Maximal r-Matching Sequences of Graphs and Hypergraphs

Adam Mammoliti

26th February 2018

Definitions and Examples

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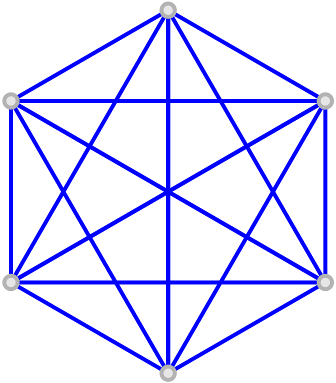

K_6

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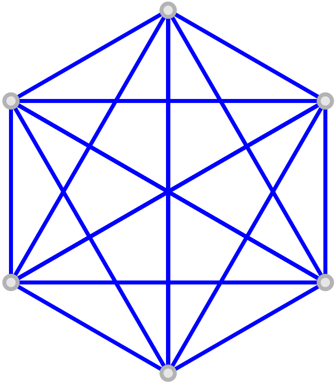

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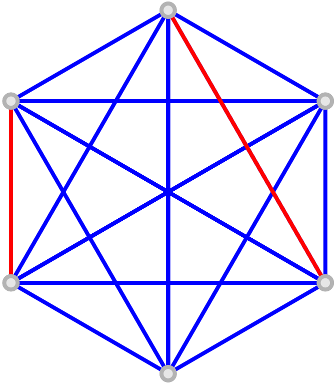

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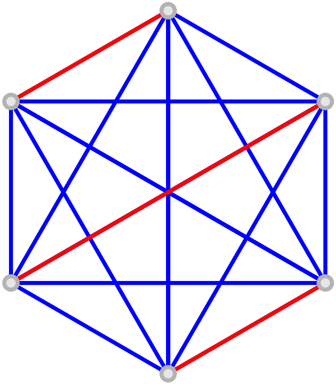
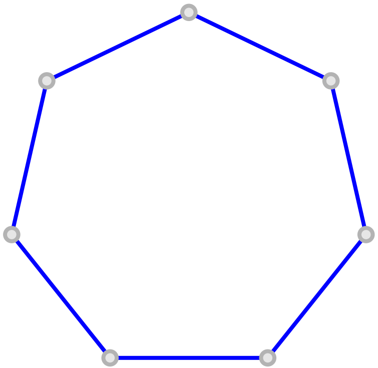


K_6

Matching Sequencibility



C_7

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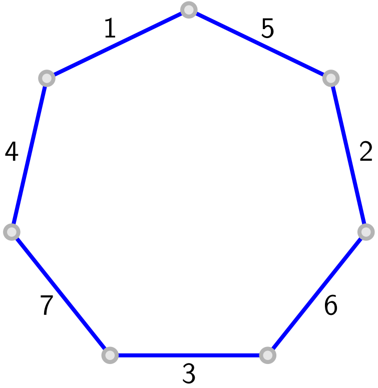

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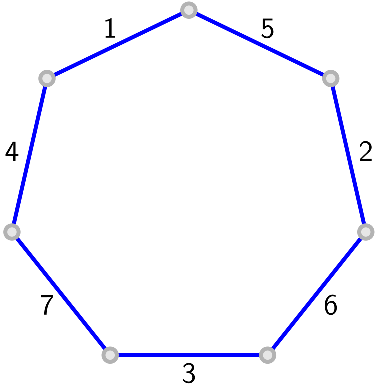

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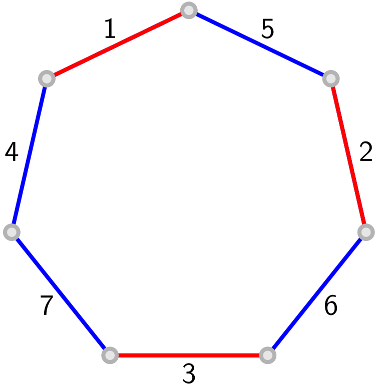

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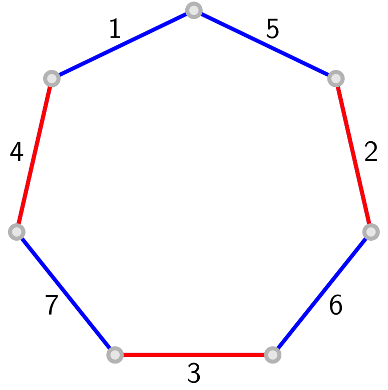

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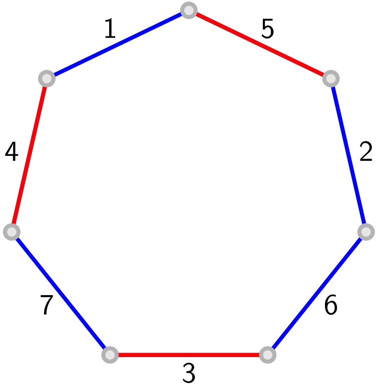

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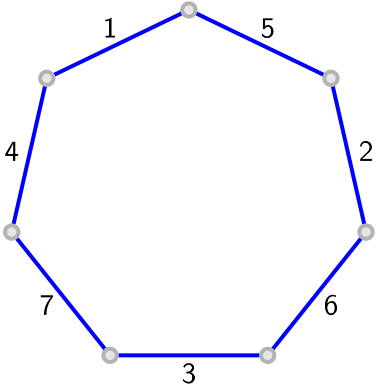

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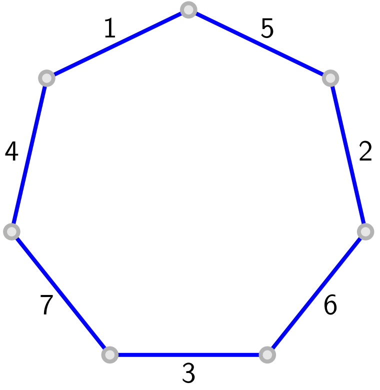

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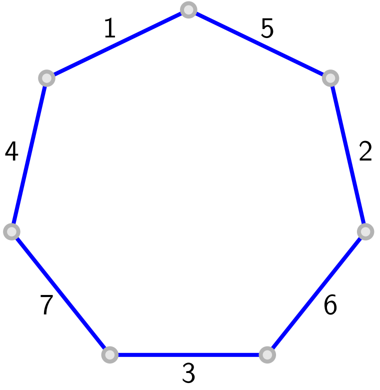

$ms(C_7) = 3$

C_7

Cyclic Matching Sequencibility

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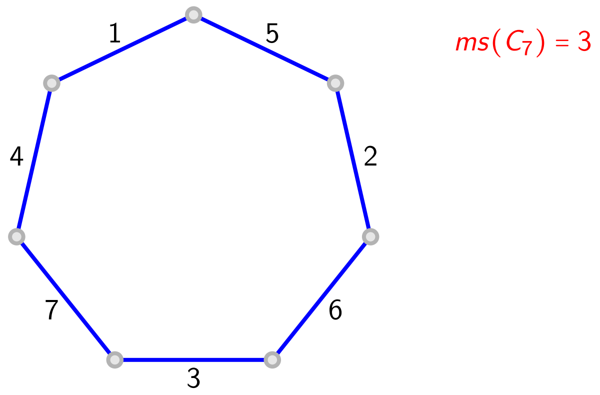

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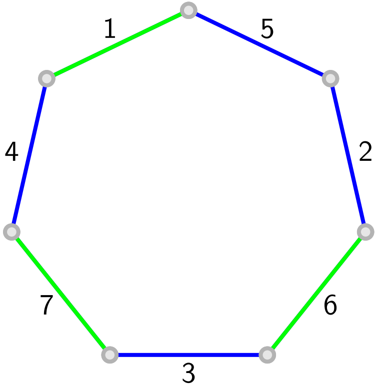

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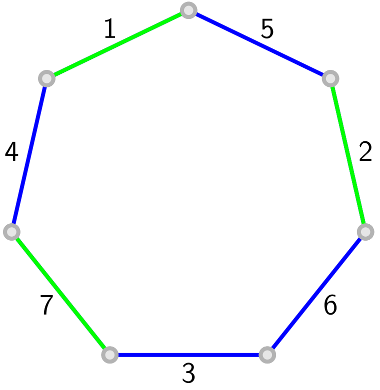

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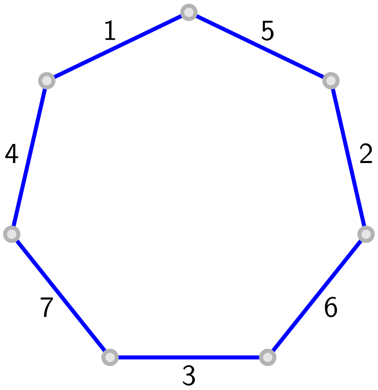

C_7

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Known Results

Alspach (2008)

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$$ms(P_n) = \left\lfloor \frac{n-1}{2} \right\rfloor \quad and \quad cms(P_n) = \left\lfloor \frac{n-2}{2} \right\rfloor$$

Known Results

Kreher et al. (2015)

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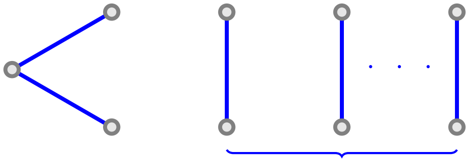

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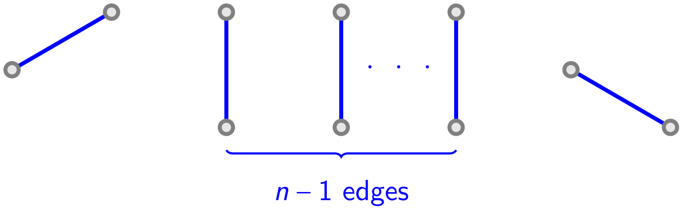

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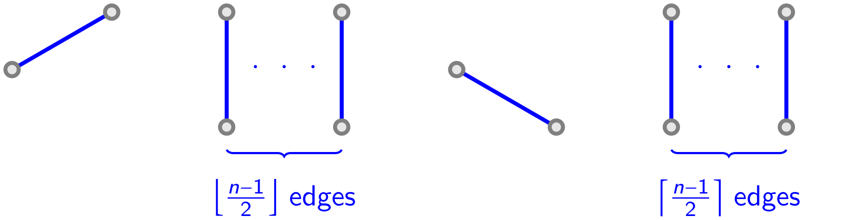
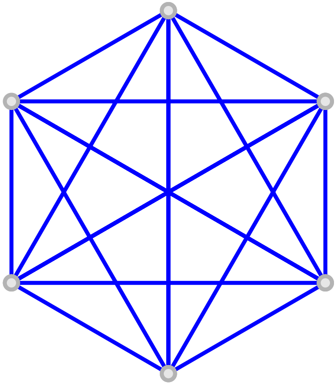


Generalising Matching Sequencibility



K_6

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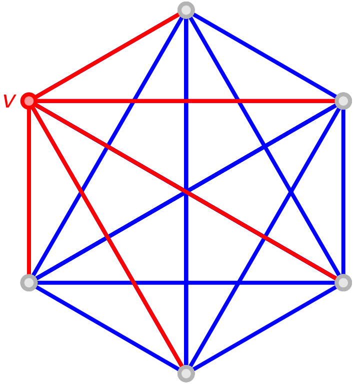
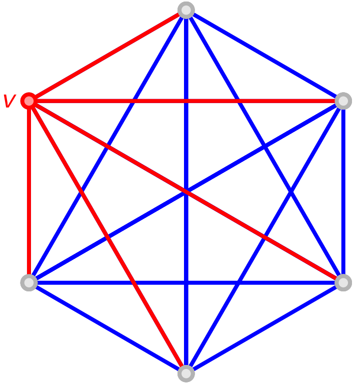


K_6

Generalising Matching Sequencibility

$deg(v) = 5$



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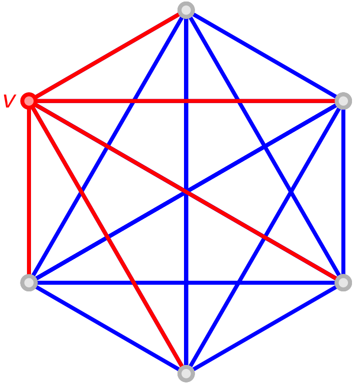

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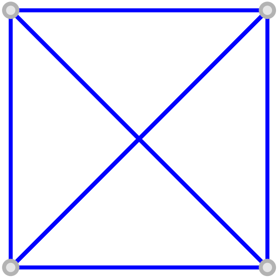

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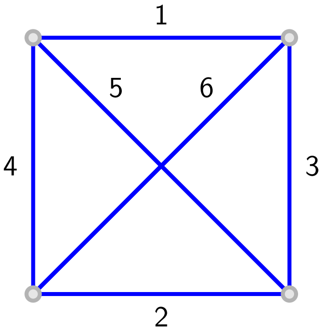

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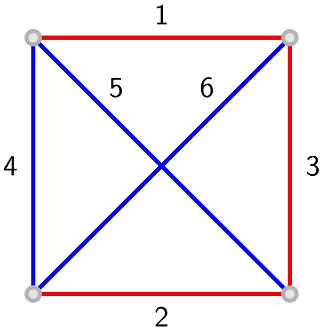

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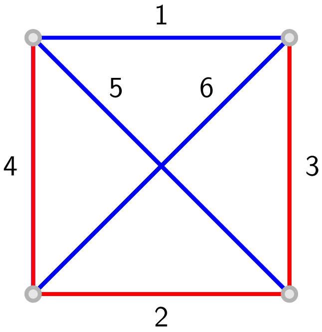

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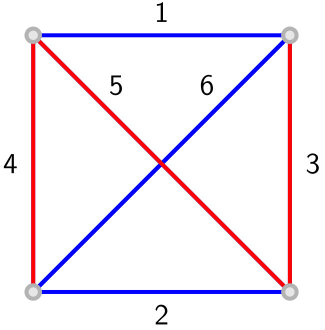

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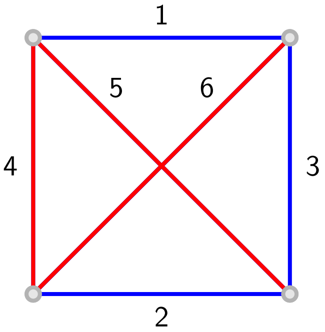

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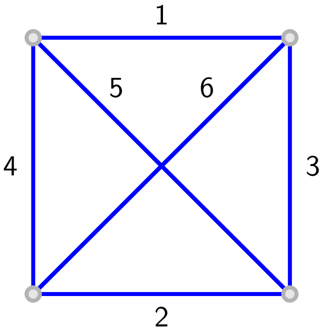

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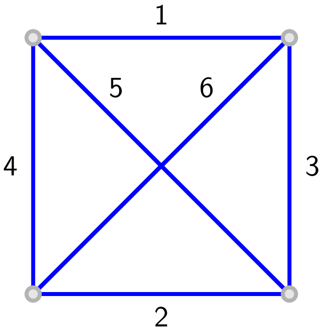

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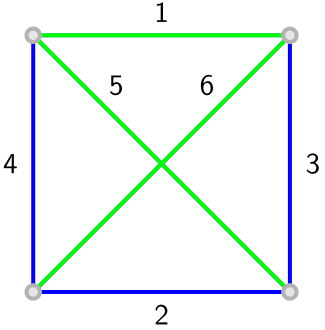

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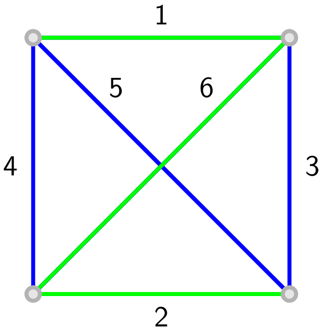

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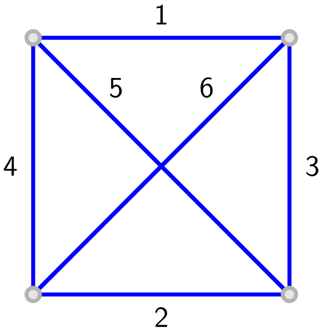

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Generalisations

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If rn is even or n is odd and either $r \geq \frac{n-1}{2}$ or $\gcd(r, n-1) = 1$, then

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Theorem

If r is even and n is odd, then

$$\left\lfloor \frac{rn-1}{2} \right\rfloor - 1 \leq cms_r(K_n) \leq \left\lfloor \frac{rn-1}{2} \right\rfloor$$

Remaining cases?

Conjecture

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For what odd r and n does

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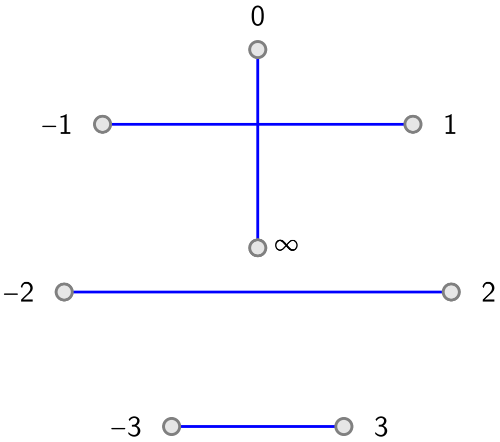
Theorem

If r and n are odd, then

$$cms_r(K_n) = \left\lfloor \frac{rn-1}{2} \right\rfloor \quad iff \quad cms_{n-1-r}(K_n) = \left\lfloor \frac{(n-1-r)n-1}{2} \right\rfloor$$

Sketch of the Construction: $r = 1$

- $cms(K_8) = 3$



K_8

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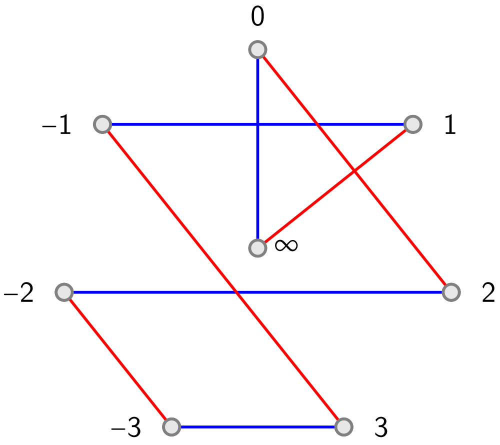

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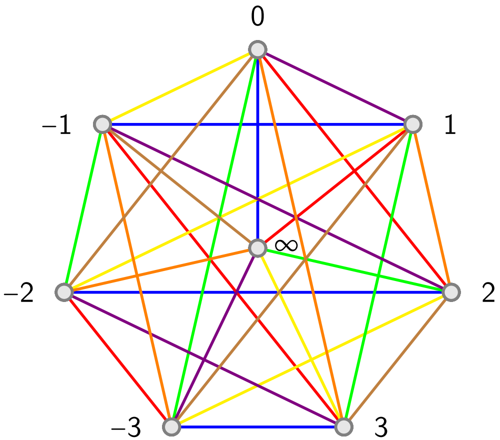

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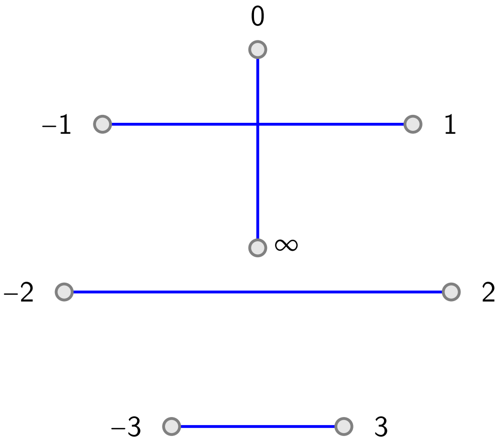

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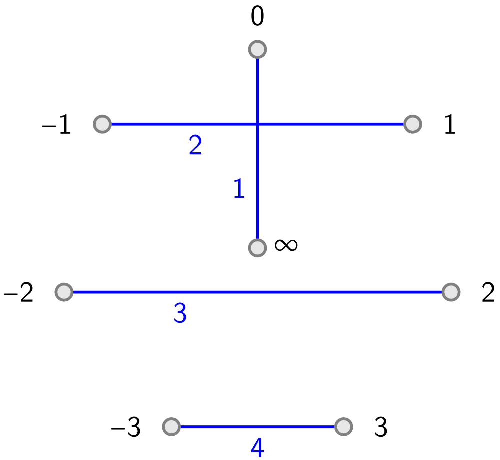

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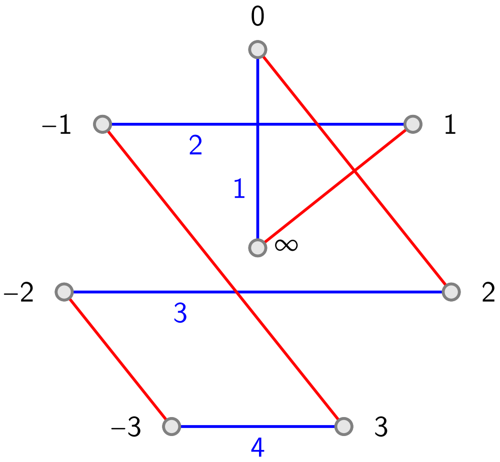

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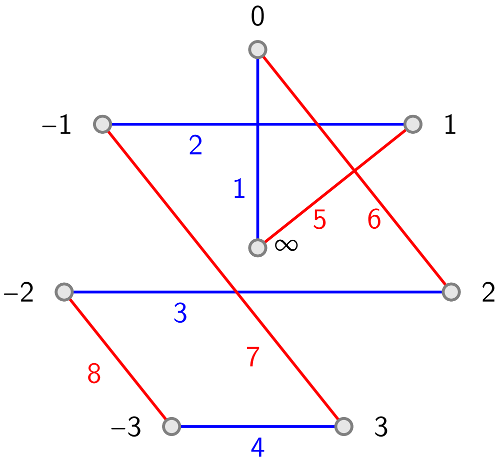

K_8

Sketch of the Construction: General r

- $cms_r(K_n) = \frac{rn}{2} - 1$

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- The matchings r spaces apart form the collections

$$M_1, M_{r+1}, \ldots \qquad M_2, M_{r+2}, \ldots \qquad \cdots \qquad M_d, M_{r+d}, \ldots$$

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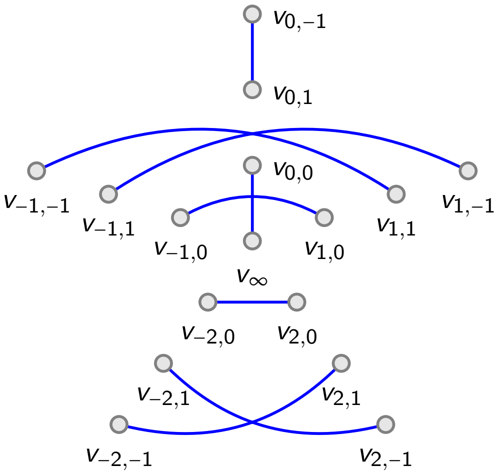

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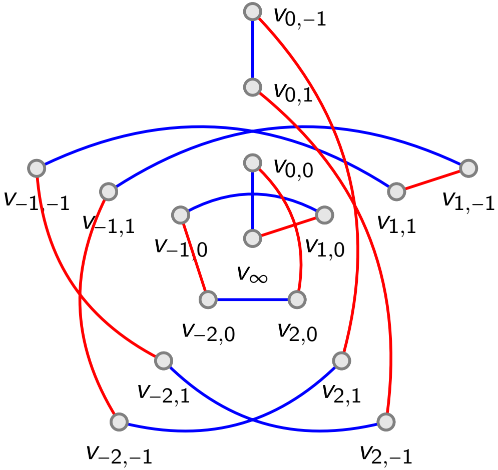

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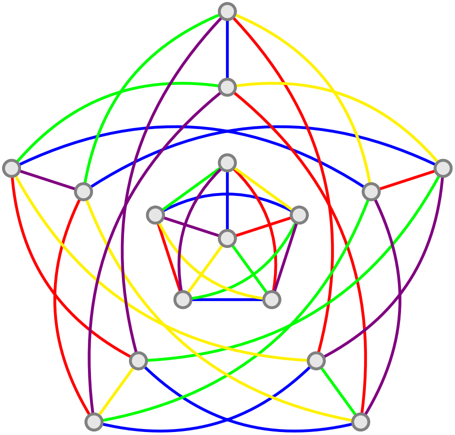

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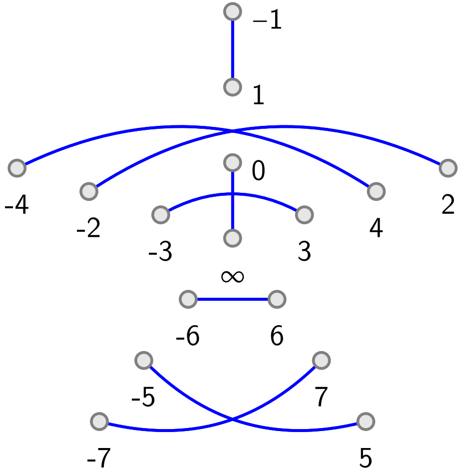

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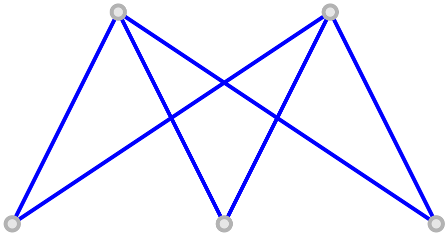
Complete Bipartite Graphs



$K_{2,3}$

Complete Bipartite Graphs

Brualdi et al. (2012)

If $n \leq m$, then

$$ms(K_{n,m}) = cms(K_{n,m}) = \begin{cases} n & \text{if } n < m \\ n-1 & \text{if } n = m \end{cases}$$

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Theorem

If $n_1 < n_2 \leq \cdots \leq n_k$, then

$$ms_r(\mathcal{K}_{n_1,n_2,\ldots,n_k}) = cms_r(\mathcal{K}_{n_1,n_2,\ldots,n_k}) = rn_1$$

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where

$$\left(\left\lfloor \frac{r}{n_1^{u-1}} \right\rfloor + 1\right) \left\lfloor \frac{1}{r} \prod_{i=2}^{k} n_i \right\rfloor \leq \prod_{i=u+1}^{k} n_i \leq \left\lfloor \frac{r}{n_1^{u-1}} \right\rfloor \left(\left\lfloor \frac{1}{r} \prod_{i=2}^{k} n_i \right\rfloor + 1\right) \tag{1}$$

The End

Thanks for listening!

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