Mixing Times for the Random Cluster Model

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A Markov chain on state space $\Omega$ is a sequence of random elements $X_1, X_2, \ldots$ of $\Omega$ such that $\mathbb{P}(X_{t+1} = y | X_t = x) = p_{xy}$.

- The day of the week
- Rolling a dice
- Random walk on a $\mathbb{Z}_n$ (cycle)
- Random walk on $S_{52}$ (card shuffling)
The day of the week: $\pi$ does not exist.

Rolling a dice: $\pi = \text{uniform distribution}$

Random walk on a $\mathbb{Z}_n$ (cycle): $\pi = \text{uniform distribution}$ (only when $n$ is odd)

Random walk on $S_{52}$ (card shuffling): $\pi = \text{uniform distribution}$
Markov chain Monte Carlo

- Sample from difficult distributions.

\[ d(\mu, \nu) = \sup_{A \subseteq \Omega} |\mu(A) - \nu(A)| \]

\[ \tau_\epsilon = \inf \{ t : \forall x \in \Omega, d(\delta_x P^t, \pi) \leq \epsilon \} \]
Mixing Times

- The day of the week: no stationary distribution
- Rolling a $n$-sided die: $\tau_\epsilon = 1$
- Random walk on a $\mathbb{Z}_n$: $\tau_\epsilon \geq n$
- Random walk on $S_n$: $\tau_\epsilon = ??$
Let $G = (V, E)$ be a graph.

$$
\pi^{RC}(A) \propto q^{k(A)} \prod_{ij \in A} p \prod_{ij \notin A} (1 - p), \quad A \in 2^E
$$

$p \in [0, 1]$ and $q > 0$ are parameters.
\[ \pi^{RC} \propto \prod_{ij \in A} p \prod_{ij \notin A} (1 - p), \quad A \in 2^E \]
Random Cluster Model (Fortuin and Kastelyn, 1969)

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\]
Random Cluster Model (Fortuin and Kastelyn, 1969)

\[ \pi^{RC} \propto q_k(A) \prod_{ij \in A} p \prod_{ij \notin A} (1 - p), \quad A \in 2^E \]
Potts Model (1951)

\[ \pi^P(\sigma) \propto \exp[-\mathcal{H}_T(\sigma)], \quad \sigma \in \{1, 2, \ldots, q\}^V \]

\[ \mathcal{H}_T(\sigma) = \sum_{\substack{ij \in E \\sigma_i \neq \sigma_j}} \frac{1}{T}, \quad \sigma \in \{1, 2, \ldots, q\}^V \]

\[ \pi^P(\sigma) \propto \prod_{\substack{ij \in E \\sigma_i \neq \sigma_j}} (1 - p), \quad \sigma \in \{1, 2, \ldots, q\}^V \]
Potts Model (1951)

\[ \pi^P(\sigma) \propto \prod_{ij \in E, \sigma_i \neq \sigma_j} (1 - p), \quad \sigma \in \{1, 2, \ldots, q\}^V \]
Potts Model (1951)

\[
\pi^P(\sigma) \propto \prod_{ij \in E, \sigma_i \neq \sigma_j} (1 - p), \quad \sigma \in \{1, 2, \ldots, q\}^V
\]
Potts Model (1951)

\[
\pi^P(\sigma) \propto \prod_{ij \in E, \sigma_i \neq \sigma_j} \left(1 - p\right), \quad \sigma \in \{1, 2, \ldots, q\}^V
\]
Edwards-Sokal Coupling (1989)

Let $X \sim \pi^{RC}$. 
Edwards-Sokal Coupling (1989)

To each cluster of \((V, X_t)\), assign a colour in \(\{1, \ldots, q\}\), uniformly and randomly...
Edwards-Sokal Coupling (1989)

To each cluster of $(V, X_t)$, assign a colour in $\{1, \ldots, q\}$, uniformly and randomly...
Edwards-Sokal Coupling (1989)

The resulting $\sigma \in \{1, 2, \ldots, q\}^V$ has distribution $\pi^P$. 
Edwards-Sokal Coupling (1989)

On the other hand, suppose \( \sigma \sim \pi^P \).
Edwards-Sokal Coupling (1989)

Draw all edges with same coloured endpoints...
Edwards-Sokal Coupling (1989)

Draw all edges with same coloured endpoints...
Edwards-Sokal Coupling (1989)

Keep each edge with probability $p$. 
Edwards-Sokal Coupling (1989)

Keep each edge with probability $p$. 
Edwards-Sokal Coupling (1989)

The result is an element of $2^E$ distributed like $\pi^{RC}$. 

![Diagram of a grid of points with some lines connecting them, illustrating the concept of the Edwards-Sokal Coupling.](image-url)
Swendsen-Wang Chain (1987)

Current state is $X_t \subseteq E$. 
Swendsen-Wang Chain (1987)

To each cluster of \((V, X_t)\), assign a colour in \(\{1, \ldots, q\}\), uniformly and randomly.
Swendsen-Wang Chain (1987)

To each cluster of \((V, X_t)\), assign a colour in \(\{1, \ldots, q\}\), uniformly and randomly.
Swendsen-Wang Chain (1987)

Add in edges with the same colour.
Swendsen-Wang Chain (1987)

Add in edges with the same colour.
Swendsen-Wang Chain (1987)

Keep each edge with probability $p$. 

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Swendsen-Wang Chain (1987)

Keep each edge with probability $p$. 
Swendsen-Wang Chain (1987)

New state $X_{t+1}$:
What if $q$ is non-integer?

Set $q \in \{1, 2, \ldots \}$ to be the integer part and $\delta > 0$ to be the non-integer part.

$$\pi^{RC}(A) \propto (q + \delta)^{k(A)} \prod_{ij \in A} p \prod_{ij \notin A} (1 - p), \quad A \in 2^E$$
Chayes-Machta Chain (1996)

Current state is $X_t \subseteq E$. 
Chayes-Machta Chain (1996)

To each cluster of \((V, X_t)\), assign a colour in \(\{1, \ldots, q\}\), uniformly and randomly, with probability \(1 - \frac{\delta}{q+\delta}\).
Chayes-Machta Chain (1996)

To each cluster of $(V, X_t)$, assign a colour in $\{1, \ldots, q\}$, uniformly and randomly, with probability $1 - \frac{\delta}{q+\delta}$. 
Chayes-Machta Chain (1996)

Add in the edges that have endpoints with the same colour.
Chayes-Machta Chain (1996)

Add in the edges that have endpoints with the same colour.
Chayes-Machta Chain (1996)

Add in the edges that have endpoints with the same colour.
Chayes-Machta Chain (1996)

Keep each edge with probability $p$. 
Chayes-Machta Chain (1996)

New state $X_{t+1}$:
Lower bounds for the Swendsen-Wang Chain

**Theorem (Li,Sokal 1988)**

*On a lattice $[1, \ldots, L]^d$, at $p = p_c$, the mixing time of the Swendsen-Wang chain is bounded below by*

$$
\tau_{\varepsilon} \geq C(\varepsilon) C_H
$$

Here $C_H$ is the *specific heat* and $C_H \sim L^{\alpha/\nu}$. Proof Idea:

If

$$
\sum_{x,y \in \Omega} \pi(x) P(x \rightarrow y) |f(y)^2 - f(x)^2| \quad \text{Var}_\pi f
$$

is small for some $f : \Omega \rightarrow \mathbb{R}$, then the chain mixes slowly. Taking $f(X) := |X|$ gives the result.

Extends to $\delta > 0$ case.
Theorem (Huber, 2004)

For $q \in \{1, 2, \ldots \}$, $\delta = 0$, if $G$ has bounded degree $\Delta$, and $p \leq \frac{1}{2\Delta - 2}$, then $\tau_\epsilon = O(\log |E|)$.

Theorem (Huber, 2004)

For $q \in \{1, 2, \ldots \}$, $\delta = 0$, and $G$ a tree, $\tau_\epsilon = O(\log |E|)$.

Does not extend to Chayes-Machta ($\delta > 0$).
Single Bond Chain

To each cluster of \((V, X_t)\), assign a colour from \(\{1, \ldots, q\}\) uniformly and randomly, with probability \(1 - \frac{\delta}{q+\delta}\).
Single Bond Chain

To each cluster of \((V, X_t)\), assign a colour from \(\{1, \ldots, q\}\) uniformly and randomly, with probability \(1 - \frac{\delta}{q+\delta}\).
Single Bond Chain

Choose an edge $e \in E$ uniformly at random.
Single Bond Chain

Choose an edge $e \in E$ uniformly at random.
Single Bond Chain

Add the edge $e$ if it has the same coloured endpoints.
Single Bond Chain

Keep the edge $e$ with probability $p$. 
Single Bond Chain

New state $X_{t+1}$:
Comparing the Chayes-Machta Chain to Single Bond Chain

Single Bond chain

1. Current state is $X_t$.

2. Choose random edge $e \in E$.

3. If $e_1 \xleftrightarrow{X_t} e_2$, keep $e$ with probability $p$.

4. If $e_1 \not\xleftrightarrow{X_t} e_2$, keep $e$ with probability $p/(q + \delta)$.

Theorem (Ullrich, 2012)

$\tau_{SW} \leq \text{poly}(|E|) \iff \tau_{SB} \leq \text{poly}(|E|)$. 
Comparing the Chayes-Machta Chain to Single Bond Chain

Theorem
\[ \tau_{CM} \leq \text{poly}(|E|) \iff \tau_{SB} \leq \text{poly}(|E|). \]

Theorem
\[ \tau_{SB} \leq \text{poly}(|E|) \text{ on tree graphs}. \]

Corollary
\[ \tau_{CM} \leq \text{poly}(|E|) \text{ on tree graphs}. \]
Further Work

- Prove things about the single bond chain.