Closed walks in a regular graph

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1 Prelude
   - Introduction
   - A Motivating Set of Equivalences

2 Fugue
   - An Extension of These Equivalences
   - A Related Method

3 Descant
   - The Plan
Outline

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Some Definitions
Graph, Spectra, Adjacency Matrix

- The spectrum of a graph with respect to its adjacency matrix consists of the eigenvalues of its adjacency matrix with their multiplicity.
- For this talk, let G be a simple graph with vertex set, \( V(G) \) of size \( n \).
- The adjacency matrix, \( A = [a_{ij}] \), of \( G \), is the \( n \times n \) matrix defined as

\[
a_{ij} = \begin{cases} 
1 & \text{if } i \text{ is adjacent to } j \\
0 & \text{otherwise}
\end{cases}
\]
This matrix, $A$, is real and symmetric, thus:
- $A$ is similar to a diagonal matrix $B$ with diagonal consisting of the eigenvalues of $A$.

Similar matrices have the same trace, so:
- the trace of $A$,

$$\text{Tr}(A) = \text{Tr}(B) = \sum \lambda_k$$

where $\lambda_k$ are the $n$ eigenvalues of $A$. 
So considering our entries of $A$,

$$a_{i,j} = 1 \text{ when we have } i \text{ adjacent to } j$$

If we consider the matrix $A^2$ and look at one entry:

$$a_{i,j}^2 = a_{i,1}a_{1,j} + a_{i,2}a_{2,j} + ... + a_{i,n}a_{n,j}$$

We get that

$$a_{i,j}^2 = \# \text{ walks of length 2 from } i \text{ to } j$$

And if you carry on in this way, and consider one entry of $A^r$:

$$a_{i,j}^r = \# \text{ walks of length } r \text{ from } i \text{ to } j$$
Walks and Adjacency Matrices
Considering the adjacency algebra of $G$.

- So considering our entries of $A$, $a_{i,j} = 1$ when we have $i$ adjacent to $j$

- If we consider the matrix $A^2$ and look at one entry:

$$a^2_{i,j} = a_{i,1}a_{1,j} + a_{i,2}a_{2,j} + \ldots + a_{i,n}a_{n,j}$$

- We get that

$$a^2_{i,j} = \# \text{ walks of length 2 from } i \text{ to } j$$

- And if you carry on in this way, and consider one entry of $A^r$:

$$a^r_{i,j} = \# \text{ walks of length } r \text{ from } i \text{ to } j$$
Closed Walks and Adjacency Matrices.
The trace acting on the adjacency algebra of $G$.

What about the diagonal?

- The entries along the diagonal in $A^r$ give the number of walks of length $r$ from a given vertex to itself.
- $Tr(A^r)$ gives the total number of closed walks of length $r$ in $G$.
- Considering our diagonal matrix $B$:

$$Tr(A^r) = Tr(B^r) = \sum_{k=1}^{n} \lambda_k^r$$
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Closed Walks and Adjacency Matrices.  
The trace acting on the adjacency algebra of $G$.

It can be shown that for $n$ as before, $e$ edges, and $t$ triangles or 3-cycles,

$$\sum_{k=1}^{n} \lambda_k^1 = Tr(A^1) = 0$$
$$\sum_{k=1}^{n} \lambda_k^2 = Tr(A^2) = 2e$$
$$\sum_{k=1}^{n} \lambda_k^3 = Tr(A^3) = 6t$$

Or simply given the spectrum of $G$
Closed Walks and Adjacency Matrices.
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Or simply given the spectrum of $G$
Can these results be extended for higher powers of $A$?

- $K_{1,4}$ and $K_1 \cup C_4$ have the same same spectrum:
  \[ \{-2^1, 0^3, 2^1\} \]
  but they don’t have the same number of 4-cycles.

- We need to look further than the sole contribution of $n$-cycles to the number of closed walks of length $n$ in $G$. 
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Has any other work been done to extend these results?

- A paper by Stevanovic et al., stated that for 4-regular bipartite graphs; where \( n \) is again the number of vertices, \( q \) the number of 4-cycles, and \( h \) the number of 6-cycles,

\[
\begin{align*}
\text{Tr}(A^0) &= n \\
\text{Tr}(A^2) &= 4n \\
\text{Tr}(A^4) &= 28n + 8q \\
\text{Tr}(A^6) &= 232n + 144q + 12h \\
\text{Tr}(A^8) &\geq 2092n + 2024q + 288h
\end{align*}
\]
Closed Walks For Higher Powers of A
Walking in the corresponding tree

These results are based on an equivalence established between the number of closed walks in $k$-regular graphs and infinite $k$-regular trees.
We can look at walks in trees recursively

- Let $w_k(d, l)$ denote the number of walks of length $l$ between the vertices at a distance $d$ in an infinite $k$-regular tree.
- $w_k(d, l) = w_k(d - 1, l - 1) + (k - 1)w_k(d + 1, l - 1)$

The authors do not find a closed form except when $d = 0$

$$w_k(0, l) = \frac{2k - 2}{k - 2 + k\sqrt{1 - 4kx}}$$
Counting Closed Walks in the Corresponding Tree

Conceptually

- What closed walks of $G$ correspond with walks where $d = 0$ in our tree?
- Which don’t?
Summary Of This Extension By Stevanovic et al.

The authors managed to

- find a recursive formula to count the number of closed walks of length $l$ containing the cycle $C$ in a $k$-regular graph
- let $k = 4$ and find the number of closed walks for $l \leq 6$ of bipartite graphs in terms of $n$ and the number of various cycles
- find a bound on walks of length 8:

$$Tr(A^8) \geq 2092n + 2024q + 288h$$

with note that they need to account for not only 8-cycles but also subgraphs like two 4-cycles sharing a common vertex.
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Curiously, the same closed form for generating closed walks in an infinite rooted *nearly-regular* tree is derived in a soon to be published paper by an **AMS 2009 medal** winning author, Wanless.

- Let $T_r$ count closed rooted walks in an infinite tree with root, degree $r$, and every other vertex, degree $k + 1$.

  $$T_r = \frac{2k}{2k - r + r \sqrt{1 - 4(k)x}}$$

- Resulting is a polynomial in $x$ with the coefficient of $x^l$ corresponding to the number of walks of length $2l$. 
This generating function, $T_r$, is used to count closed walks in a graph $G$ which are specifically:

- **totally-reducible**: back-tracks itself completely
- and **not tree-like**: contains a cycle at some intermediate step of the back-tracking process

The author recognizes that all of the *desired* closed walks contain a particular kind of walk about a cycle.
Closed Walks That Extend A Given Walk

The generating function, $T_r$ is used to craft a generating function that takes a certain walk around a cycle of length $2l$ that induces a certain subgraph in $G$ and

- adds totally-reducible bits
- moves the start/end point of the walk
Summary Of This Method by Wanless

The author managed to:

- Obtain a generating function for all totally-reducible walks about a given closed walk
- Express the number of totally-reducible not tree-like walks of length $2l$ as polynomials in $n$, $k$, and the number of certain subgraphs of the $(k + 1)$-regular graph $G$
- Confirm some known results for $l \leq 5$ and publish results for $l \leq 8$
- Confirm $l \leq 6$ and publish $l \leq 10$ in the bipartite case
Examples, where $\epsilon_l$ denotes the number of totally-reducible not tree-like walks of length $2l$:

$$\epsilon_4 = 48kC_3 + 8C_4$$
$$\epsilon_5 = 270k^2 C_3 + 80kC_4 + 10C_5 - 40\theta_{2,2,1}$$
$$\epsilon_6 = (1320k^3 - 6)C_3 + 528k^2 C_4 + 120kC_5 + 12C_6 + 192K_4 - (480k + 12)\theta_{2,2,1} - 48(\theta_{3,2,1} + \theta_{2,2,2} + C_{3,3})$$
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The number of closed walks with $n$ as before, $e$ edges, and $t$ 3-cycles,

$$\sum_{k=1}^{n} \lambda_k^1 = 0$$

$$\sum_{k=1}^{n} \lambda_k^2 = 2e$$

$$\sum_{k=1}^{n} \lambda_k^3 = 6t$$
The number of closed walks with $n$ as before, $q$ 4-cycles, and $h$ 6-cycles,

\[
\sum_{k=1}^{n} \lambda_k^0 = n \\
\sum_{k=1}^{n} \lambda_k^2 = 4n \\
\sum_{k=1}^{n} \lambda_k^4 = 28n + 8q \\
\sum_{k=1}^{n} \lambda_k^6 = 232n + 144q + 12h
\]
I plan to extend these results for $k$-regular graphs and to consider $k$-regular bipartite graphs for general $k$.

The number of closed walks will be given by sets of polynomials, a polynomial for each length of walk.

These will be polynomials on $n$, $k$, and the number of certain subgraphs of the original graph.

A series of equations will be formed from the two ways of counting closed walks on regular graphs of length $l$: the trace of the $l$-th power of the adjacency matrix of the graph and the above polynomials.
Further Possibilities

- These equations have unknowns on n, k, the number of various subgraphs of the graph, and the eigenvalues of the graph.
- Thus given different knowns, the possibilities for the unknowns could be determined.
- For example: Stevanovic et al. used the equations determined here to
  - refine their list of feasible spectra of 4-regular bipartite integral graphs
  - extend the list of known 4-regular integral graphs
Further Possibilities

In this way, using known properties of regular graph spectrum for families of graphs
- on certain numbers of vertices
- or of certain subgraph configurations
- or with certain properties;

I hope to obtain results relating these graphs to their algebraic properties
FINE
For Further Reading I

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C. D. Godsil, G. Royle.  
_Algebraic Graph Theory._  

Dragan Stevanovic and Nair M.M. de Abreu and Maria A.A. de Freitas and Renata Del-Veccchio.  
Walks and regular integral graphs.  

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Counting matchings and tree-like walks in regular graphs.  
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