ANNULAR AND PANTS THRACKLES

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Introduction

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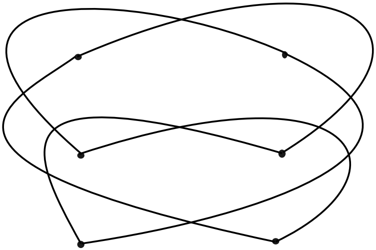

Figure 1: Thrackled 6-cycle

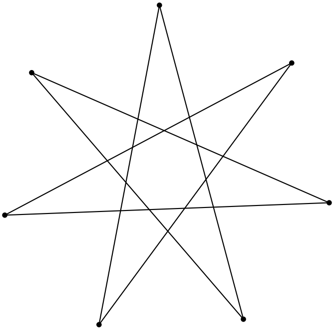

Figure 2: Thrackled 7-cycle

The Musquash

An *n-gonal musquash* is a thrackled n-cycle whose successive edges $e_0, \ldots, e_{n-1}$ intersect in the following manner: if the edge e_0 intersects the edges $e_{k_1}, \ldots, e_{k_{n-3}}$ in that order, then for all $j = 1, \ldots, n-1$, the edge e_j intersects the edges $e_{k_1+j}, \ldots, e_{k_{n-3}+j}$ in that order, where the edge subscripts are computed modulo n [Woodall, 1969].

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Every musquash is either isotopic to a standard n-musquash, or is a thrackled six-cycle [CK, 1999, 2001].

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The Conjecture is however known to be true for some classes of thrackles such as

(i) straight line thrackles,

(ii) spherical thrackles,

(iii) outerplanar thrackles.

Outerplanar Thrackles

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Theorem 1

Suppose a graph G admits an outerplanar thrackle drawing. Then

a. *any cycle in G is odd [CN 2012];*
b. *the number of edges of G does not exceed the number of vertices [PS 2011];*
c. *if G is a cycle, then the drawing is Reidemeister equivalent to a standard odd musquash [CN 2012].*

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We say that a thrackle drawing belongs to the class T_d, $d \geq 1$, if all the vertices of the drawing lie on the boundaries of d disjoint discs $D_1, \ldots, D_d$.

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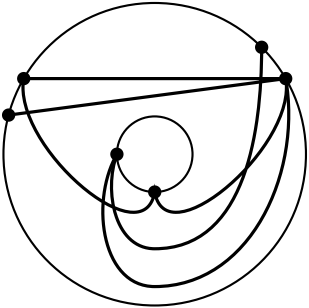

Figure 3: An annular thrackle drawing.

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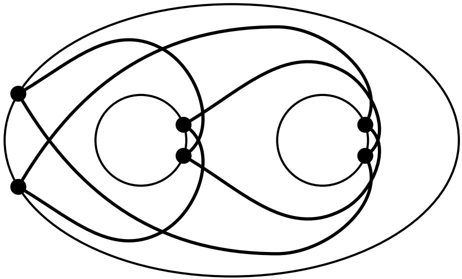
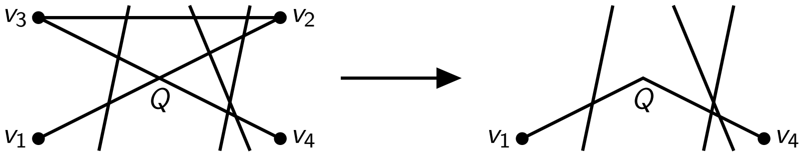


Figure 4: Pants thrackle drawing of a six-cycle.

Edge removal operation



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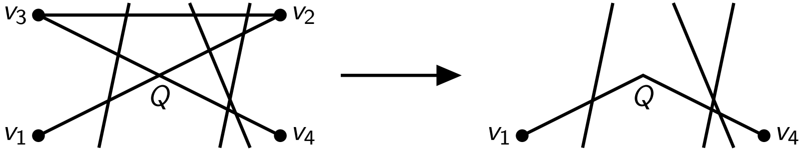

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Edge removal does not necessarily result in a thrackle drawing. Consider the triangular domain △ bounded by the arcs v_2v_3, Qv_2 and v_3Q and not containing the vertices v_1 and v_4 (if we consider the drawing on the plane, △ can be unbounded).

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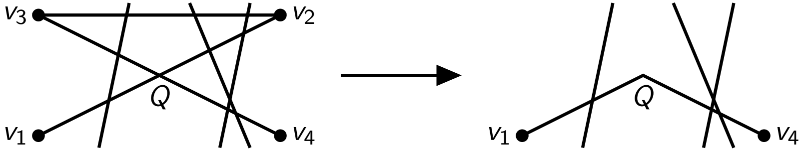

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Edge removal does not necessarily result in a thrackle drawing. Consider the triangular domain $\triangle$ bounded by the arcs v_2v_3, Qv_2 and v_3Q and not containing the vertices v_1 and v_4 (if we consider the drawing on the plane, $\triangle$ can be unbounded).

Lemma 1

Edge removal results in a thrackle drawing if and only if $\triangle$ contains no vertices of $\mathcal{T}(G)$.

Edge removal ctd

For a thrackle drawing of class T_d;

(a) the condition of Lemma 1 is satisfied if $\triangle$ contains none of the d circles bounding the discs D_k;

(b) edge removal on an n-cycle, if possible, produces a thrackle drawing of the same class T_d of an $(n-2)$-cycle.

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We call a thrackle drawing irreducible if it admits no edge removals and reducible otherwise.

Theorem- annular thrackles

Theorem 2

Suppose a graph G admits an annular thrackle drawing. Then

a. *any cycle in G is odd;*
b. *the number of edges of G does not exceed the number of vertices;*
c. *if G is a cycle, then the drawing is, in fact, outerplanar (and as such, is Reidemeister equivalent to a standard odd musquash).*

Theorem- pants thrackles

Theorem 3

Suppose a graph G admits a pants thrackle drawing. Then

a. *any even cycle in G is a six-cycle, and its drawing is Reidemeister equivalent to the one in Figure 4;*
b. *if G is an odd cycle, then the drawing can be obtained from a pants drawing of a three-cycle by a sequence of edge insertions;*
c. *the number of edges of G does not exceed the number of vertices.*

The word W

To a path in a thrackle drawing of class T_d we associate a word W in the alphabet $X = \{x_1, \ldots, x_d\}$ in such a way that the i-th letter of W is x_k if the i-th vertex of the path lies on the boundary of the disc D_k.

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Lemma 2

For a thrackle drawing of a graph G of class T_d,

a. *For no two different $i, j = 1, \dots, d$, may a thrackle drawing of class T_d contain two edges with the words x_i^2 and x_j^2.*
b. *Suppose that for some $i = 1, \dots, d$, a thrackle drawing of class T_d contains a two-path with the word x_i^3 the first two vertices of which have degree 2. Then the drawing is reducible.*

Proof of the Theorems

To prove Theorem 2(a) and Theorem 3(a, b) we need Lemma 3 and Lemma 4, respectively:

Lemma 3

If an n-cycle admits an irreducible annular thrackle drawing, then $n = 3$.

Lemma 4

If a cycle C admits an irreducible pants thrackle drawing, then C is either a three-cycle or a six-cycle, and in the latter case, the drawing is Reidemeister equivalent to the one in Figure 4.

To deduce Theorem 3(a) from Lemma 4 we look at all the thrackled 8-cycles.

Up to isotopy and Reidemeister moves, there exist exactly three thrackled eight-cycles [MY 2016], each of which can be obtained by edge insertion in a thrackled six-cycle but none of them is a pants thrackle.

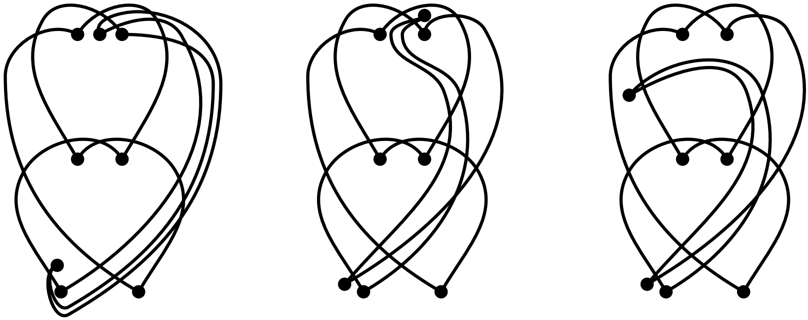

Figure 6: All thrackled eight-cycles up to Reidemeister equivalency.

To prove Theorem 2(c), we analyse short thrackled paths and show that any annular thrackled cycle is alternating; i.e, for every edge e and every two-path fg vertex-disjoint from e, the crossings of e by f and g have opposite orientations.

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Finally to prove Conways Thackle Conjecture for the class T_2 and T_3, i.e, Theorem 2(b) and Theorem 3(c) respectively, we annalyse the forbidden configurations.

Forbidden configurations

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It follows that to prove Conway's Thrackle Conjecture for thrackle drawings in a class T_d it is sufficient to prove that no dumbbell and no theta-graph admit a thrackle drawing of class T_d.

Proof continued

For Theorem 2(b), the proof follows from Lemma 3 and the fact that we must have at least one even cycle [LPS, 1997].

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For Theorem 3(c), we have a dumb-bell graph or a theta graph consisting of a six-cycle and another graph.

By analysing small trees attached to the standard pants thrackled six-cycle we get a contradiction.