Two sharp sufficient conditions

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14.4.2014

Overview

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6. A sharp sufficient condition for bipartite graphic sequences

The definition

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A sequence $\underline{d} = (d_1, \ldots, d_n)$ of nonnegative integers is *graphic* if there exists a simple, finite graph with degree sequence $\underline{d}$.

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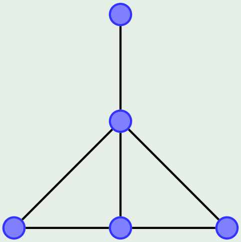

More examples of graphic sequences

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Example

The sequence $(4, 3, 2, 1)$ is not graphic. Neither is the sequence (3^5).

A fundamental result

The Erdős–Gallai Theorem is a fundamental, classic result that tells you when a sequence of integers occurs as the sequence of degrees of a simple graph.

Erdős–Gallai Theorem (1960)

A sequence $\underline{d} = (d_1, \ldots, d_n)$ *of nonnegative integers in decreasing order is graphic iff its sum is even and, for each integer* k *with* $1 \leq k \leq n$,

$$\sum_{i=1}^{k} d_i \leq k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}. \qquad (*)$$

There are several proofs of the Erdős–Gallai Theorem.

A fundamental result

Theorem (Li 1975)

A decreasing sequence of nonnegative integers is graphic if and only if it has even sum and for every index k with $d_k \geq k$ the Erdős–Gallai inequalities hold.

A sufficient condition for graphic sequences

Theorem (Zverovich and Zverovich 1992 [6])

Let a, b be positive integers and $\underline{d} = (d_1, \ldots, d_n)$ a decreasing sequence of integers with even sum and $d_1 \leq a$, $d_n \geq b$. If

$$nb \geq \frac{(a+b+1)^2}{4},$$

then $\underline{d}$ is graphic.

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Corollary

Let $\underline{d} = (d_1, \ldots, d_n)$ be a decreasing sequence of positive integers with even sum. If

$$n \geq \frac{d_1^2}{4} + d_1 + 1,$$

then $\underline{d}$ is graphic.

The case $d_n = 1$

Theorem (Cairns and Mendan 2012 [3])

Suppose that $\underline{d} = (d_1, \ldots, d_n)$ is a decreasing sequence of positive integers with even sum. If

$$n \geq \left\lfloor \frac{d_1^2}{4} + d_1 \right\rfloor ,$$

then $\underline{d}$ is graphic.

An equivalent theorem

Theorem

Let $\underline{d}$ be a decreasing sequence of positive integers with even sum and maximal element a, minimal element b and length n. If

$$nb \geq \frac{(a+b+1)^2}{4},$$

then $\underline{d}$ is graphic.

A sharp version of the Zverovich–Zverovich bound

Theorem (Cairns, Mendan and Nikolayevsky 2013 [5])

Suppose that $\underline{d}$ is a decreasing sequence of positive integers with even sum. Let a (resp. b) denote the maximal (resp. minimal) element of $\underline{d}$. Then $\underline{d}$ is graphic if

$$nb \geq \begin{cases} \left\lfloor \dfrac{(a+b+1)^2}{4} \right\rfloor - 1 & : \textit{ if } b \textit{ is odd, or } a+b \equiv 1 \pmod 4, \\ \\ \left\lfloor \dfrac{(a+b+1)^2}{4} \right\rfloor & : \textit{ otherwise,} \end{cases} \tag{1}$$

where $\lfloor . \rfloor$ denotes the integer part. Moreover, for any triple (a, b, n) of positive integers with $b < a < n$ that fails (1), there is a non-graphic sequence of length n having even sum with maximal element a and minimal element b.

Four cases

The inequality (1) can be conveniently expressed according to the following four disjoint, exhaustive cases:

(I) If $a+b+1 \equiv 2bn \pmod 4$, then $(a+b+1)^2 \leq 4bn$.

(II) If $a+b+1 \equiv 2bn+2 \pmod 4$, then $(a+b+1)^2 \leq 4bn+4$.

(III) If $a+b$ is even and bn is even, then $(a+b+1)^2 \leq 4bn+1$.

(IV) If n, a, b are all odd, then $(1+a+b)^2 \leq 4bn+5$.

Sharp examples

Theorem (Cairns, Mendan and Nikolayevsky 2013 [5])

Consider natural numbers $b < a < n$ and suppose that $as + b(n - s)$ is even. Then the sequence (a^s, b^{n-s}) is graphic if and only if $s^2 - (a + b + 1)s + nb \geq 0$.

The proof of this result is an application of the Erdős–Gallai Theorem.

Sharp examples

We need to find examples of non-graphic sequences.

(I) Suppose $a+b+1 \equiv 2bn \mod 4$. Assume $(a+b+1)^2 > 4bn$. Choose $s = \frac{a+b+1}{2}$.

(II) Suppose $a+b+1 \equiv 2bn+2 \mod 4$. Assume $(a+b+1)^2 > 4bn+4$. Choose $s = \frac{a+b+3}{2}$.

(III) Suppose $a+b$ is even and bn is even. Assume $(a+b+1)^2 > 4bn+1$. Choose $s = \frac{a+b}{2}$.

(IV) Suppose a, b, n are all odd. Assume $(a+b+1)^2 > 4bn+5$. Choose $s = \frac{a+b}{2}$ and $d_{s+1} = b+1$.

The proof

Theorem (Zverovich and Zverovich 1992)

A decreasing sequence $\underline{d}$ of nonnegative integers with even sum is graphic if and only if for every integer $k \leq d_k$ we have

$$\sum_{i=1}^{k}(d_i + in_{k-i}) \leq k(n-1),$$

where n_j is the number of elements in $\underline{d}$ equal to j.

The proof

Define K to be the maximum index such that $d_k \geq k$ and let $k > b$.

Lemma

Let $\underline{d} = (d_1, \ldots, d_n)$ be a decreasing sequence of integers. We have

$$\sum_{i=1}^{k} (d_i + i n_{k-i}) \leq k(n-1) + K(a+b+1) - K^2 - bn,$$

with equality only possible when $k = K$ and the sequence $\underline{d}$ has the form $\underline{d} = (a^K, b^{n-K})$.

The definition of bipartite graphic

Definition

A pair $(\underline{d_1}, \underline{d_2})$ of sequences is *bipartite graphic* if there exists a simple, finite bipartite graph whose parts have $\underline{d_1}, \underline{d_2}$ as their respective lists of vertex degrees.

Definition

A sequence $\underline{d}$ is *bipartite graphic* if there exists a simple, bipartite graph whose two parts each have $\underline{d}$ as their list of vertex degrees.

The Gale–Ryser Theorem gives a characterisation of bipartite graphic sequences.

A sufficient condition for bipartite graphic sequences

Theorem (Alon, Ben–Shimon and Krivelevich 2010 [1])

Let $a \geqslant 1$ be a real. If $\underline{d} = (d_1, \ldots, d_n)$ is a list of integers in decreasing order and

$$d_1 \leqslant \min\left\{ ad_n, \frac{4an}{(a+1)^2} \right\},$$

then $\underline{d}$ is bipartite graphic.

Theorem

A decreasing list of positive integers $\underline{d}$ with maximal element a and minimal element b is bipartite graphic if

$$nb \geq \frac{(a+b)^2}{4}. \tag{2}$$

A sharp sufficient condition for bipartite graphic sequences

Theorem (Cairns, Mendan and Nikolayevsky 2014 [4])

Suppose that $\underline{d}$ is a decreasing sequence of positive integers. Let a (resp. b) denote the maximal (resp. minimal) element of $\underline{d}$. Then $\underline{d}$ is bipartite graphic if

$$nb \geq \begin{cases} \frac{(a+b)^2}{4} & : \text{ if } a \equiv b \pmod 2, \\ \left\lfloor \frac{(a+b)^2}{4} \right\rfloor & : \text{ otherwise,} \end{cases} \tag{3}$$

where $\lfloor . \rfloor$ denotes the integer part. Moreover, for any triple (a, b, n) of positive integers with $b < a < n + 1$ that fails (3), there is a non-bipartite-graphic sequence of length n with maximal element a and minimal element b.

Sharp examples

Theorem (Cairns, Mendan and Nikolayevsky 2013 [4])

Let $a, b, n, s \in \mathbb{N}$ *with* $b < a \leq n$ *and* $s < n$. *Then the sequence* (a^s, b^{n-s}) *is bipartite graphic if and only if* $s^2 - (a+b)s + nb \geq 0$.

Graphs with loops and bipartite graphs

Proposition (Cairns and Mendan 2012 [2])

A sequence $\underline{d} = (d_1, \ldots, d_n)$ of nonnegative integers in decreasing order is the sequence of reduced degrees of the vertices of a graph-with-loops if and only if $\underline{d}$ is bipartite graphic.

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