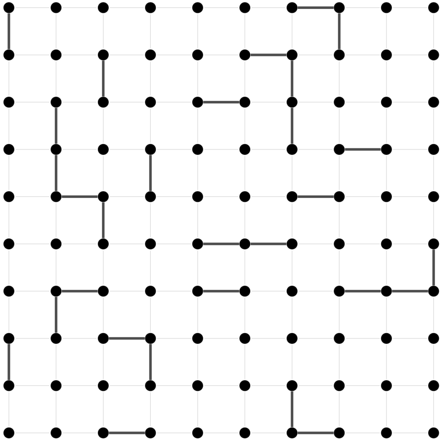
Chayes-Machta Chain (1996)

Current state is $X_t \subseteq E$.



Chayes-Machta Chain (1996)

To each cluster of (V, X_t), assign a colour in $\{1, \ldots, q\}$, uniformly and randomly, with probability $1 - \frac{\delta}{q+\delta}$.



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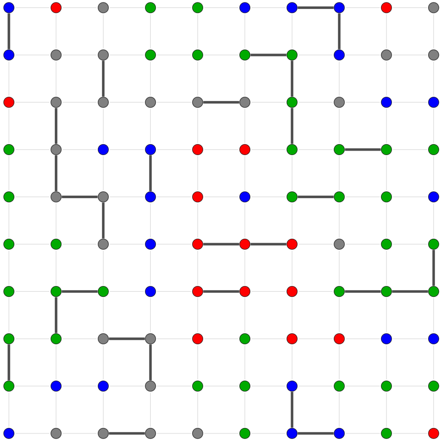
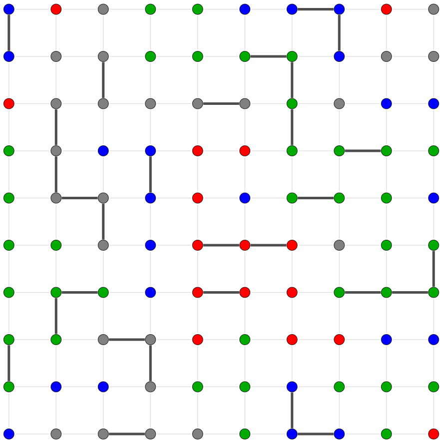


Chayes-Machta Chain (1996)

Add in the edges that have endpoints with the same colour.



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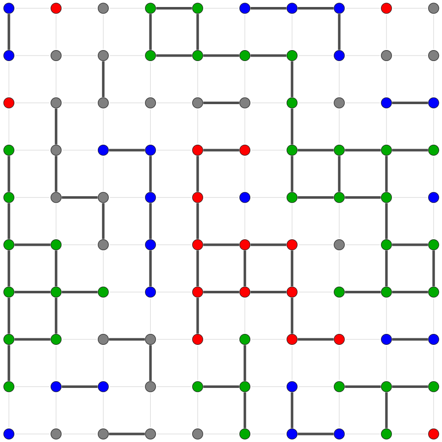

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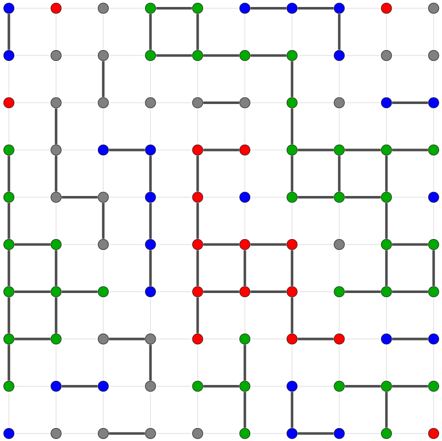

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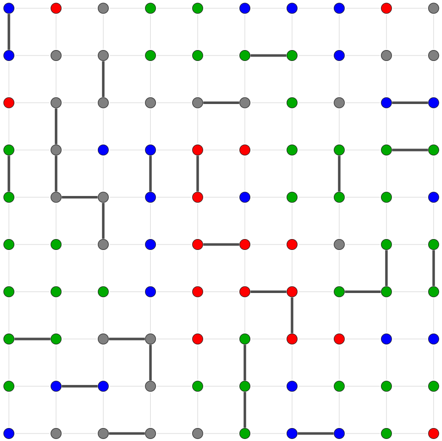
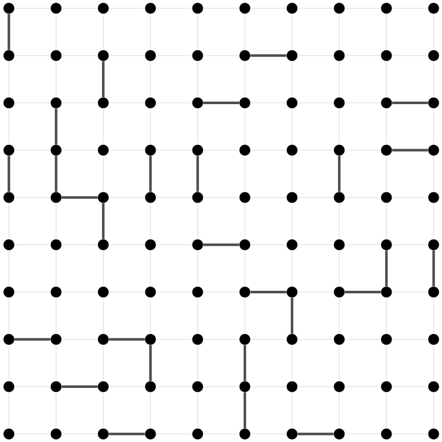
Keep each edge with probability p.



Chayes-Machta Chain (1996)

New state X_{t+1}:



Lower bounds for the Swendsen-Wang Chain

Theorem (Li,Sokal 1988)

On a lattice $[1, \ldots, L]^d$, at $p = p_c$, the mixing time of the Swendsen-Wang chain is bounded below by

$$\tau_\epsilon \geq C(\epsilon) C_H$$

Here C_H is the *specific heat* and $C_H \sim L^{\alpha/\nu}$. Proof Idea:

If

$$\frac{\sum_{x,y \in \Omega} \pi(x) P(x \to y) |f(y)^2 - f(x)^2|}{\mathrm{Var}_\pi f}$$

is small for some $f : \Omega \to \mathbb{R}$, then the chain mixes slowly. Taking $f(X) := |X|$ gives the result.

Extends to $\delta > 0$ case.

Rapid Mixing for the Swendsen-Wang Chain

Theorem (Huber, 2004)

For $q \in \{1, 2, \dots\}$, $\delta = 0$, if G has bounded degree Δ, and $p \leq \frac{1}{2\Delta - 2}$, then $\tau_\epsilon = O(\log |E|)$.

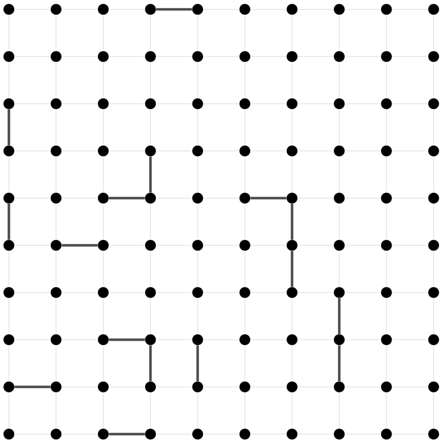
Theorem (Huber, 2004)

For $q \in \{1, 2, \dots\}$, $\delta = 0$, and G a tree, $\tau_\epsilon = O(\log |E|)$.

Does not extend to Chayes-Machta ($\delta > 0$).

Single Bond Chain

To each cluster of (V, X_t), assign a colour from $\{1, \dots, q\}$ uniformly and randomly, with probability $1 - \frac{\delta}{q+\delta}$.



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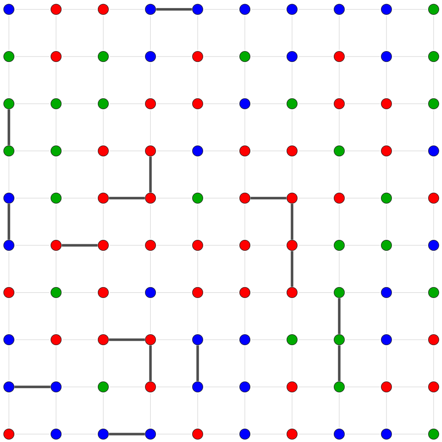
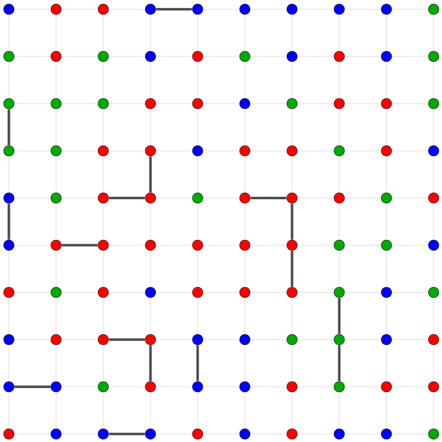


Single Bond Chain

Choose an edge $e \in E$ uniformly at random.



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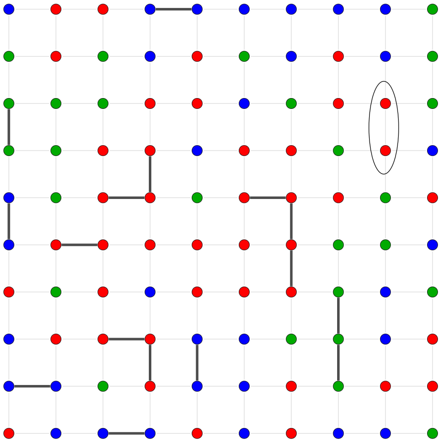
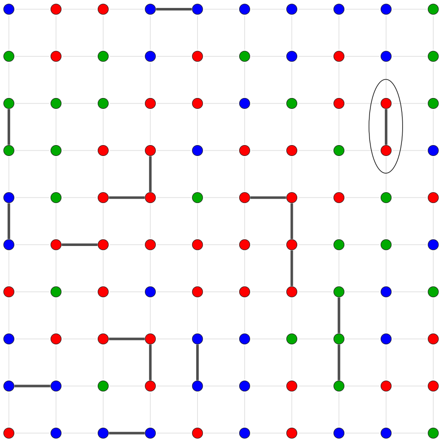


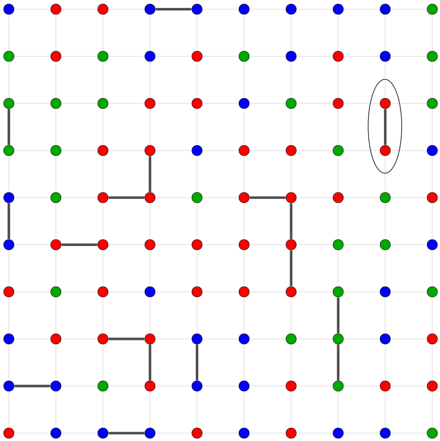
Single Bond Chain

Add the edge e if it has the same coloured endpoints.



Single Bond Chain

Keep the edge e with probability p.



Single Bond Chain

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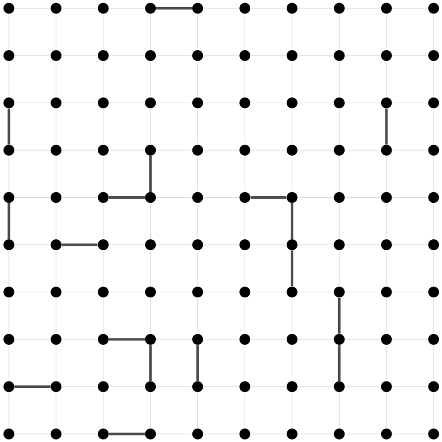

Comparing the Chayes-Machta Chain to Single Bond Chain

Single Bond chain

1. Current state is X_t.
2. Choose random edge $e \in E$.
3. If $e_1 \overset{X_t}{\leftrightarrow} e_2$, keep e with probability p.
4. If $e_1 \overset{X_t}{\not\leftrightarrow} e_2$, keep e with probability $p/(q+\delta)$.

Theorem (Ullrich, 2012)

$\tau_{SW} \leq poly(|E|) \iff \tau_{SB} \leq poly(|E|)$.

Comparing the Chayes-Machta Chain to Single Bond Chain

Theorem

$\tau_{CM} \leq poly(|E|) \iff \tau_{SB} \leq poly(|E|)$.

Theorem

$\tau_{SB} \leq poly(|E|)$ *on tree graphs.*

Corollary

$\tau_{CM} \leq poly(|E|)$ *on tree graphs.*

Further Work

- Prove things about the single bond chain.