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Introduction

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Figure 1: Thrackled 6-cycle

Figure 2: Thrackled 7-cycle
An *n*-gonal musquash is a thrackled *n*-cycle whose successive edges $e_0, \ldots, e_{n-1}$ intersect in the following manner: if the edge $e_0$ intersects the edges $e_{k_1}, \ldots, e_{k_{n-3}}$ in that order, then for all $j = 1, \ldots, n - 1$, the edge $e_j$ intersects the edges $e_{k_1+j}, \ldots, e_{k_{n-3}+j}$ in that order, where the edge subscripts are computed modulo $n$ [Woodall, 1969].
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A **standard odd musquash** is the simplest example of a thrackled cycle: for *n* odd, distribute *n* vertices evenly on a circle and then join by an edge every pair of vertices at the maximal distance from each other.
An \textit{n-gonal musquash} is a thrackled \textit{n}-cycle whose successive edges $e_0, \ldots, e_{n-1}$ intersect in the following manner: if the edge $e_0$ intersects the edges $e_{k_1}, \ldots, e_{k_{n-3}}$ in that order, then for all $j = 1, \ldots, n - 1$, the edge $e_j$ intersects the edges $e_{k_1+j}, \ldots, e_{k_{n-3}+j}$ in that order, where the edge subscripts are computed modulo $n$ [Woodall, 1969].

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Every musquash is either isotopic to a standard $n$-musquash, or is a thrackled six-cycle [CK, 1999, 2001].
Conjecture

Conway’s Thrackle Conjecture [1967]

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The Conjecture is however known to be true for some classes of thrackles such as

(i) straight line thrackles,
(ii) spherical thrackles,
(iii) outerplanar thrackles.
Outerplanar Thrackles

**Outerplanar thrackles** are thrackles whose vertices all lie on the boundary of a single disc $D_1$. Such thrackles are very well understood.
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**Theorem 1**

Suppose a graph $G$ admits an outerplanar thrackle drawing. Then

(a) any cycle in $G$ is odd [CN 2012];

(b) the number of edges of $G$ does not exceed the number of vertices [PS 2011];

(c) if $G$ is a cycle, then the drawing is Reidemeister equivalent to a standard odd musquash [CN 2012].
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We say that a thrackle drawing **belongs to the class** $T_d$, $d \geq 1$, if all the vertices of the drawing lie on the boundaries of $d$ disjoint discs $D_1, \ldots, D_d$. 
Thrackles of class $T_2$

A thrackle drawing of class $T_2$ is called annular thrackle. This is a thrackle whose vertices lie on the boundary of 2 discs, $D_1$ and $D_2$. 
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Figure 3: An annular thrackle drawing.
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Figure 4: Pants thrackle drawing of a six-cycle.
Edge removal operation

Figure 5: The edge removal operation.
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Edge removal does not necessarily result in a thrackle drawing. Consider the triangular domain $\triangle$ bounded by the arcs $v_2v_3$, $Qv_2$ and $v_3Q$ and not containing the vertices $v_1$ and $v_4$ (if we consider the drawing on the plane, $\triangle$ can be unbounded).
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**Lemma 1**

*Edge removal results in a thrackle drawing if and only if $\triangle$ contains no vertices of $T(G)$.*
For a thrackle drawing of class $T_d$;

(a) the condition of Lemma 1 is satisfied if $\triangle$ contains none of the $d$ circles bounding the discs $D_k$;

(b) edge removal on an $n$-cycle, if possible, produces a thrackle drawing of the same class $T_d$ of an $(n - 2)$-cycle.
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We call a thrackle drawing irreducible if it admits no edge removals and reducible otherwise.
Theorem 2

Suppose a graph $G$ admits an annular thrackle drawing. Then

(a) any cycle in $G$ is odd;

(b) the number of edges of $G$ does not exceed the number of vertices;

(c) if $G$ is a cycle, then the drawing is, in fact, outerplanar (and as such, is Reidemeister equivalent to a standard odd musquash).
Theorem 3

Suppose a graph $G$ admits a pants thrackle drawing. Then

(a) any even cycle in $G$ is a six-cycle, and its drawing is Reidemeister equivalent to the one in Figure 4;

(b) if $G$ is an odd cycle, then the drawing can be obtained from a pants drawing of a three-cycle by a sequence of edge insertions;

(c) the number of edges of $G$ does not exceed the number of vertices.
To a path in a thrackle drawing of class $T_d$ we associate a word $W$ in the alphabet $X = \{x_1, \ldots, x_d\}$ in such a way that the $i$-th letter of $W$ is $x_k$ if the $i$-th vertex of the path lies on the boundary of the disc $D_k$. 
The word $W$

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For a word $w$ and an integer $m$, $w^m$ denote the word obtained by $m$ consecutive repetitions of $w$. 
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For a word $w$ and an integer $m$, $w^m$ denote the word obtained by $m$ consecutive repetitions of $w$.

Lemma 2

For a thrackle drawing of a graph $G$ of class $T_d$,

(a) For no two different $i, j = 1, \ldots, d$, may a thrackle drawing of class $T_d$ contain two edges with the words $x_i^2$ and $x_j^2$.

(b) Suppose that for some $i = 1, \ldots, d$, a thrackle drawing of class $T_d$ contains a two-path with the word $x_i^3$ the first two vertices of which have degree 2. Then the drawing is reducible.
Proof of the Theorems

To prove Theorem 2(a) and Theorem 3(a, b) we need Lemma 3 and Lemma 4, respectively:

**Lemma 3**

If an $n$-cycle admits an irreducible annular thrackle drawing, then $n = 3$.

**Lemma 4**

If a cycle $C$ admits an irreducible pants thrackle drawing, then $C$ is either a three-cycle or a six-cycle, and in the latter case, the drawing is Reidemeister equivalent to the one in Figure 4.
To deduce Theorem 3(a) from Lemma 4 we look at all the thrackled 8-cycles.

Up to isotopy and Reidemeister moves, there exist exactly three thrackled eight-cycles [MY 2016], each of which can be obtained by edge insertion in a thrackled six-cycle but none of them is a pants thrackle.

Figure 6: All thrackled eight-cycles up to Reidemeister equivalency.
To prove Theorem 2(c), we analyse short thrackled paths and show that any annular thrackled cycle is alternating; i.e., for every edge $e$ and every two-path $fg$ vertex-disjoint from $e$, the crossings of $e$ by $f$ and $g$ have opposite orientations.

The claim then follows from the fact that every alternating thrackle is outerplanar [CN, 2012].

Finally to prove Conways Thackle Conjecture for the class $T_2$ and $T_3$, i.e., Theorem 2(b) and Theorem 3(c) respectively, we analyse the forbidden configurations.
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Forbidden configurations

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Given a thrackle drawing of class $T_d$ of a figure-8 graph, one can always perform vertex-splitting operation [MY2016] on the vertex of degree 4 to obtain a thrackle drawing of the same class $T_d$ of a dumbbell graph.
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It follows that to prove Conway’s Thrackle Conjecture for thrackle drawings in a class $T_d$ it is sufficient to prove that no dumbbell and no theta-graph admit a thrackle drawing of class $T_d$. 
For Theorem 2(b), the proof follows from Lemma 3 and the fact that we must have at least one even cycle [LPS, 1997].
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For Theorem 3(c), we have a dumb-bell graph or a theta graph consisting of a six-cycle and another graph.

By analysing small trees attached to the standard pants thrackled six-cycle we get a contradiction.