Discrete Contact Geometry

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Overview

Introduction

What is contact geometry?

History

Motivation

Discrete aspects of contact geometry

Combinatorics of surfaces and dividing sets

Contact-representable automata
Contact geometry

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- Knot theory
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This talk is about some interesting recent applications that are discrete and combinatorial:

- Arrangements & combinatorics of curves on surfaces
- “Topological computation"
- Finite state automata
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Non-integrable: tangent curves (car-parking) but no tangent surfaces!

Such $\xi$ can be given as $\ker \alpha$ where $\alpha$ is a differential 1-form satisfying $\alpha \wedge d\alpha \neq 0$ everywhere.

E.g. $\mathbb{R}^3$ with $\alpha = dz - y \, dx$. 
Flexible vs discrete

The definition of a contact structure is:

- Very *differential-geometric* (non-integrability)
Flexible vs discrete

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- Very *flexible*: A small perturbation of a contact structure is again a contact structure. $(\alpha \wedge d\alpha \neq 0)$

But it’s also a surprisingly *rigid* type of geometry.

- Any other "nontrivial" contact structure $\xi$ on $\mathbb{R}^3$ is *isotopic* to the standard one $\xi_{std}$.
  (i.e. $\xi$ can be continuously deformed through contact structures to $\xi_{std}$.)
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- Hamiltonian mechanics, *symplectic geometry*.
- “Contact geometry = odd-dim symplectic geometry".
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Modern period:
- Eliashberg (1989): Distinction — *tight* (non-trivial) and *overtwisted* (trivial) contact structures.
- Gromov (1986), Eliashberg (1990s), ...:
  Development of *pseudoholomorphic curve* methods.
- Ozsváth-Szabó (2004), many others... : Development of *Floer homology* methods.
Why we contactify

Some motivations for the study of contact geometry:

- **Topology:** One way to understand the topology of a manifold is to study the contact structures on it.

- **Dynamics:** There are natural *vector fields* on contact manifolds and their dynamics have important applications to classical mechanics.
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- **Pure mathematical / Structural**: Mathematical structures found in contact geometry connect to other fields...
  - Combinatorics
  - Information theory
  - Discrete mathematics
Outline

1. Overview

2. Discrete aspects of contact geometry
   - 4 discrete facts about contact geometry

3. Combinatorics of surfaces and dividing sets

4. Contact-representable automata
Fact #1: Dividing sets

Consider a generic *surface* $S$ in a contact 3-manifold $M$, possibly with boundary $\partial S$. (In this talk, $S = \text{disc or annulus}$.)

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A contact structure $\xi$ near $S$ is described exactly by a finite set $\Gamma$ of non-intersecting smooth curves on $S$, called its *dividing set*. 
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Moreover, *isotopy* (continuous deformation) of contact structures near $S$ corresponds to *isotopy* of dividing sets $\Gamma$.

Interested in the *combinatorial/topological arrangement of the curves* $\Gamma$. 

**Chord diagrams**

Consider a disc $D$ with some points $F$ marked on $\partial D$. A chord diagram is a pairing of the points of $F$ by non-intersecting curves on $D$. E.g. Note: We can shade alternating regions of a chord diagram. Colour = visible side of contact plane.
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- On a *sphere*, when there is *more than one* dividing curve.
Boundary conditions

Examine what contact planes look like near boundary $\partial S$:
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- Always tangent to $\partial S$
- Perpendicular to $S$ along $\Gamma$.  

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\text{Fixing points of } F \text{ fixes boundary conditions for } \xi.
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E.g.: Consider contact structures $\xi$ near a disc $D$. Fix boundary conditions $F$ with $|F| = 2n$.  

# (isotopy classes of) (tight) contact structures on $D = C_n$.  

Here $C_n$ is the $n$'th Catalan number $= \frac{1}{n+1} \left( \begin{array}{c} 2n \\ n \end{array} \right)$.  

E.g. $n = 3$:  

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**Fact #3 (Honda 2000)**

When surfaces intersect transversely, dividing sets *interleave*. Rounding corners, “turn right to dive” and “turn left to climb”.

This leads to interesting combinatorics of curves...
Fact #4: Bypasses

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Naturally obtain bypass triples of dividing sets.
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Bypass surgery is a natural order-3 operation on dividing sets.
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Fact #2: Giroux's criterion
Dividing sets detect trivial contact structures (OT discs).
- On a *disc* $D$, via a *closed dividing curve*.
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Fact #3: Edge rounding (Honda 2000)
When surfaces intersect transversely, dividing sets interleave.
- Rounding edges, “turn right to dive” and “turn left to climb”.

Fact #4: Bypass surgery (Honda 2000)
*Bypass surgery* is a natural order-3 operation on dividing sets.
Outline

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2. Discrete aspects of contact geometry

3. Combinatorics of surfaces and dividing sets
   - Chord diagrams and cylinders
   - A vector space of chord diagrams
   - Slalom basis
   - A partial order on binary strings

4. Contact-representable automata
Cylinders

A combinatorial construction using dividing sets (fact #1), edge rounding (#3) and Giroux’s criterion (#2):
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*Insert chord diagrams into the two ends of a cylinder...*

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\[ \Gamma_1 \mapsto \Gamma_0 \]
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A combinatorial construction using dividing sets (fact #1), edge rounding (#3) and Giroux’s criterion (#2):

Insert chord diagrams into the two ends of a cylinder...
...and round corners to obtain a dividing set on $S^2$.

By Giroux’s criterion, the contact structure obtained on $S^2$ is:
- **Trivial** (OT) if it is disconnected, i.e. contains $> 1$ curve.
- **Nontrivial** (tight) if it is connected, i.e. contains 1 curve.
An “inner product" on chord diagrams

Define an “inner product" function based on this construction.

**Definition**

\[
\langle \cdot | \cdot \rangle : \{ \text{Div sets on } D^2 \} \times \{ \text{Div sets on } D^2 \} \rightarrow \mathbb{Z}_2
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\[ \langle \Gamma_0 | \Gamma_1 \rangle = \begin{cases} 1 & \text{if the resulting curves on the cylinder form a single connected curve;} \\ 0 & \text{if the result is disconnected.} \end{cases} \]
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**Proposition (M.)**

*For any \(\Gamma, \Gamma', \Gamma''\) as above and any \(\Gamma_1\),*

\[
\langle \Gamma | \Gamma_1 \rangle + \langle \Gamma' | \Gamma_1 \rangle + \langle \Gamma'' | \Gamma_1 \rangle = 0.
\]
A vector space of chord diagrams

Idea of proof:

\[ \begin{array}{ccc}
\text{\includegraphics[width=0.2\textwidth]{chord_diagram1}} & + & \text{\includegraphics[width=0.2\textwidth]{chord_diagram2}} \\
= 1 + 0 + 1 = 0
\end{array} \]
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These ideas lead us to define a *relation* on chord diagrams: three chord diagrams forming a bypass triple sum to 0.

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Leads to the definition of a vector space (over \( \mathbb{Z}_2 \)).

Definition

\[ V_n = \frac{\mathbb{Z}_2 \langle \text{Chord diagrams with } n \text{ chords} \rangle}{\text{Bypass relation}} \]

(One can show \( V_n \) is a rudimentary form of Floer homology...)
A vector space of chord diagrams

Theorem (M.)

$V_n$ has *dimension* $2^{n-1}$, *with natural bases indexed by binary strings of length* $n - 1$. 
A vector space of chord diagrams

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1. The **Slalom basis** $\{S_b\}_{b \in B_{n-1}}$
2. The **Turing tape basis** $\{T_b\}_{b \in B_{n-1}}$
The slalom basis

Construction of the *slalom chord diagram* of a binary string.
The slalom basis

Construction of the \textit{slalom} chord diagram of a binary string.

\begin{align*}
1011
\end{align*}
The slalom basis

Construction of the *slalom* chord diagram of a binary string.

1011 ↔

In this basis, the bilinear form $\langle \cdot | \cdot \rangle$ has a simple description:

\[
\langle S_a | S_b \rangle = \begin{cases} 
1 & \text{if } a \preceq b \\
0 & \text{otherwise}, 
\end{cases}
\]

where $\preceq$ is a certain partial order on binary strings.
The slalom basis

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1011 ↔

\[ = S_{1011} \]
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In this basis, the bilinear form $\langle \cdot | \cdot \rangle$ has a simple description:

**Theorem (M.)**

$$\langle S_a | S_b \rangle = \begin{cases} 1 & \text{if } a \leq b \\ 0 & \text{otherwise,} \end{cases}$$

where $\leq$ is a certain *partial order* on binary strings.
A partial order on binary strings

Definition

For two binary strings $a, b$, the relation $a \preceq b$ holds if

1. $a$ and $b$ both contain the same number of 0s and 1s
A partial order on binary strings

**Definition**

For two binary strings $a, b$, the relation $a \preceq b$ holds if

1. *a and b both contain the same number of 0s and 1s*

2. *Each 0 in a occurs to the left of, or same position as, the corresponding 0 in b.*
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E.g.

\[
\begin{align*}
0011 & \preceq 1001 & \preceq 1010 & \preceq 1100 \\
\preceq & 0110 & \preceq &
\end{align*}
\]
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Note $\preceq$ is a *sub-order* of the lexicographic/numerical order $\leq$. 
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Note $\preceq$ is a sub-order of the lexicographic/numerical order $\preceq$.

Inserting chord diagrams into a cylinder is a “topological machine” for comparing binary strings with respect to $\preceq$. 
Properties of $\preceq$

Recall we said the slalom chord diagrams form a *basis* for $V_n$. E.g.
Properties of $\preceq$

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E.g.

$$\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
+ 
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}$$
Properties of $\preceq$

Recall we said the slalom chord diagrams form a *basis* for $V_n$. E.g.

$$
\begin{array}{c}
\begin{array}{c}
\includegraphics{example1.png}
\end{array}
\end{array}
= \begin{array}{c}
\begin{array}{c}
\includegraphics{example2.png}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics{example3.png}
\end{array}
\end{array}

= \begin{array}{c}
\begin{array}{c}
\includegraphics{example4.png}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics{example5.png}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics{example6.png}
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{c}
\includegraphics{example7.png}
\end{array}
\end{array}
\end{array}$$
Recall we said the slalom chord diagrams form a *basis* for $V_n$. E.g.,

\[
\begin{align*}
\text{= } & \begin{array}{c}
\text{Diagram 1}\n\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{= } & \begin{array}{c}
\text{Diagram 2}\n\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{= } & \begin{array}{c}
\text{Diagram 3}\n\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{= } & \begin{array}{c}
\text{Diagram 4}\n\end{array}
\end{align*}
\]

\[
\begin{align*}
\text{= } & S_{0011} + S_{0110} + S_{1001} + S_{1010}
\end{align*}
\]
Properties of \( \preceq \)

Recall we said the slalom chord diagrams form a basis for \( V_n \).

E.g.

\[
\begin{align*}
\text{=} & \quad \text{+} \\
\text{=} & \quad \text{+} \\
\text{=} & \quad \text{+} \\
\text{=} & \quad S_{0011} \quad + \quad S_{0110} \quad + \quad S_{1001} \quad + \quad S_{1010}
\end{align*}
\]

- We say the *component strings* of \( \Gamma \) are 0011, 0110, 1001, 1010.
- Given a chord diagram \( \Gamma \), let \( b_-(\Gamma) \) denote the *numerically least*, and \( b_+(\Gamma) \) the *numerically greatest*, component string.

So for the example \( \Gamma \) above, \( b_-(\Gamma) = 0011 \) and \( b_+(\Gamma) = 1010 \).
Partial order $\preceq$ and Catalan numbers

The partial order $\preceq$ has interesting combinatorics...
Partial order $\preceq$ and Catalan numbers

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Theorem (M.)

1. For any chord diagram $\Gamma$, $b_-(\Gamma) \preceq b_+(\Gamma)$. 

... and produces the Catalan numbers again.

Corollary

The number of pairs of strings $s^-, s^+$ of length $n$ such that $s^- \preceq s^+$ is $C_{n+1}$. 
Partial order $\preceq$ and Catalan numbers

The partial order $\preceq$ has interesting combinatorics...

**Theorem (M.)**

1. For any chord diagram $\Gamma$, $b_-(\Gamma) \preceq b_+(\Gamma)$.
2. For any pair of strings $s_-, s_+$ satisfying $s_- \preceq s_+$, there exists a unique chord diagram $\Gamma$ such that $b_-(\Gamma) = s_-$ and $b_+(\Gamma) = s_+$. 
Partial order $\preceq$ and Catalan numbers

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**Theorem (M.)**

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The number of pairs of strings $s_-, s_+$ of length $n$ such that $s_- \preceq s_+$ is $C_{n+1}$.
Outline

1. Overview
2. Discrete aspects of contact geometry
3. Combinatorics of surfaces and dividing sets
4. Contact-representable automata
   - Turing tape basis
   - Cubulated inner product
   - Finite state automata
The Turing tape basis

Divide the disc with $|F| = 2n$ into $n - 1$ squares:

```
  |   |   |
  |   |   |
  |   |   |
  |   |   |
```
The Turing tape basis

Divide the disc with $|F| = 2n$ into $n - 1$ squares:

On each square there are two “basic” possible sets of sutures

0: , 1: ,
The Turing tape basis

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On each square there are two “basic” possible sets of sutures

0:

1:

Draw them according to a string $b$ to obtain Turing tape basis diagrams $T_b$ — another basis for $V_n$. E.g.

$T_{1011} =$
Cubulated inner product

With chord diagrams are drawn in “Turing tape" form, the inner product $\langle \cdot | \cdot \rangle$ becomes “cubulated"...
Cubulated inner product

With chord diagrams are drawn in “Turing tape" form, the inner product $\langle \cdot | \cdot \rangle$ becomes “cubulated"...

E.g. $\langle T_{1011} | T_{1000} \rangle = \begin{array}{c}
\text{Chord diagram 1} \\
\text{Chord diagram 2}
\end{array}$
Cubulated inner product

With chord diagrams are drawn in "Turing tape" form, the inner product $\langle \cdot | \cdot \rangle$ becomes "cubulated"...

E.g. $\langle T_{1011} | T_{1000} \rangle =$

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{cubulated_inner_product_1} \\
\includegraphics[width=0.5\textwidth]{cubulated_inner_product_2}
\end{array}
\]
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\begin{align*}
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots \\
\end{array}
\end{align*}
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E.g. $\langle T_{1011} | T_{1000} \rangle =$

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{cubulated_inner_product_1} \\
\includegraphics[width=0.3\textwidth]{cubulated_inner_product_2} \\
\includegraphics[width=0.3\textwidth]{cubulated_inner_product_3}
\end{array}
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\[ \begin{array}{c}
\text{Diagram 1} \\
\text{Diagram 2} \\
\text{Diagram 3}
\end{array} \]

\[ \leftrightarrow \]

\[ \begin{array}{c}
\text{Diagram 4} \\
\text{Diagram 5} \\
\text{Diagram 6}
\end{array} \]
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\[
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\begin{array}{c}
\end{array}
\end{array}
\]
Cubulation, step by step

Draw curves curvier, and analyse this computation in step-by-step fashion.
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A finite state automaton

We can consider this process as a *finite state automaton*. 
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3 states:  

- **A**:  

- **B**:  

- **⊥**: (or anything with a closed curve)
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3 states:  
A: ![State A](image)
B: ![State B](image)
⊥: ![State ⊥](image) (or anything with a closed curve)

4 inputs:  
00
01
10
11

*Figure: A finite state automaton with 3 states and 4 inputs.*
A finite state automaton

We can consider this process as a finite state automaton.

3 states:  
- A: [diagram]
- B: [diagram]
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4 inputs:
- 00
- 01
- 10
- 11

Transitions e.g.:

A \rightarrow B 

B \rightarrow A
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Can check that the calculation of the inner product on the "cubulated cylinder" on the "Turing tape basis" computes the finite state automaton:
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Definition

A finite state automaton is contact-representable if:

- To every state $s \in S$ is associated a dividing set $\Gamma_s$ on a disc with $2n$ fixed boundary points.
- To each input $\sigma \in \Sigma$ is associated a dividing set $\Gamma_\sigma$ on an annulus with $2n$ fixed points on each boundary circle.
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- The transition function $S \times \Sigma \rightarrow S$ is achieved by gluing annuli to discs: if $(s, \sigma) \mapsto s'$ then $\Gamma_s \cup \Gamma_\sigma = \Gamma_{s'}$. 

E.g. for the previous example $n=2$, 3 states: $\Gamma_A = \Gamma_B = \Gamma_\perp$: (or anything with a closed curve) 4 inputs: $\Gamma_00 = \Gamma_01 = \ldots$
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- $\Gamma_{00}$
- $\Gamma_{01}$
- \ldots
Quantum information theory and computation

**Question**

Which finite state automata can be represented by contact geometry in this way?
Quantum information theory and computation

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Various applications:
- These constructions give linear maps $V_n \rightarrow V_n$ which form a Topological quantum field theory.
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- These constructions give linear maps $V_n \rightarrow V_n$ which form a Topological quantum field theory.
- The above is a toy model of a quantum theory which explicitly encodes information: “it from bit”.

Moreover, this is a TQFT which explicitly encodes computation.

Quantum states based on curves on surfaces and topology are considered in the physical theory of “anyons”.

A very combinatorial, geometric way of performing certain computations.

A reversible/conservative type of computation.
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Thanks for listening!

References:

