Coclass theory for nilpotent associative algebras

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(joint work with Bettina Eick)

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Introduction

What is coclass theory?

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- The *coclass* of a p-group of order p^n and class c is defined as $n - c$.
- Using coclass as primary invariant has led to a rich structure theory for finite p-groups, in particular *Theorems A-E*.
- Conjecturally the finite p-groups of a fixed coclass can be described by a finite set of data (Theorem for $p = 2$).

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- *Nilpotent semigroups* have been considered e.g. by Distler & Eick.

Goal of this project

Goal

Use ideas and methods of coclass theory to gain further insight into the structure of nilpotent associative algebras.

Associative algebras

Definition

An associative $\mathbb{F}$-algebra *is an* $\mathbb{F}$*-vector space equipped with an associative, bilinear multiplication.*

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We call an associative $\mathbb{F}$*-algebra* unital *if it contains an identity element, otherwise we call it* non-unital.

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Remark

Nilpotent algebras do not *contain an identity element, i.e. nilpotent algebras are non-unital.*

Importance of nilpotent associative algebras

Theorem (Mal'cev and Wedderburn)

Let A be a finite-dimensional unital associative algebra and let $J(A)$ denote its Jacobson radical.

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- *$A/J(A)$ is a direct sum of full matrix algebras over skewfields.*
- *$J(A)$ is a nilpotent associative algebra.*

Classification

Problem (generally wide open)

Classify the finite-dimensional nilpotent associative $\mathbb{F}$-algebras up to isomorphism.

Classification

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General classification of d-dimensional nilpotent associative $\mathbb{F}$-algebras.

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- $d = 3$: known (Kruse & Price, de Graaf)
  - $\mathbb{F}$ infinite: infinitely many
  - $\mathbb{F}$ finite: $|\mathbb{F}| + 6$ or $|\mathbb{F}| + 5$ depending on characteristic.

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Can the nilpotent associative $\mathbb{F}$-algebras of a fixed coclass r be classified?

Example (Coclass $r = 0$)

The nilpotent associative $\mathbb{F}$-algebras of coclass 0 are precisely the algebras $C_n = \langle a \mid a^{n+1} \rangle$.

Coclass graphs

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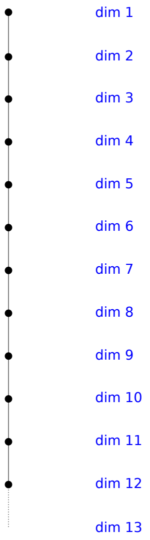
- *The* vertices *of $\mathcal{G}_{\mathbb{F}}(r)$ correspond one-to-one to the isomorphism types of nilpotent associative $\mathbb{F}$-algebras of coclass r.*
- *There is a* directed edge *$A \to B$ if B is an immediate descendant of A.*

The coclass graph $\mathcal{G}_{\mathbb{F}}(0)$ for all fields $\mathbb{F}$



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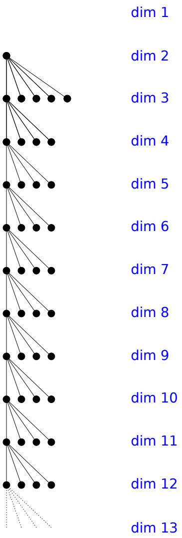
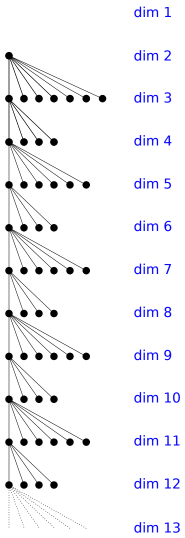
We have developed and implemented an algorithm to construct finite parts of coclass graphs over finite fields. The implementation is in GAP.

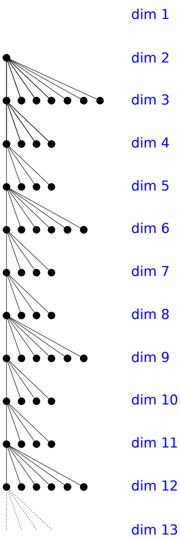
The coclass graph $\mathcal{G}_{\mathbb{F}_2}(1)$



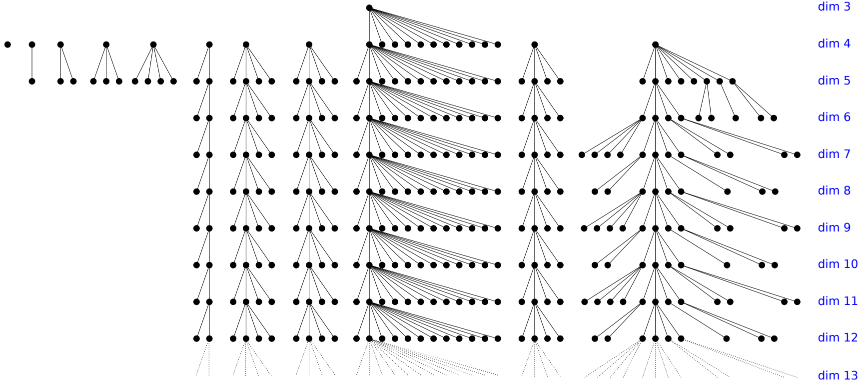
The coclass graph $\mathcal{G}_{\mathbb{F}_3}(1)$



The coclass graph $\mathcal{G}_{\mathbb{F}_4}(1)$



The coclass graph $\mathcal{G}_{\mathbb{F}_2}(2)$



Infinite paths in coclass graphs

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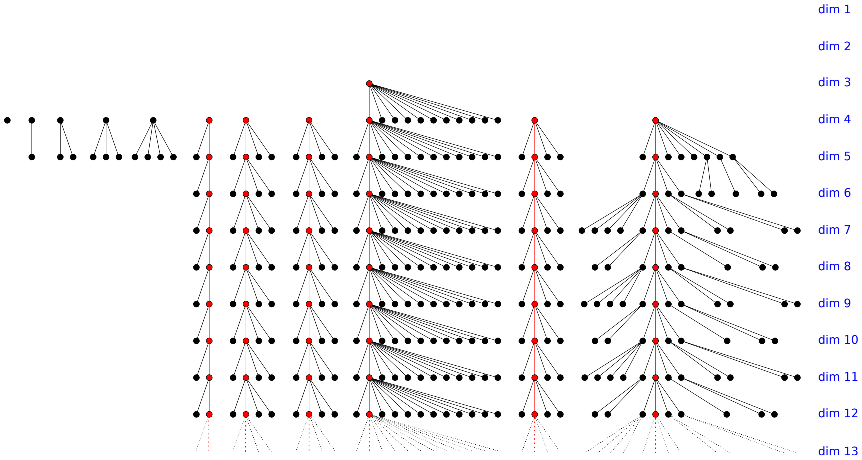

Inverse limits

Definition

Let $A_1 \to A_2 \to \ldots$ be an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$. By construction there are epimorphisms $\varphi_i : A_{i+1} \to A_i$. Hence we can form the inverse limit

$$\varprojlim_i A_i = \{(a_1, a_2, \ldots) \in \prod_i A_i \mid \varphi_i(a_{i+1}) = a_i \ \textit{for all} \ \ i \in \mathbb{N}\}.$$

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- *The inverse limit associated to an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ is a finitely generated, infinite-dimensional associative $\mathbb{F}$-algebra.*

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Remark

- *The inverse limit associated to an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ is a finitely generated, infinite-dimensional associative $\mathbb{F}$-algebra.*
- *We will use these inverse limits as a convenient description for the infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$.*

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Example (Coclass r=0)

Consider the algebras $C_i = \langle a \mid a^{i+1} \rangle$ for $i \in \mathbb{N}$. Let $\varphi_{i+1} : C_{i+1} \to C_i$ be the natural projection that maps $a \in C_{i+1}$ to $a \in C_i$. Then

$$\mathbb{F}_\circ[[t]] \cong \varprojlim_i C_i$$

via the isomorphism given by $t \mapsto (a, a, \ldots)$.

Annihilator series

Definition

For an algebra A we define its annihilator *by*

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Finally let

$$Ann_*(A) = \bigcup_{i \geq 0} Ann_i(A).$$

Structure of inv. limits associated to inf. paths

Theorem

Let $\mathbb{F}$ be an arbitrary field. An associative $\mathbb{F}$-algebra A is isomorphic to an inverse limit associated to an infinite path in $\mathcal{G}_{\mathbb{F}}(r)$ if and only if

$$A \cong T \ltimes Ann_*(A),$$

where $T \cong \mathbb{F}_\circ[[t]]$ and $dim(Ann_*(A)) = r$.

Inv. limits and equivalence classes of inf. paths

Definition

Two infinite paths in $\mathcal{G}_{\mathbb{F}}(r)$ are called equivalent *if one is a subpath of the other.*

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Remark

Equivalence classes of infinite paths in $\mathcal{G}_{\mathbb{F}}(r) \overset{1:1}{\longleftrightarrow}$ isomorphism types of algebras of the form $A \cong T \ltimes Ann_*(A)$, where again $T \cong \mathbb{F}_\circ[[t]]$ and $dim(Ann_*(A)) = r$.

Number of inv. limits associated to inf. paths

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- $n_{\mathbb{F}}(r) \leq |\mathbb{F}|^{r^3+2r^2}$ *if* $\mathbb{F}$ *is a* finite *field. In particular* $n_{\mathbb{F}}(r) < \infty$.

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- $n_{\mathbb{F}}(r) = \infty$ *if* $r \geq 2$ *and* $\mathbb{F}$ *is an* infinite *field.*

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- $n_{\mathbb{F}}(r) = \infty$ *if* $r \geq 2$ *and* $\mathbb{F}$ *is an* infinite *field.*

Corollary

The number $n_{\mathbb{F}}(r)$ is finite if and only if $r \leq 1$ or $\mathbb{F}$ is finite.

Remark

- *The structure description of the inverse limits associated to infinite paths naturally leads to an algorithm for explicitly constructing the inverse limits associated to infinite paths in* $\mathcal{G}_{\mathbb{F}}(r)$.

Construction of inv. limits associated to inf. paths

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- *This algorithm involves solving the isomorphism problem for so called annihilator extensions.*

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- *The structure description of the inverse limits associated to infinite paths naturally leads to an algorithm for explicitly constructing the inverse limits associated to infinite paths in* $\mathcal{G}_{\mathbb{F}}(r)$.
- *This algorithm involves solving the isomorphism problem for so called annihilator extensions.*
- *The isomorphism problem can be solved using an action of automorphism groups on 2-cohomology.*

Trees in coclass graphs

Descendant trees

Definition

- *The* descendant tree $\mathcal{T}(A)$ *of* A *is the full subtree of* $\mathcal{G}_{\mathbb{F}}(r)$ *consisting of* A *and all its descendants in* $\mathcal{G}_{\mathbb{F}}(r)$.

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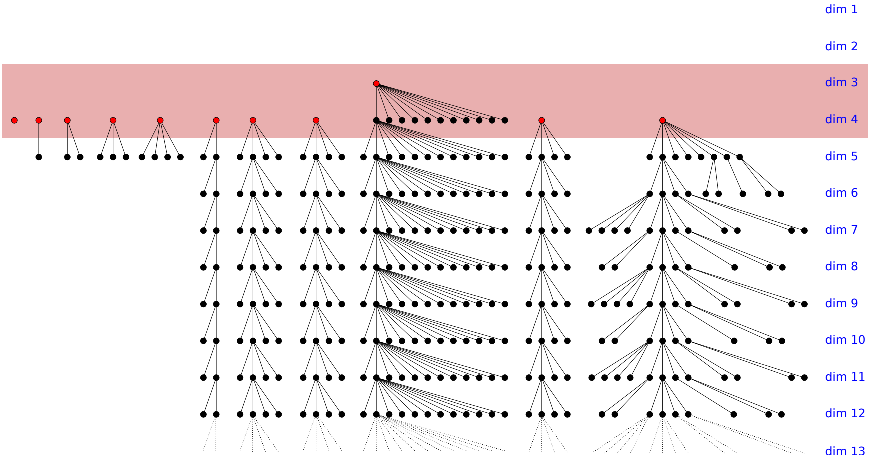
Remark

A descendant tree $\mathcal{T}(A)$ *can be finite or infinite with one or several infinite paths starting at its root* A*.*

Roots of maximal descendant trees in $\mathcal{G}_{\mathbb{F}_2}(2)$



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- $dim(R) = 1$ *if* $r = 0$,
- $r + 1 \leq dim(R) \leq 2r$ *if* $r \geq 1$.

Corollary

Let $\mathbb{F}$ be an arbitrary finite *field. Then $\mathcal{G}_{\mathbb{F}}(r)$ is a disjoint union of finitely many maximal descendant trees.*

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Theorem

Let $\mathbb{F}$ be an arbitrary finite *field. Then $\mathcal{G}_{\mathbb{F}}(r)$ is a disjoint union of finitely many maximal coclass trees and finitely many other vertices.*

Depth of coclass trees

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- $dep(\mathcal{T}) = \infty$ *otherwise.*

Depth of coclass trees

Theorem

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Theorem

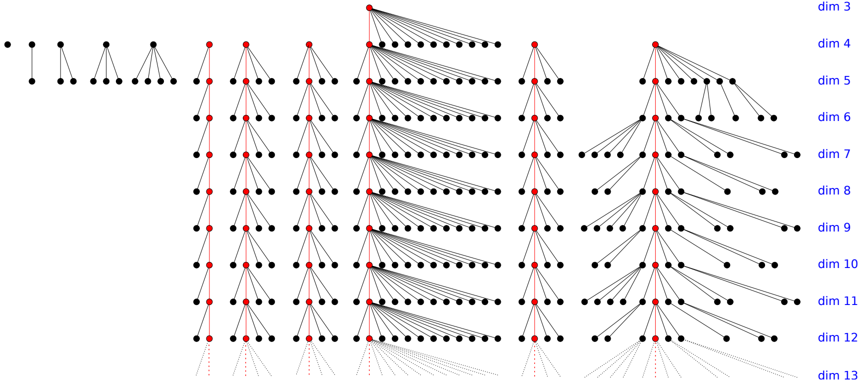
Let $\mathbb{F}$ be an arbitrary field and let r be a non-negative integer. Let $\mathcal{T}$ be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then $\mathcal{T}$ has bounded depth.

Remark

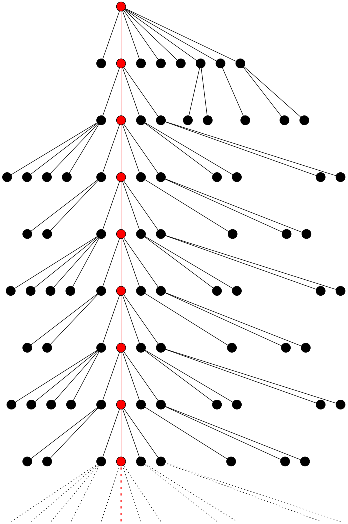
We actually prove a stronger result: For almost all branches $\mathcal{B}$ of $\mathcal{T}$ we have $\mathrm{dep}(\mathcal{B}) \leq r$. In many cases we can prove an even better bound.

Periodicity of coclass graphs

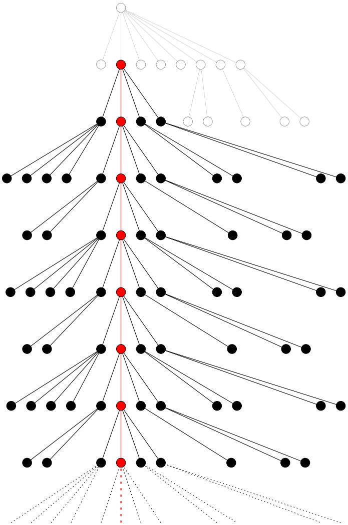
Periodicity conjecture



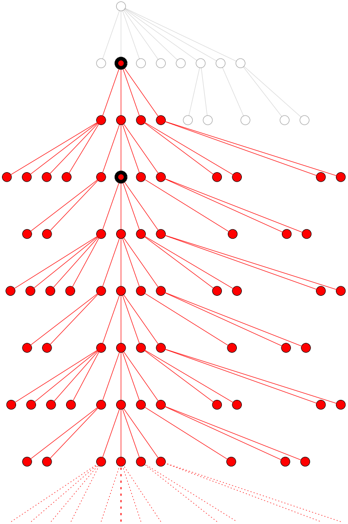
Periodicity conjecture



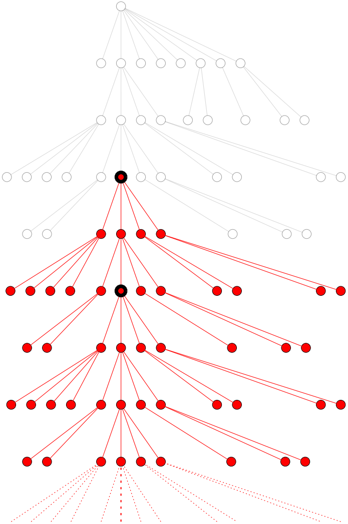
Periodicity conjecture



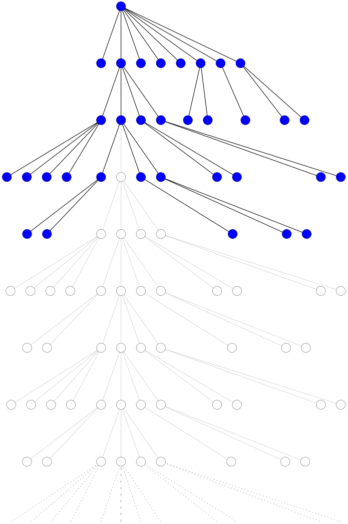
Periodicity conjecture



Periodicity conjecture



Periodicity conjecture



Virtual periodicity

Definition

Let $\mathbb{F}$ be an arbitrary finite field and r a non-negative integer. Let $\mathcal{T}$ be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$ with unique infinite path $A_1 \to A_2 \to \ldots$ starting at the root of $\mathcal{T}$. Then $\mathcal{T}$ is called virtually periodic *with* period *d and* periodic root *A_ℓ if the descendant trees $\mathcal{T}(A_i)$ and $\mathcal{T}(A_{i+d})$ are isomorphic as directed graphs for each $i \geq \ell$.*

Periodicity conjecture

Conjecture (Theorem for $r = 0$ and $r = 1$)

Let $\mathbb{F}$ be an arbitrary finite field and r a non-negative integer and let $\mathcal{T}$ be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then $\mathcal{T}$ is virtually periodic.

Periodicity conjecture

Conjecture (Theorem for $r = 0$ and $r = 1$)

Let $\mathbb{F}$ be an arbitrary finite field and r a non-negative integer and let $\mathcal{T}$ be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then $\mathcal{T}$ is virtually periodic.

Remark

Our experimental data suggests that if $\mathcal{T}$ is a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$ and $\mathbb{F}$ is a field of size $q = p^n$, then the period is of the form $p^s(q-1)$ with $0 \leq s \leq r-1$.

Periodicity conjecture

Conjecture (Theorem for $r = 0$ and $r = 1$)

Let $\mathbb{F}$ be an arbitrary finite field and r a non-negative integer and let $\mathcal{T}$ be a maximal coclass tree in $\mathcal{G}_{\mathbb{F}}(r)$. Then the infinitely *many algebras in $\mathcal{G}_{\mathbb{F}}(r)$ can be described by* finitely *many parametrized presentations.*

Thank you!