Finitely forcible graph limits are universal

Jacob Cooper
Dan Král'
Taísa Martins

University of Warwick

Monash University - Discrete Maths Research Group

Graph limits

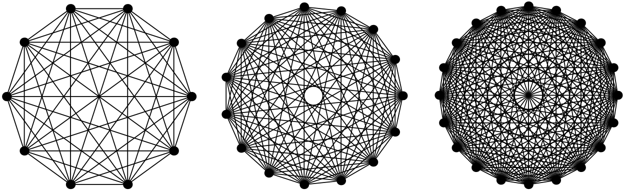
- Approximate asymptotic properties of large graphs
- Extremal combinatorics/computer science :
  flag algebra method, property testing
  large networks, e.g. the internet, social networks...
- The 'limit' of a convergent sequence of graphs
  is represented by an analytic object called a graphon

Dense graph convergence

- Convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- A sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H

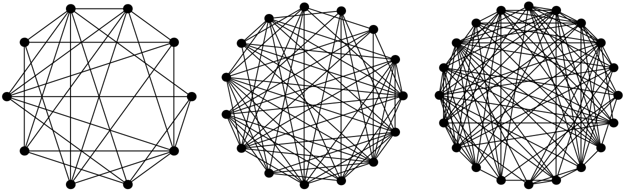
Dense graph convergence

- Convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- A sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
  - complete graphs K_n



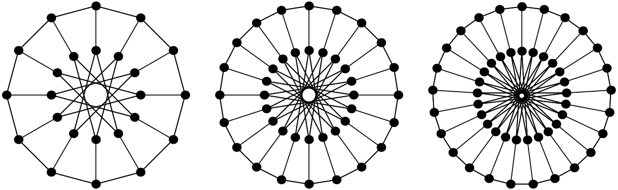
Dense graph convergence

- Convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- A sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
    - complete graphs K_n
    - Erdős-Rényi random graphs $G_{n,p}$



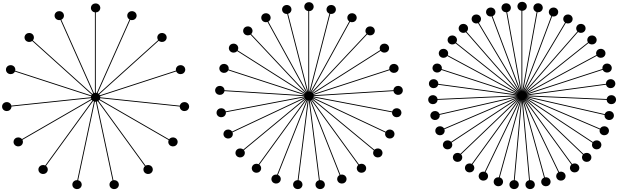
Dense graph convergence

- Convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- A sequence $(G_n)_{n \in \mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
  - complete graphs K_n
  - Erdős-Rényi random graphs $G_{n,p}$
  - any sequence of sparse graphs



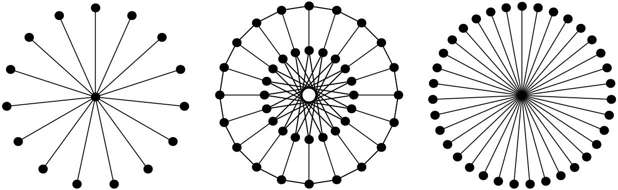
Dense graph convergence

- Convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- A sequence $(G_n)_{n \in \mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
  - complete graphs K_n
  - Erdős-Rényi random graphs $G_{n,p}$
  - any sequence of sparse graphs



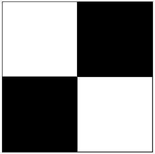
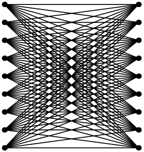
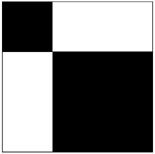
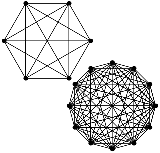
Dense graph convergence

- Convergence for dense graphs ($|E| = \Omega(|V|^2)$)
- $d(H, G)$ = probability $|H|$-vertex subgraph of G is H
- A sequence $(G_n)_{n\in\mathbb{N}}$ of graphs is convergent if $d(H, G_n)$ converges for every H
  - complete graphs K_n
  - Erdős-Rényi random graphs $G_{n,p}$
  - any sequence of sparse graphs



Limit object: graphon

- Graphon: measurable function $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)\ \forall x,y \in [0,1]$
- W-random graph of order n: n random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$

Limit object: graphon

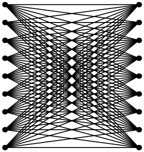
- Graphon: measurable function $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)\ \forall x,y \in [0,1]$
- W-random graph of order n:
  n random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $d(H,W)$ = probability W-random graph of order $|H|$ is H

Limit object: graphon

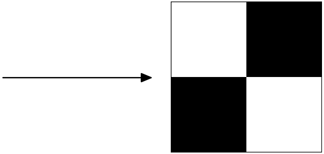
- Graphon: measurable function $W : [0,1]^2 \to [0,1]$, s.t. $W(x,y) = W(y,x)\ \forall x,y \in [0,1]$
- W-random graph of order n: n random points $x_i \in [0,1]$, edge probability $W(x_i, x_j)$
- $d(H,W) =$ probability W-random graph of order $|H|$ is H
- W is a limit of $(G_n)_{n\in\mathbb{N}}$ if $d(H,W) = \lim_{n\to\infty} d(H,G_n)\ \forall\ H$
  - Every convergent sequence of graphs has a limit
  - W-random graphs converge to W with probability one

Examples of graph limits

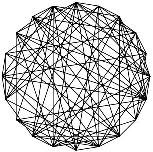
- The sequence of complete bipartite graphs, $(K_{n,n})_{n\in\mathbb{N}}$



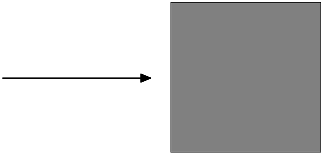
$$\begin{matrix}
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1\\
0&0&0&0&0&0&1&1&1&1&1&1\\
1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0\\
1&1&1&1&1&1&0&0&0&0&0&0
\end{matrix}$$



- The sequence of random graphs, $(G_{n,1/2})_{n\in\mathbb{N}}$

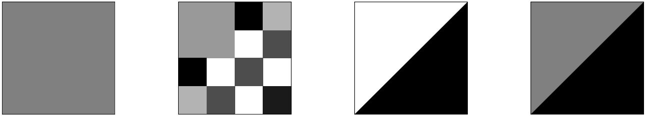


$$\begin{matrix}
0&0&1&0&0&0&0&1&1&1&1&0\\
1&1&0&0&0&1&1&0&0&1&1&0\\
1&0&0&1&0&1&0&0&0&0&1&0\\
1&0&1&0&0&1&1&0&1&1&0&1\\
1&0&1&1&1&1&1&0&0&0&0&1\\
0&1&0&0&1&1&1&1&1&1&0&0\\
1&0&0&1&1&1&1&0&0&0&0&0\\
1&1&1&1&0&1&0&0&0&0&1&0\\
1&1&0&0&0&0&1&1&1&0&1&0\\
1&0&1&0&0&1&1&0&1&1&0&1\\
0&1&1&0&0&0&0&1&1&0&0&1\\
1&1&0&0&0&0&1&1&1&0&0&0
\end{matrix}$$



Finitely forcible graphons

- A graphon W is finitely forcible if $\exists\ H_1 \ldots H_k$ s.t $d(H_i, W') = d(H_i, W) \implies d(H, W') = d(H, W)\ \forall\ H$



1. Thomason (87), Chung, Graham and Wilson (89)
2. Lovász and Sós (2008)
3. Diaconis, Holmes and Janson (2009)
4. Lovász and Szegedy (2011)

Motivation

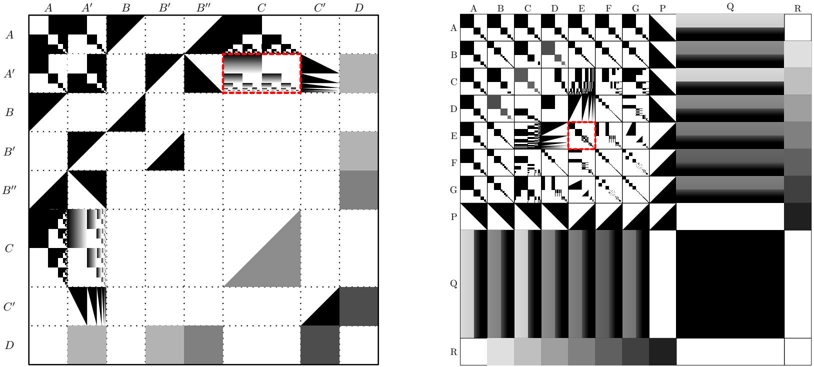
- Conjecture (Lovász and Szegedy, 2011)
  The space of typical vertices of a finitely forcible graphon is compact.

- Conjecture (Lovász and Szegedy, 2011)
  The space of typical vertices of a finitely forcible graphon is finite dimensional.

Motivation

- Conjecture (Lovász and Szegedy, 2011)
  The space of typical vertices of a finitely forcible graphon is compact.
  - Theorem (Glebov, Král', Volec, 2013)
    $T(W)$ can fail to be locally compact
- Conjecture (Lovász and Szegedy, 2011)
  The space of typical vertices of a finitely forcible graphon is finite dimensional.
  - Theorem (Glebov, Klimošová, Král', 2014)
    $T(W)$ can have a part homeomorphic to $[0,1]^\infty$
  - Theorem (Cooper, Kaiser, Král', Noel, 2015)
    $\exists$ finitely forcible W such that every ε-regular partition has at least $2^{\varepsilon^{-2}/\log\log\varepsilon^{-1}}$ parts (for inf. many $\varepsilon \to 0$).

Previous Constructions



Universal Construction Theorem

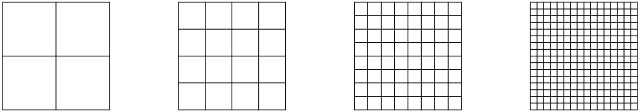
- Theorem (Cooper, Král', M.)
  Every graphon is a subgraphon of a finitely forcible graphon.
  - Existence of a finitely forcible graphon that is non-compact, infinite dimensional, etc
  - For every non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ tending to ∞, $\exists$ finitely forcible W and positive reals ε_i tending to 0 such that every weak ε_i-regular partition of W has at least $2^{\Omega\left(\frac{\varepsilon_i^{-2}}{f(\varepsilon_i^{-1})}\right)}$ parts.

Ingredients of the proof

- Partitioned graphons
  - vertices with only finitely many degrees
  - parts with vertices of the same degree
- Decorated constraints
  - method for constraining partitioned graphons
  - density constraints rooted in the parts
  - based on notions related to flag algebras
- Encoding a graphon as a real number in $[0, 1]$
  - forcing W by fixing its density in dyadic subsquares

A graphon as a real number

- Unique representation by densities on dyadic squares

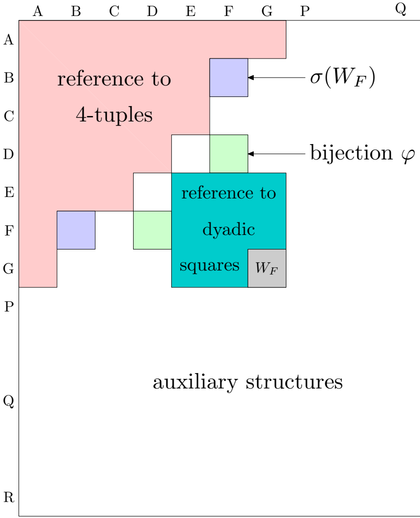


- 4-tuple map $\delta : (d, s, t, k) \to \{0, 1\}$
  - dyadic square: $\left[\frac{s}{2^d}, \frac{s+1}{2^d}\right] \times \left[\frac{t}{2^d}, \frac{t+1}{2^d}\right]$
  - k-th bit in the standard binary representation of the density of W in the dyadic square
  - 0, otherwise

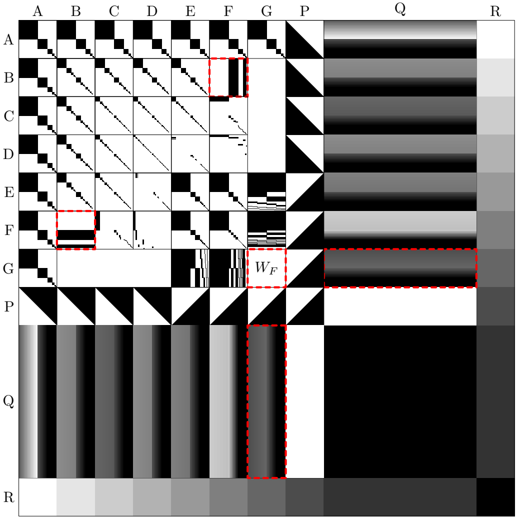
- $\varphi : \mathbb{N}^4 \to \mathbb{N}$ (bijection), $\sigma : \mathcal{W} \to [0, 1]$

$$\sigma(W)\ j\text{-th bit} = \delta(\varphi^{-1}(j))$$

Sketch of the construction



Universal construction



Universality $\times$ Meager set

- Theorem (Cooper, Král', M.)
  Every graphon is a subgraphon of a finitely forcible graphon.

- Theorem (Lovász and Szegedy, 2011)
  Finitely forcible graphons form a meager set in the space of all graphons.

- Analogy:
  - $\phi : \mathcal{W} \to [0,1]^{\mathbb{N}}$ (injection)
  - $S \subseteq [0,1]$ measurable
  - $\phi(W[S \times S])$: projection of $\phi(W)$ in a subspace of $[0,1]^{\mathbb{N}}$

e.g. $H = \{(\mathcal{C}(x,y), z) \mid (x,y,z) \in \mathbb{R}^3, \mathcal{C} \text{ is a space-filling curve}\}$

Thank you for your attention!