# Transitivity of properties of two-generator subgroups of finite groups

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# Commutativity as a relation on $G \setminus \{1\}$

Let $G$ be a group and
$$G^{\times} = G \setminus \{1\}.$$

Consider the **commutativity relation** on $G^{\times}$:

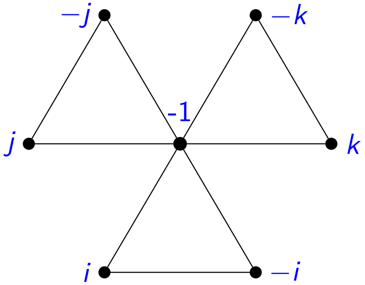
$$x \leftrightarrow y \iff xy = yx.$$

The relation $\leftrightarrow$ is reflexive and symmetric on $G^{\times}$.

## Definition

$G$ is a **CT-group** if commutativity is a transitive relation on $G^{\times}$.

# $Q_8$ is not a CT-group



# Role of the center

Let $G$ be a group. Then

$$Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$$

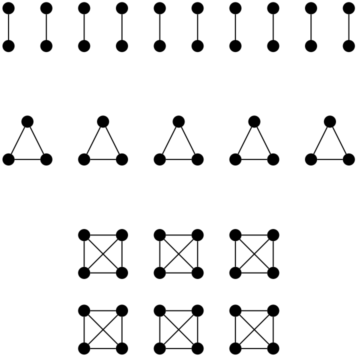
is called the **center** of $G$.

### Proposition

*Let $G$ be a non-abelian CT-group. Then*

$$Z(G) = \{1\}.$$

# $A_5$ is a CT-group



# Main questions

- Classification of finite CT-groups?
- What can be said about infinite CT-groups?
- Possible generalizations?

# Characterizations of CT-groups

### Proposition

*Let $G$ be a group. The following are equivalent:*

1. *$G$ is a CT-group.*
2. *$C_G(g)$ is abelian for every $g \in G^\times$.*
3. *The connected components of the relation graph of $\leftrightarrow$ on $G^\times$ are complete graphs.*

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Y.F. Wu (1998). $G$ finite non-abelian solvable $C\mathrm{T}$-group $\Longleftrightarrow$ $G$ finite Frobenius group with abelian kernel and cyclic complement.

# Commutative-transitive Lie algebras

A Lie algebra $L$ is called **commutative transitive (CT)** if for all $x, y, z \in L \setminus \{0\}$, $[x, y] = [y, z] = 0$ imply $[x, z] = 0$.

# Actions in Lie algebras

Let $L$ be a Lie algebra, $N$ any ideal in $L$ and $U$ a subalgebra in $L$. Then $U$ acts on $N$ by derivations, that is,

$$(u, n) \mapsto [u, n],$$

where $u \in U$ and $n \in N$. Each action of $U$ induces **conjugation**

$$(u, n) \mapsto n + [u, n].$$

An action of an algebra $U$ on an ideal $N$ of $L$ is said to be **fixed-point-free** if the stabilizer of any nonzero element of $N$ in $U$ under conjugation is trivial.

# Solvable CT Lie algebras

### Theorem

*Let $L$ be a finite dimensional solvable CT Lie algebra over $k$. If $L$ is nonabelian, then:*

- *$L$ is a semidirect product of its nil radical $N$ which is abelian, and an abelian Lie algebra that acts fixed-point-freely on $N$.*
- *If $U$ and $V$ are two complements to $N$ in $L$, then there exists $a \in N$ such that $V = (1 + \mathrm{ad}\, a)(U)$.*
- *If $k$ is algebraically closed, then the complements are one-dimensional.*

# Simple CT Lie algebras and general case

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**Theorem**

*Let $k$ be algebraically closed. Then every finite dimensional CT Lie algebra over $k$ is either solvable or simple.*

# Graph $\Gamma_{\mathfrak{X}}(G)$

Let $\mathfrak{X}$ be a class of groups, and let $G$ be any group.
Define a graph $\Gamma_{\mathfrak{X}}(G)$:

- *vertices*: all non-trivial elements of $G$;
- *edges*: different vertices $a$ and $b$ are connected by an *edge* iff $\langle a, b \rangle \in \mathfrak{X}$.

# $\mathfrak{X}$-transitive groups

A group $G$ is said to be $\mathfrak{X}$-**transitive** (briefly: an $\mathfrak{X}\mathrm{T}$-group) if

$$\langle a, b\rangle \in \mathfrak{X} \text{ and } \langle b, c\rangle \in \mathfrak{X} \text{ imply } \langle a, c\rangle \in \mathfrak{X}$$

for all $a$, $b$, $c \in G \setminus \{1\}$.

# Three important classes of groups

- A group $G$ is called **solvable** if it has a subnormal series whose factor groups are all abelian.
- A group $G$ is called **supersolvable** if it has a normal series whose factors are all cyclic.
- A group $G$ is called **nilpotent** if it has a normal series whose factors are central.

# Bigenetic properties

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(ii) supersolvability [R.W. Carter, B. Fischer and T. Hawkes (1968)];

(iii) nilpotency [M. Zorn (1936)].

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- $\mathfrak{X}$ contains all finite abelian groups;
- $\mathfrak{X}$ is bigenetic in the class of all finite groups.

# The $\mathfrak{X}$-radical of a group

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If $R_{\mathfrak{X}}(G) = 1$ the group $G$ is said to be $\mathfrak{X}$-**semisimple**.

Lemma

*Let $\mathfrak{X}$ be a good class of groups, and let $G$ be a finite $\mathfrak{X}\mathrm{T}$-group. Then $R_{\mathfrak{X}}(G) \in \mathfrak{X}$.*

# $\mathfrak{X}\mathrm{T}$-groups – three cases

**Theorem**

*Let $\mathfrak{X}$ be a good class of groups, and let $G$ be a finite $\mathfrak{X}\mathrm{T}$-group. Then one of the following holds:*

(i) $G \in \mathfrak{X}$;

(ii) $G$ *is* $\mathfrak{X}$*-semisimple;*

(iii) $G$ *is a Frobenius group with kernel and complement both in* $\mathfrak{X}$.

# The $\mathfrak{X}$-centralizers of a group

Let $\mathfrak{X}$ be any class of groups, and let $H$ be any subgroup of a group $G$. The subset

$$C_G^{\mathfrak{X}}(H) = \{x \in G : \langle x, h \rangle \in \mathfrak{X}, \text{ for some } h \in H \setminus \{1\}\}$$

is called the $\mathfrak{X}$-**centralizer of** $H$ **in** $G$.

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## Lemma

*Let $\mathfrak{X}$ be a good class of groups. Let $G$ be a finite $\mathfrak{X}\mathrm{T}$-group, and let $H$ be an $\mathfrak{X}$-subgroup of $G$. Then $C_G^{\mathfrak{X}}(H)$ is an $\mathfrak{X}$-subgroup of $G$ containing $H$.*

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### Proposition

*Let $\mathfrak{X}$ be a good class of groups, and let $G$ be a finite Frobenius group with kernel $F$ and complement $H$. Then $G$ is an $\mathfrak{X}\mathrm{T}$-group if and only if $C_G^{\mathfrak{X}}(F)$ and $C_G^{\mathfrak{X}}(H)$ are $\mathfrak{X}$-groups.*

# Lack of $\mathfrak{X}$-semisimple groups

**Theorem**

*Let $\mathfrak{X}$ be a good class of groups, and suppose the following:*

- *$\mathfrak{X}$ contains all finite dihedral groups,*
- *Every finite $\mathfrak{X}$-group is solvable.*

*If $G$ is a finite $\mathfrak{X}\mathrm{T}$-group which is not in $\mathfrak{X}$, then $G$ is a Frobenius group with complement belonging to $\mathfrak{X}$. In particular, $G$ is solvable.*

# Solvable-transitive groups, supersolvable-transitive groups

**Corollary**

*Every finite solvable-transitive group is solvable.*

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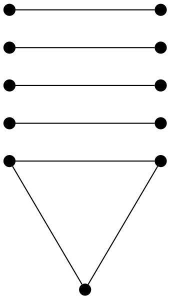
*Let $G$ be a finite supersolvable-transitive group. If $G$ is not supersolvable, then $G$ is a Frobenius group with supersolvable complement. In particular, $G$ is solvable.*

# The supersolvable graph of $A_4$

Supersolvable-transitive $\not\Rightarrow$ supersolvable:



# An example

The following is an example of a Frobenius group with supersolvable complement, which is not supersolvable-transitive:

### Example

Let $A = \langle x \rangle \oplus \langle y \rangle$ be an elementary group of order 9 and let $\alpha$ be the automorphism of $A$ given by the matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix}.$$

Let $G = A \rtimes \langle \alpha \rangle$. This is a group of order 36 which is not supersolvable-transitive. For, $\langle \alpha^2, (\alpha y)^2 \rangle$ is a dihedral group, $\langle (\alpha y)^2, \alpha y \rangle$ is cyclic, whereas $\langle \alpha^2, \alpha y \rangle = G$ is not supersolvable. Note that $C_G^{\mathfrak{S}}(\langle \alpha \rangle)$ has 20 elements, so it is not a subgroup of $G$.

# The nilpotent graph of $PSL(2, 9)$

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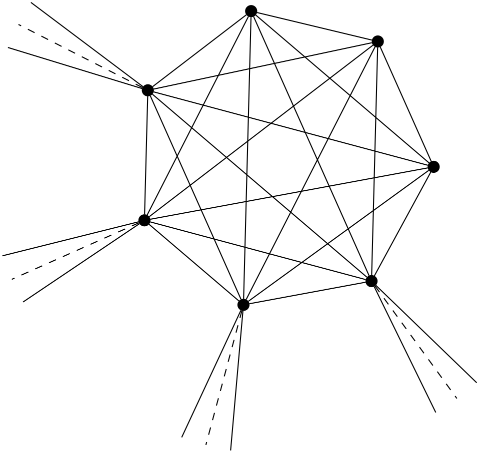
This graph contains a component of the form



# Nilpotent-transitive groups

**Theorem**

*Let $G$ be a finite $\mathfrak{N}\mathrm{T}$-group. Then one of the following holds:*

(i) *$G$ is nilpotent;*

(ii) *$G$ is a Frobenius group with nilpotent complement;*

(iii) *$G \cong \mathsf{PSL}(2, 2^f)$ for some $f > 1$;*

(iv) *$G \cong \mathsf{Sz}(q)$ with $q = 2^{2n+1} > 2$.*

*Conversely, every finite group under (i)–(iv) is an $\mathfrak{N}\mathrm{T}$-group.*