Transitivity of properties of two-generator subgroups of finite groups

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Let $G$ be a group and

$$G^\times = G \setminus \{1\}.$$ 

Consider the **commutativity relation** on $G^\times$:

$$x \leftrightarrow y \iff xy = yx.$$ 

The relation $\leftrightarrow$ is reflexive and symmetric on $G^\times$.

**Definition**

$G$ is a **CT-group** if commutativity is a transitive relation on $G^\times$. 
$Q_8$ is not a CT-group
Role of the center

Let $G$ be a group. Then

$$Z(G) = \{ g \in G \mid gx = xg \text{ for all } x \in G \}$$

is called the center of $G$.

**Proposition**

Let $G$ be a non-abelian CT-group. Then

$$Z(G) = \{ 1 \}.$$
$A_5$ is a CT-group
Main questions

- Classification of finite CT-groups?
- What can be said about infinite CT-groups?
- Possible generalizations?
Characterizations of CT-groups

Proposition

Let $G$ be a group. The following are equivalent:

1. $G$ is a CT-group.
2. $C_G(g)$ is abelian for every $g \in G^\times$.
3. The connected components of the relation graph of $\leftrightarrow$ on $G^\times$ are complete graphs.
Commutative-transitive groups

L. Weisner (1925).

\[ G \text{ finite CT-group} \Rightarrow G \text{ solvable or simple.} \]

M. Suzuki (1957).

\[ G \text{ finite non-abelian simple CT-group} \iff G \sim = \text{PSL}(2, f), f > 1. \]


\[ G \text{ finite non-abelian solvable CT-group} \iff G \text{ finite Frobenius group with abelian kernel and cyclic complement.} \]
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Y.F. Wu (1998). $G$ finite non-abelian solvable $CT$-group $\iff G$ finite Frobenius group with abelian kernel and cyclic complement.
A Lie algebra $L$ is called **commutative transitive (CT)** if for all $x, y, z \in L \setminus \{0\}$, $[x, y] = [y, z] = 0$ imply $[x, z] = 0$. 
Let $L$ be a Lie algebra, $N$ any ideal in $L$ and $U$ a subalgebra in $L$. Then $U$ acts on $N$ by derivations, that is,

$$(u, n) \mapsto [u, n],$$

where $u \in U$ and $n \in N$. Each action of $U$ induces \textbf{conjugation}

$$(u, n) \mapsto n + [u, n].$$

An action of an algebra $U$ on an ideal $N$ of $L$ is said to be \textbf{fixed-point-free} if the stabilizer of any nonzero element of $N$ in $U$ under conjugation is trivial.
Solvable CT Lie algebras

Theorem

Let $L$ be a finite dimensional solvable CT Lie algebra over $k$. If $L$ is nonabelian, then:

- $L$ is a semidirect product of its nil radical $N$ which is abelian, and an abelian Lie algebra that acts fixed-point-freely on $N$.
- If $U$ and $V$ are two complements to $N$ in $L$, then there exists $a \in N$ such that $V = (1 + \text{ad } a)(U)$.
- If $k$ is algebraically closed, then the complements are one-dimensional.
Simple CT Lie algebras and general case

Theorem

If $k$ is algebraically closed, then the only finite dimensional simple CT Lie algebra over $k$ is $\mathfrak{sl}_2$. 
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Theorem

Let $k$ be algebraically closed. Then every finite dimensional CT Lie algebra over $k$ is either solvable or simple.
Let \( \mathcal{X} \) be a class of groups, and let \( G \) be any group. Define a graph \( \Gamma_{\mathcal{X}}(G) \):

- **vertices**: all non-trivial elements of \( G \);
- **edges**: different vertices \( a \) and \( b \) are connected by an edge iff \( \langle a, b \rangle \in \mathcal{X} \).
A group $G$ is said to be \textbf{$\mathcal{X}$-transitive} (briefly: an $\mathcal{XT}$-group) if

$$\langle a, b \rangle \in \mathcal{X} \text{ and } \langle b, c \rangle \in \mathcal{X} \text{ imply } \langle a, c \rangle \in \mathcal{X}$$

for all $a, b, c \in G \setminus \{1\}$. 
Three important classes of groups

- A group $G$ is called **solvable** if it has a subnormal series whose factor groups are all abelian.
- A group $G$ is called **supersolvable** if it has a normal series whose factors are all cyclic.
- A group $G$ is called **nilpotent** if it has a normal series whose factors are central.
A group theoretical property \( \mathcal{X} \) is **bigenetic** in the class of all finite groups when a finite group \( G \) is in \( \mathcal{X} \) if and only if all its two-generator subgroups are in \( \mathcal{X} \).
Bigenetic properties

A group theoretical property $\mathcal{X}$ is **bigenetic** in the class of all finite groups when a finite group $G$ is in $\mathcal{X}$ if and only if all its two-generator subgroups are in $\mathcal{X}$.

The following properties are bigenetic in the class of all finite groups:

- (i) solvability [J.G. Thompson (1968)];
- (ii) supersolvability [R.W. Carter, B. Fischer and T. Hawkes (1968)];
- (iii) nilpotency [M. Zorn (1936)].
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- $\mathcal{X}$ is subgroup closed;
- $\mathcal{X}$ contains all finite abelian groups;
- $\mathcal{X}$ is bigenetic in the class of all finite groups.
The $\mathcal{X}$-radical of a group

Let $\mathcal{X}$ be any class of groups. The $\mathcal{X}$-radical of a group $G$ is the product $R_\mathcal{X}(G)$ of all normal $\mathcal{X}$-subgroups of $G$. 
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If $R_{\mathcal{X}}(G) = 1$ the group $G$ is said to be $\mathcal{X}$-semisimple.
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If $R_\mathcal{X}(G) = 1$ the group $G$ is said to be $\mathcal{X}$-semisimple.

**Lemma**

Let $\mathcal{X}$ be a good class of groups, and let $G$ be a finite $\mathcal{X}T$-group. Then $R_\mathcal{X}(G) \in \mathcal{X}$. 
**Theorem**

Let $\mathcal{X}$ be a good class of groups, and let $G$ be a finite $\mathcal{X}_T$-group. Then one of the following holds:

(i) $G \in \mathcal{X}$;

(ii) $G$ is $\mathcal{X}$-semisimple;

(iii) $G$ is a Frobenius group with kernel and complement both in $\mathcal{X}$. 

The $\mathcal{X}$-centralizers of a group

Let $\mathcal{X}$ be any class of groups, and let $H$ be any subgroup of a group $G$. The subset

$$C^\mathcal{X}_G(H) = \{ x \in G : \langle x, h \rangle \in \mathcal{X}, \text{ for some } h \in H \setminus \{1\} \}$$

is called the $\mathcal{X}$-centralizer of $H$ in $G$. 
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Lemma

Let $\mathcal{X}$ be a good class of groups. Let $G$ be a finite $\mathcal{X}T$-group, and let $H$ be an $\mathcal{X}$-subgroup of $G$. Then $C^\mathcal{X}_G(H)$ is an $\mathcal{X}$-subgroup of $G$ containing $H$. 

**The $\mathcal{X}$-centralizers of a group**

Let $\mathcal{X}$ be any class of groups, and let $H$ be any subgroup of a group $G$. The subset

$$C_G^\mathcal{X}(H) = \{x \in G : \langle x, h \rangle \in \mathcal{X}, \text{ for some } h \in H \setminus \{1\}\}$$

is called the **$\mathcal{X}$-centralizer of $H$ in $G$.**

**Lemma**

Let $\mathcal{X}$ be a good class of groups. Let $G$ be a finite $\mathcal{X}T$-group, and let $H$ be an $\mathcal{X}$-subgroup of $G$. Then $C_G^\mathcal{X}(H)$ is an $\mathcal{X}$-subgroup of $G$ containing $H$.

**Proposition**

Let $\mathcal{X}$ be a good class of groups, and let $G$ be a finite Frobenius group with kernel $F$ and complement $H$. Then $G$ is an $\mathcal{X}T$-group if and only if $C_G^\mathcal{X}(F)$ and $C_G^\mathcal{X}(H)$ are $\mathcal{X}$-groups.
Lack of $\mathcal{X}$-semisimple groups

Theorem

Let $\mathcal{X}$ be a good class of groups, and suppose the following:

- $\mathcal{X}$ contains all finite dihedral groups,
- Every finite $\mathcal{X}$-group is solvable.

If $G$ is a finite $\mathcal{X}T$-group which is not in $\mathcal{X}$, then $G$ is a Frobenius group with complement belonging to $\mathcal{X}$. In particular, $G$ is solvable.
Corollary

Every finite solvable-transitive group is solvable.
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Corollary

Let $G$ be a finite supersolvable-transitive group. If $G$ is not supersolvable, then $G$ is a Frobenius group with supersolvable complement. In particular, $G$ is solvable.
The supersolvable graph of $A_4$

Supersolvable-transitive $\nRightarrow$ supersolvable:
An example

The following is an example of a Frobenius group with supersolvable complement, which is not supersolvable-transitive:

Example

Let $A = \langle x \rangle \oplus \langle y \rangle$ be an elementary group of order 9 and let $\alpha$ be the automorphism of $A$ given by the matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

Let $G = A \rtimes \langle \alpha \rangle$. This is a group of order 36 which is not supersolvable-transitive. For, $\langle \alpha^2, (\alpha y)^2 \rangle$ is a dihedral group, $\langle (\alpha y)^2, \alpha y \rangle$ is cyclic, whereas $\langle \alpha^2, \alpha y \rangle = G$ is not supersolvable. Note that $C_G^G(\langle \alpha \rangle)$ has 20 elements, so it is not a subgroup of $G$. 
The nilpotent graph of $\text{PSL}(2, 9)$
The nilpotent graph of $\text{PSL}(2, 9)$

This graph contains a component of the form
Nilpotent-transitive groups

Theorem

Let $G$ be a finite $\mathfrak{NT}$-group. Then one of the following holds:

(i) $G$ is nilpotent;

(ii) $G$ is a Frobenius group with nilpotent complement;

(iii) $G \cong \text{PSL}(2, 2^f)$ for some $f > 1$;

(iv) $G \cong \text{Sz}(q)$ with $q = 2^{2n+1} > 2$.

Conversely, every finite group under (i)–(iv) is an $\mathfrak{NT}$-group.