Closed walks in a regular graph

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Outline

1. Background
   - The Set Up
   - Need To Be Knowns
2. Related Results
   - Stevanovic et al.
   - Wanless
3. The Best Is Yet To Come
   - Present
   - Future

Outline

Definitions: Adjacency Matrix, Spectrum

- For this talk, G is a simple graph with $|V(G)| = n$ vertices.
- The `adjacency matrix`, $A = [a_{ij}]$, of G, is the $n \times n$ matrix defined as

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ is adjacent to } j \\ 0 & \text{otherwise} \end{cases}$$

- The `spectrum` of a graph with respect to its adjacency matrix consists of the *eigenvalues* of its adjacency matrix with their *multiplicity*.

Integral Graphs

When are the eigenvalues of a graph integers?

- `integral graphs` are graphs that have integer eigenvalues
- Ex// C_3, C_4, C_6, K_n, P_2
- $\exists$ operations closed under integrality: $\times$, $+$

n	1	2	3	4	5	6	7	8	9	10	11	12	13
#	1	1	1	2	3	6	7	22	24	83	113	?	?

Definitions: Regular graph,Closed walk

Limit ourselves to...
Integral Graphs
$\rightarrow$ `regular` - G is k-regular if $deg(v) = k \forall v \in V(G)$
$\rightarrow$ `bipartite` - G is `bipartite` if $V(G)$ can be partitioned into two subsets X and Y such that each edge has one end in X and one end in Y
Look at...
Counting Closed Walks

- A `walk` in G is a finite sequence $W = v_0 v_1 ... v_l$ of vertices such that v_i is adjacent to v_{i+1}.
- W is `closed` if $v_0 = v_l$.

In this talk, I present a preliminary report on how we might go about searching for **regular bipartite integral graphs** by **counting closed walks**.

Outline

Closed Walks and Adjacency Matrices

- **Lemma:** For $a^r_{i,j}$ the i,jth entry of the matrix A^r,

$$a^r_{i,j} = \#\text{ walks of length } r \text{ from } i \text{ to } j$$

- It follows that,

$$\sum_{i=1}^{n} a^r_{i,i} = \text{total } \# \text{ closed walks of length } r \text{ in } G$$
$$= Tr(A^r)$$
$$= \sum_{i=1}^{n} \lambda_i^r$$

Closed Walks Relating Eigenvalues To Graph Info

It follows that for n vertices, e edges, and t 3-cycles,

$$\sum_{i=1}^{n} \lambda_i^1 = \#\text{ closed walks of length 1 in } G = 0$$

$$\sum_{i=1}^{n} \lambda_i^2 = \#\text{ closed walks of length 2 in } G = 2e$$

$$\sum_{i=1}^{n} \lambda_i^3 = \#\text{ closed walks of length 3 in } G = 6t$$

Closed Walks Relating Eigenvalues To Graph Info

It follows that for n vertices, e edges, and t 3-cycles,

$$\sum_{i=1}^{n} \lambda_i^1 = 0$$

$$\sum_{i=1}^{n} \lambda_i^2 = 2e$$

$$\sum_{i=1}^{n} \lambda_i^3 = 6t$$

Thus edges and 3-cycles are completely determined by the spectrum of G.

Outline

Using the Trace Equations to Refine Graph Eigenvalue Lists

This has been done for integral graphs when G is 4-**regular bipartite.**

- $Sp(G) = \{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$
- Stevanovic et al. (2007) adjusted and added to the former trace equations for this special case: for n vertices, q 4-cycles, and h 6-cycles,

$$Tr(A^0) = n$$
$$Tr(A^2) = 4n$$
$$Tr(A^4) = 28n + 8q$$
$$Tr(A^6) = 232n + 144q + 12h$$
$$Tr(A^8) \geq 2092n + 2024q + 288h$$

Stevanovic et al. Results

The authors

- used the equations to determine 1888 feasible spectra of the 4-regular bipartite integral graphs
- used the inequality to reduce this list to 828, $n \leq 280$
- added the inequality via a recurrence relation that counted the closed walks containing a given cycle:
  - 4-cycles
  - 6-cycles

n	x	y	z	q	h
5 0	0	4	0	30	130
6 0	1	4	0	27	138

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There's More To Be Done!

I plan to take this further

- WHAT?
  $\longrightarrow$ Get equality rather than a bound for $Tr(A^8)$
  $\longrightarrow$ Add more equations to the Stevanovic set
- HOW? Consider subgraphs other than cycles: bound is a result of this
- WHY? More equations means
  $\longrightarrow$ more information
  $\longrightarrow$ enough to make lists of feasible spectra
  $\longrightarrow$ less candidates (refine obtainted lists)

Outline

Counting Around Subgraphs Other Than Cycles

Wanless (2009) recently submitted a paper that counted certain closed walks to find approximations for the matching polynomial of a graph.

- the graphs are regular
- these closed walks are counted based on
  $\longrightarrow$ the cycles AND
  $\longrightarrow$ the polycyclic subgraphs
- an algorithm is given that counts these walks up to a given length

Wanless Algorithm

The mentioned algorithm counts certain closed walks in **regular graphs**, using

- enumeration - find/collect *base walks* about subgraphs
- generating functions - count all desired closed walks around base walks
- inclusion/exclusion principles - resolve overcounting

Resulting Expression Examples

For G, $(k+1)$-regular bipartite:

$$\begin{aligned}\epsilon_5 &= 80kC_4\\ \epsilon_6 &= 528k^2C_4 + 12C_6 - 48\theta_{2,2,2}\\ \epsilon_7 &= 2912k^3C_4 + 168kC_6 - 672k\theta_{2,2,2} - 56\theta_{3,3,1}\end{aligned}$$

where ϵ_l denotes the *desired* closed walks of length $2l$

Outline

A Work In Progress

Closed walks are

- totally-reducible - generating function already existed
- closed containing a cycle - have a generating function for the number containing a single cycle of arbitrary length
- closed containing a polycyclic subgraph - have a generating function for the number containing a closed walk around a subgraph

Note: these generating functions require that G is regular

Counting Closed Walks

So for regular bipartite graphs G:

- Determine the subgraphs that matter
- Devise an algorithm that considers each subgraph and
  - takes *base walks* that induce it - defined
  - counts walks containing base walks - uses polycyclic generating function
  - adds counts of all base walks together - the all encompassing generating function for the subgraph is ready
- Produce polynomials for each length that depend on n, regularity, and the number of certain subgraphs of G

Outline

What's Next?

- Use equations to find/refine lists of feasible spectra for k-regular bipartite integral graphs with $k \leq 4$
- Consider integral graphs that are regular non-bipartite; add other pertinent subgraphs, equations
- Apply the same methodology to strongly regular graphs
  - Find possible configurations of the missing *Moore* graph?

Dragan Stevanovic and Nair M.M. de Abreu and Maria A.A. de Freitas and Renata Del-Vecchio.
Walks and regular integral graphs.
Linear Algebra and its Applications, 423(1):119–135, 2007.

I. M. Wanless.
Counting matchings and tree-like walks in regular graphs.
Combinatorics, Probability and Computing, Accepted, 2009.

THE END

Using Closed Walk Polynomials

- Take the polynomials and build a system of equations for regular bipartite graphs
- Let $k = 4$, since G is k-regular
- Apply it to the list of feasible spectra for 4-regular bipartite integral graphs
- Obtain shorter lists of the form:
- Obtain a new count < 828 for graphs with spectra of the form:

$$Sp(G) = \{4, 3^x, 2^y, 1^z, 0^{2w}, -1^z, -2^y, -3^x, -4\}$$