

Courcelle's Theorem, tree automata, hypergraphs, and matroids

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Courcelle's Theorem

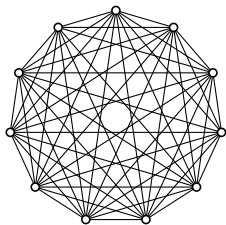
Courcelle's Theorem (1990)

Let φ be a sentence in MS_2 , the monadic second-order logic of graphs.

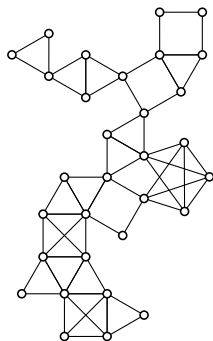
Let \mathcal{G} be a class of graphs with bounded tree-width.

There is a polynomial-time algorithm that tests graphs in \mathcal{G} and decides whether they satisfy φ .

Tree-width



High tree-width



Low tree-width

Monadic second-order logic

$$\begin{aligned} & \exists E_1 \forall v \exists e_1 \exists e_2 (e_1 \in E_1 \wedge e_2 \in E_1 \wedge e_1 \neq e_2 \\ & \quad \wedge \text{inc}(v, e_1) \wedge \text{inc}(v, e_2) \\ & \quad \wedge \forall e_3 (e_3 \in E_1 \wedge e_3 \neq e_1 \wedge e_3 \neq e_2 \rightarrow \neg \text{inc}(v, e_3))) \\ & \quad \wedge \forall V_1 \forall V_2 \\ & \quad \quad (\exists v_1 \exists v_2 (v_1 \in V_1 \wedge v_2 \in V_2) \\ & \quad \quad \wedge \forall v (v \in V_1 \vee v \in V_2 \wedge \neg (v \in V_1 \wedge v \in V_2))) \\ & \quad \rightarrow \\ & \quad (\exists e (e \in E_1 \wedge \exists v_1 \exists v_2 \\ & \quad \quad (v_1 \in V_1 \wedge v_2 \in V_2 \wedge \text{inc}(v_1, e) \wedge \text{inc}(v_2, e)))) \end{aligned}$$

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“There exists a set of edges, E_1 , such that every vertex is incident with exactly two edges in E_1 , and whenever (V_1, V_2) is a partition of the vertices, there is an edge in E_1 that is incident with vertices in both V_1 and V_2 .”

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In other words, the graph is hamiltonian.

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There is a polynomial-time algorithm that tests graphs in \mathcal{G} and decides whether they satisfy φ .

Courcelle's Theorem means that we can test intractable properties efficiently, if we limit the structural complexity of the input graph.

Matroids

A matroid is a structured hypergraph (**set-system**).

Definition

A **matroid** is a pair, (E, \mathcal{I}) , where E is a finite set (the **ground set**), and \mathcal{I} is a family of subsets (the **independent sets**), satisfying:

- ▶ $\emptyset \in \mathcal{I}$,
- ▶ $I_1 \in \mathcal{I}$ and $I_2 \subseteq I_1$ implies $I_2 \in \mathcal{I}$,
- ▶ $I_1, I_2 \in \mathcal{I}$ and $|I_2| < |I_1|$ implies there exists $e \in I_1 - I_2$ such that $I_2 \cup \{e\} \in \mathcal{I}$.

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Definition

A maximal independent set is a **basis**. A subset of E that is not independent is **dependent**. A minimal dependent set is a **circuit**.

Graphic matroids

Example

Let G be a graph with edge set E . Then

$$(E, \{I \subseteq E : G[I] \text{ does not contain a cycle}\})$$

is a **graphic** matroid.

The bases of a graphic matroid are maximal forests. The circuits are cycles in the graph.

Representable matroids

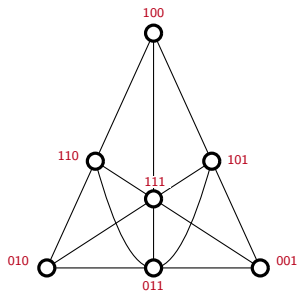
Example

Let \mathbb{F} be a field. Let E be a finite subset of the vector space \mathbb{F}^n .

Then

$$(E, \{I \subseteq E : I \text{ is linearly independent}\})$$

is an \mathbb{F} -representable matroid.



Matroids and the Robertson-Seymour project

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The **Graphs Minors Project** of Robertson and Seymour provides a qualitative structural description of any proper minor-closed class of graphs.

Among other consequences, we can deduce that in any infinite collection of graphs, one is isomorphic to a minor of another.

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Geelen, Gerards, and Whittle have now established a qualitative structural description of any proper minor-closed class of \mathbb{F} -representable matroids, when \mathbb{F} is a finite field.

We can deduce that in any infinite collection of \mathbb{F} -representable matroids, one is isomorphic to a minor of another.

Hliněný's Theorem

Hliněný's Theorem (2006)

Let φ be a sentence in MS_0 , the **monadic second-order logic** of matroids.

Let \mathcal{M} be a class of \mathbb{F} -representable matroids with bounded **branch-width**, where \mathbb{F} is a finite field.

There is a polynomial-time algorithm that tests matroids in \mathcal{M} and decides whether they satisfy φ .

Monadic second-order logic

$$\forall X_1 \forall X_2 (\text{Ind}(X_1) \wedge X_2 \subseteq X_1 \rightarrow \text{Ind}(X_2))$$

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$$\forall X_1 \forall X_2 (\text{Ind}(X_1) \wedge X_2 \subseteq X_1 \rightarrow \text{Ind}(X_2))$$

This expresses the second matroid axiom: every subset of an independent set is independent.

Tree automata

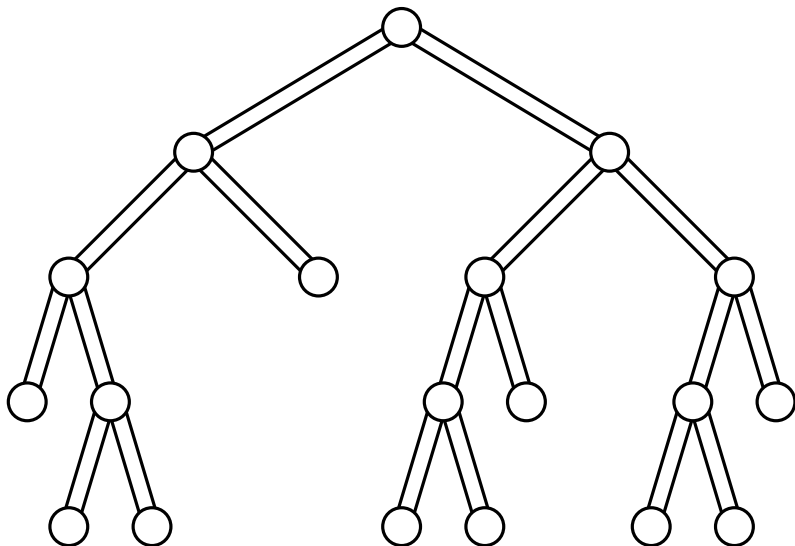
A **tree automaton** consists of a set of **states** (colours)



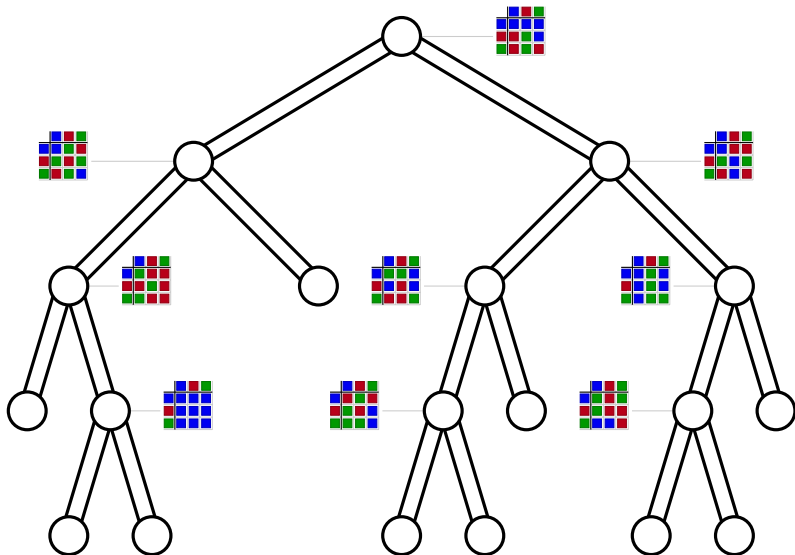
and a distinguished subset of **accepting** states.



Tree automata

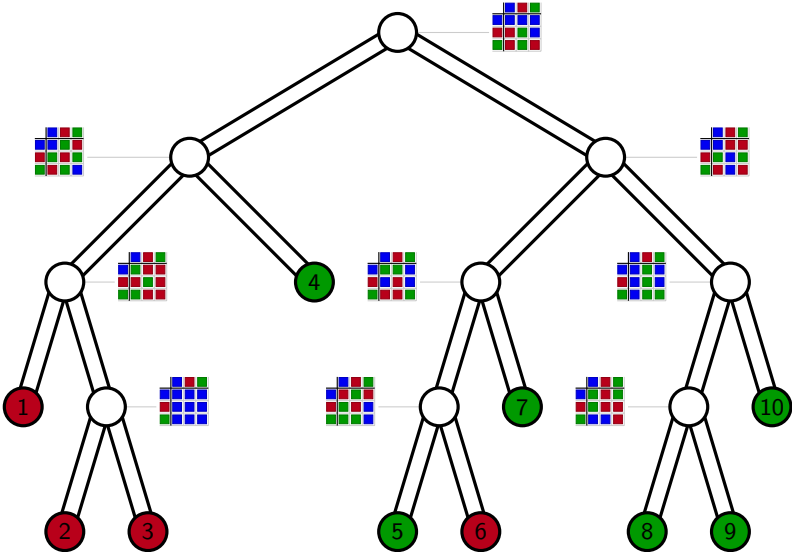


Tree automata

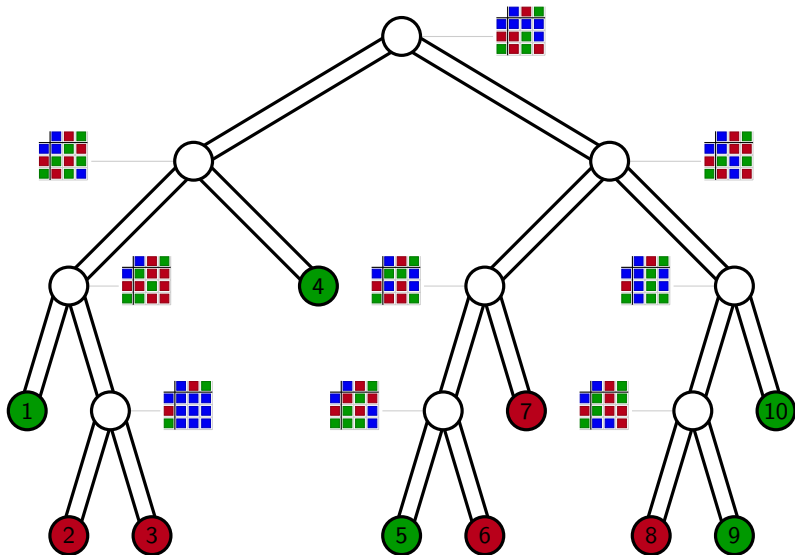


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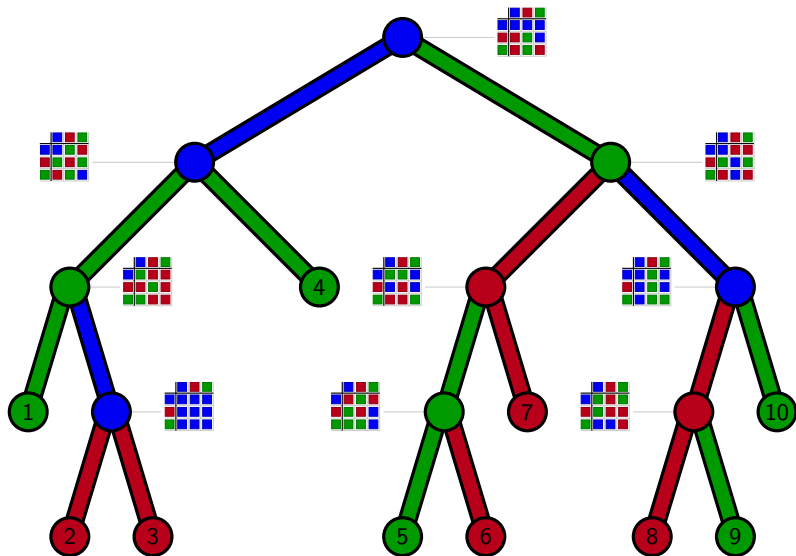
Tree automata



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Automatic set-systems

Given an automaton with two distinguished colours that encode subsets, there is a corresponding set-system on the leaf-set of any tree.

What families of set-systems arise in this way?

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What families of set-systems arise in this way?

Definition

Let \mathcal{M} be family of set-systems.

Assume there is an automaton, A , and for every $(E, \mathcal{I}) \in \mathcal{M}$, there is a tree, T_M , with leaf-set E , such that A accepts the subsets in \mathcal{I} and rejects the subsets not in \mathcal{I} .

Then we say that \mathcal{M} is **automatic**.

What families of set-systems are automatic?

Characterising automatic set-systems

Theorem

Let \mathcal{M} be a family of set-systems. Then \mathcal{M} is automatic if and only if it has bounded decomposition-width.

Decomposition-width

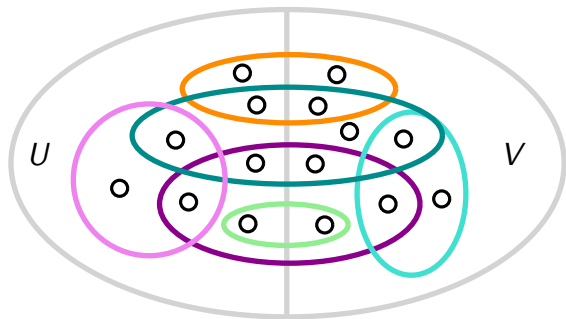
Definition

Let (E, \mathcal{I}) be a set-system, and let (U, V) be a partition of E .

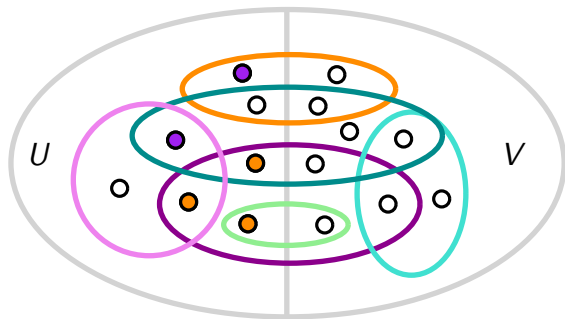
We define \sim_U , an equivalence relation on subsets of U .

Subsets $X, X' \subseteq U$ satisfy $X \sim_U X'$ if, whenever Z is a subset of V , both $X \cup Z$ and $X' \cup Z$ are in \mathcal{I} , or neither are.

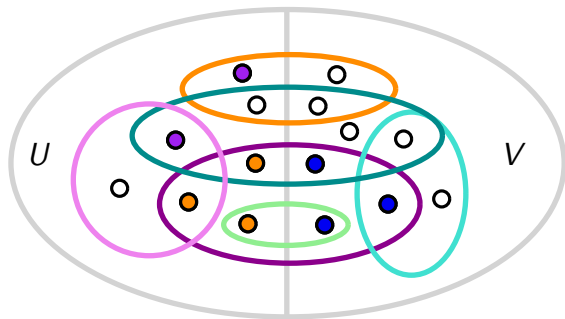
Decomposition-width



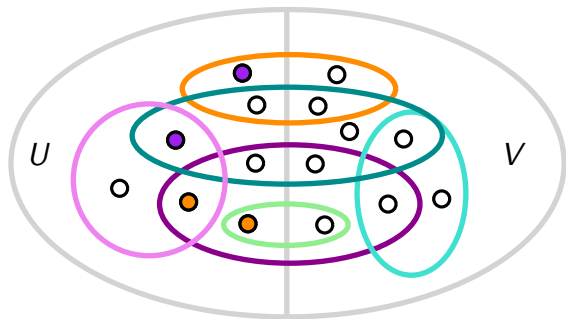
Decomposition-width



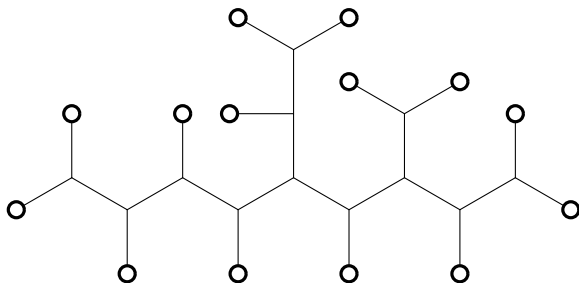
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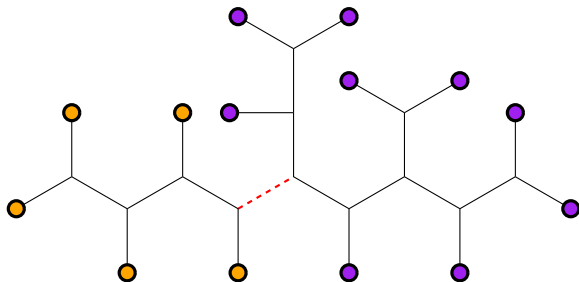


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A **decomposition** of a set-system (E, \mathcal{I}) is a bijection between E and the leaves of a tree where every non-leaf vertex has degree three.

A **displayed** set is any set corresponding to the leaves in a connected component created by deleting an edge of the tree.

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Definition

If \mathcal{M} has bounded decomposition-width, then there is an integer, K , and for every $M = (E, \mathcal{I}) \in \mathcal{M}$, we have a decomposition of M such that whenever U is a displayed set, then \sim_U has at most K equivalence classes.

An equivalent notion of decomposition-width was discussed by Král and Strozecki.

Tree automata and Hliněný's Theorem

If \mathcal{M} is an automatic family of set-systems, then there is an automaton, A , and for every $(E, \mathcal{I}) \in \mathcal{M}$, there is a tree, T_M , with leaf-set E , such that A accepts the subsets in \mathcal{I} and rejects the subsets not in \mathcal{I} .

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We can use a bootstrapping procedure to build an automaton that will quickly test any given MS_0 sentence for set-systems in \mathcal{M} .

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We can use a bootstrapping procedure to build an automaton that will quickly test any given MS₀ sentence for set-systems in \mathcal{M} .

This explains why tree automata are at the heart of Hliněný's Theorem.

The challenge is constructing T_M , given $M \in \mathcal{M}$.

Our conclusions

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We have done this for the following classes.

- ▶ Gain-graphic matroids (with gain labels from a finite group)
- ▶ Bicircular matroids
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We have some negative results: if \mathbb{F} is an infinite field, then Hliněný's Theorem cannot extend to the class of \mathbb{F} -representable matroids. Nor can it extend to the class of gain-graphic matroids over an infinite group (assuming $P \neq NP$).