A Refinement of Cayley Graphs Associated to Rings

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Outline

Introduction

Some notations and structure for commutative rings

Examples of commutative rings

(1) $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.
(2) $\mathbb{F}_n$= field of order n.

Ideals and Maximal ideals

Let R be a commutative Ring with identity.
(1) An *ideal* in R is an additive subgroup $I \subseteq R$ such that $Ix \subseteq I$ for all $x \in R$.
(2)I is called *maximal ideal* if there is no ideal J with $I \subsetneq J \subsetneq R$.

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Local rings

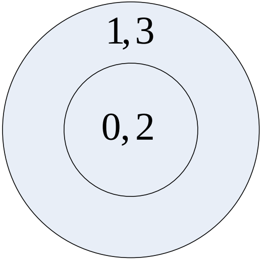
Definition of local ring

Call a ring R local if R has exactly one maximal ideal.

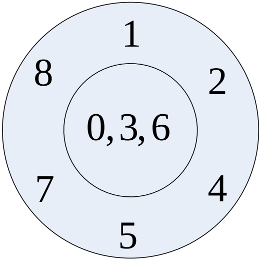
Examples of local rings

(1) $\mathbb{Z}_4$.
(2) $\mathbb{Z}_9$.
(3) $\mathbb{Z}_{p^2}$, where p is a prime number.
(4) $\frac{\mathbb{Z}_p[X]}{(X^2)}$, where p is a prime number.

Local rings ($\mathbb{Z}_4$ and $\mathbb{Z}_9$)



$\mathbb{Z}_4 = \{0, 1, 2, 3\}$



$\mathbb{Z}_9 = \{0, 1, 2, \ldots, 8\}$

Structure of finite commutative rings

(1)

Let R be a finite commutative ring. Then

$$R = R_1 \times R_2 \times \cdots \times R_k,$$

where R_i is a local ring.

(2)

Let R be a finite commutative ring. Then

$$R/J(R) = F_1 \times F_2 \times \cdots \times F_k,$$

where F_i is a Field.
(Here $J(R)$, the Jacobson radical of R, is the intersection of all maximal ideals of R)

Some facts about Jacobson radical

Structure of Jacobson radical

$J(R) = \{r \in R | 1 + rx \quad \text{is unit for all} \quad x \in R\}.$

Theorem

u is a unit in R if and only if $u + J(R)$ is a unit in $R/J(R)$.

Some important graphs associated to rings

(1) Zero divisor Graph of a ring.
(2) Cayley Garph of a ring.
(3) Unit Graph of a ring.

(1) Zero divisor graph of a ring

The concept of a zero-divisor graph of a commutative ring was first introduced by Beck. (In his work all elements of the ring were vertices of the graph).
$V(\Gamma(R) = Z(R) \setminus \{0\}$ and two distinct vertices x and y are adjacent if and and only if $xy = 0$.

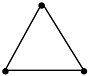
[Beck] I. Beck, Coloring of commutative rings, J. Algebra 116 (1988), 208-226.

Some examples of Zero divisor graphs

$\mathbb{Z}_4$ or $\dfrac{\mathbb{Z}_2[x]}{(x^2)}$

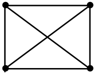
$\mathbb{Z}_9, \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\dfrac{\mathbb{Z}_3[x]}{(x^2)}$

$\mathbb{Z}_6, \mathbb{Z}_8$ or $\dfrac{\mathbb{Z}_2[x]}{(x^3)}$



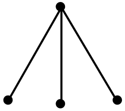
$\dfrac{\mathbb{Z}_2[x,y]}{(x^2, xy, y^2)}$ or $\dfrac{\mathbb{F}_4[x]}{(x^2)}$

$\mathbb{Z}_3 \times \mathbb{Z}_3$

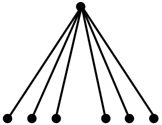


$\mathbb{Z}_{25}$ or $\dfrac{\mathbb{Z}_5[x]}{(x^2)}$

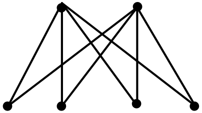
Some examples of Zero divisor Graphs



$\mathbb{Z}_2 \times \mathbb{F}_4$



$\mathbb{Z}_2 \times \mathbb{Z}_7$



$\mathbb{Z}_3 \times \mathbb{Z}_5$

(2) Cayley graph of a ring

The *Cayley graph* $\Gamma(R)$ is the graph with vertex set R such that two distinct vertices x and y are adjacent if and only if $x - y$ is unit in R. Unitary Cayley graphs are introduced in:

Lucchini, et al.

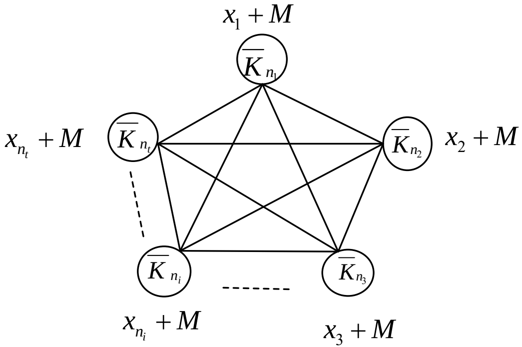
A. Lucchini, A. Maroti, Some results and questions related to the generating graph of a finite group, Proceedings of the Ischia Group Theory Conference, 2008.

Akhtar, et al.

R. Akhtar, M. Boggess, T. Jackson-Henderson, I. Jim´enez, R. Karpman, A. Kinzel and D. Pritikin, On the unitary Cayley graph of a finite ring, *Electron. J. Combin.* 16 (2009) #R117.

A general example of Cayley graph

(R, M) is a local ring, $R/M = \{x_1 + M, x_2 + M, \ldots, /x_t + M\}$ and $|x_i + M| = n_i|$ for all $1 \leq i \leq t$.



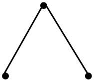
(3) Unit graph of a ring

The *unit graph* $\Gamma(R)$ is the graph with vertex set R such that two distinct vertices x and y are adjacent if and only if $x + y$ is unit in R. The unit graphs are introduced in:

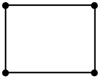
Fuchs

E. Fuchs, Longest induced cycles in circulant graphs, *Electron. J. Combin.* 14 (2005) #R52.

Some examples unit graphs

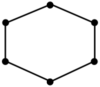


$\mathbb{Z}_3$

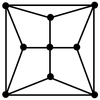


$\mathbb{Z}_4$

$\mathbb{Z}_2 \times \mathbb{Z}_2$



$\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$



$\mathbb{Z}_3 \times \mathbb{Z}_3$

Definition in this talk
A Refinement of Cayley Graphs Associated to Rings

Let R be a finite ring and $U(R)$ be the set of all unit elements of R. The Unit graph $\Gamma(R)$ is the graph with vertex set R such that two distinct vertices x and y are adjacent if and only if there exists a unit element u of R such that $x + uy$ is unit in R.

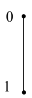
If we omit the word "distinct", we obtain the graph $\Gamma_\ell(R)$; this graph may have loops.

Motivation

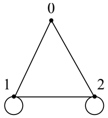
(1) The study of algebraic structures using the properties of graphs,
(2) Some result about unit 1-stable range rings.

We recall that a ring R is said to have unit 1-stable range if, whenever $Rx + Ry = R$, there exists $u \in U(R)$ such that $x + uy \in U(R)$.

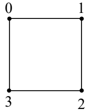
Some examples



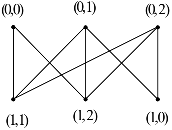
$\Gamma_\ell(\mathbb{Z}_2) = \Gamma(\mathbb{Z}_2)$



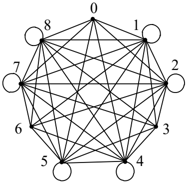
$\Gamma_\ell(\mathbb{Z}_3)$



$\Gamma_\ell(\mathbb{Z}_4) = \Gamma(\mathbb{Z}_4)$



$\Gamma_\ell(\mathbb{Z}_2 \times \mathbb{Z}_3) = \Gamma(\mathbb{Z}_2 \times \mathbb{Z}_3)$



$\Gamma_\ell(\mathbb{Z}_9)$

Outline

Some Properties of $\Gamma(R)$

When $\Gamma(R)$ is a Complete Bipartite Graph

Let R be a ring and let M be a maximal ideal of R such that $|R/M| = 2$. Then $\Gamma(R)$ is a complete bipartite graph if and only if R is a local ring.

Degree of Vertices

Let R be a local ring with maximal ideal M such that $|R/M| > 2$ and let $x \in R$. Then

$$deg(x) = \begin{cases} |R| - 1 & \text{if } x \in U(R), \\ |U(R)| & \text{otherwise.} \end{cases}$$

Some Properties of $\Gamma(R)$

Connectedness (1)

$\Gamma(R)$ is connected if and only if $R/J(R)$ has at most one $\mathbb{Z}_2$ as a summand.

Connectedness (2)

$\Gamma(R)$ is not connected if and only if
$R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times R_3 \times \cdots \times R_n$. ($R_i$ is a local ring)

Some Properties of $\Gamma(R)$

Clique and Chromatic number

Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a ring, where R_i is a local ring with maximal ideal M_i. Then

$$\chi(\Gamma(R)) = \omega(\Gamma(R)) = \begin{cases} 2 & \text{if } |R_i/M_i| = 2 \text{ for some } i \\ |U(R)| + n & \text{otherwise.} \end{cases}$$

Proof

Let $|R_i/M_i| = 2$, for some $1 \leq i \leq n$. Then $M := R_1 \times \cdots \times R_{i-1} \times M_i \times R_{i+1} \times \cdots \times R_n$ is a maximal ideal of R such that $|R/M| = 2$. Therefore $\omega(\Gamma(R)) = 2$. Now suppose that $|R_i/M_i| > 2$ for all $1 \leq i \leq n$. We set:

$$\begin{aligned} S_i &:= U(R_1) \times \cdots \times U(R_{i-1}) \times M_i \times R_{i+1} \times \cdots \times R_n. \\ S_{n+1} &:= U(R_1) \times U(R_2) \times \cdots \times U(R_n). \end{aligned}$$

Some Properties of $\Gamma(R)$

Proof

It is easy to see that $S_i \bigcap S_j = \emptyset$, for all $i \neq j$, and $\bigcup_{i=1}^{i=n+1} S_i = R$. S_{n+1} is a clique. Set

$$C := S_{n+1} \cup \{(0,1,1,\ldots,1),(1,0,1,\ldots,1),(1,1,\ldots,1,0)\}.$$

It is easy to see that C is a clique of $\Gamma(R)$. Since $S_i\ (1 \leq i \leq n)$ is a coclique, then every arbitrary clique of $\Gamma(R)$ contains at most one element of $S_i\ (1 \leq i \leq n)$. Therefore $\omega(\Gamma(R)) = |U(R_1)| \times |U(R_2)| \times \cdots \times |U(R_n)| + n = |U(R)| + n$. This argument also shows that $\chi(\Gamma(R)) = |U(R_1)| \times |U(R_2)| \times \cdots \times |U(R_n)| + n = |U(R)| + n$.

Some Properties of $\Gamma(R)$

Vertex-connectivity and Edge-connectivity

Let $\kappa(G)$ and $\kappa'(G)$ denote the vertex-connectivity and edge-connectivity of a graph G,

Theorem

$\kappa(\Gamma(R)) = \kappa'(\Gamma(R)) = |U(R)| = \delta(\Gamma(R))$.

Some Properties of $\Gamma(R)$

Hamiltonian cycle

Let R be a ring such that $R \neq \mathbb{Z}_2$. Then $\Gamma(R)$ is a connected graph if and only if $\Gamma(R)$ is Hamiltonian.

Matching

$\Gamma(R)$ has perfect matching if and only if $|R|$ is an even number.

Some properties of $\Gamma(R)$

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Planarity for $\Gamma(R)$

$\Gamma(R)$ is planar if and only if R is one of the following rings

(1) $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}}$.

(2) $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}} \times \mathbb{Z}_4$.

(3) $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}} \times \mathbb{F}_4$.

(4) $\underbrace{\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2}_{n \text{ times}} \times \frac{\mathbb{Z}_2[x]}{(x^2)}$.

Some Properties of $\overline{\Gamma}(R)$

Connectedness

Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a ring, where R_i is a local ring with maximal ideal M_i. Then $\overline{\Gamma}(R)$ is connected if and only if R satisfies one of the following:
(1) $|R_1/M_1| = |R_2/M_2| = 2$ and $n \geq 2$.
(2) $|R_1/M_1| = 2$ and $|R_k/M_k| > 2$ for every $2 \leq k \leq n$.

Hamiltonian Cycle

Let $R = R_1 \times R_2 \times \cdots \times R_n$ be a ring, where R_i is a local ring with maximal ideal M_i. Then $\overline{\Gamma}(R)$ is Hamiltonian if and only if R satisfies any one of the following three cases:
(1) $|R_1/M_1| = |R_2/M_2| = 2$ and $n \geq 2$.
(2) $n = 2$ with $|R_1/M_1| = 2$, $|R_2/M_2| > 2$ and at least one of R_1 or R_2 is not field.
(3) $n \geq 3$, $|R_2/M_2| = 2$ and $|R_k/M_k| > 2$ for every $2 \leq k \leq n$.

Planarity for $\overline{\Gamma}(R)$

$\overline{\Gamma}(R)$ is planar if and only if R is one of the following rings

(1) $\mathbb{Z}_2, \mathbb{Z}_4, \frac{\mathbb{Z}_2[x]}{(x^2)}, \mathbb{Z}_8, \frac{\mathbb{Z}_2[x]}{(x^3)}, \frac{\mathbb{Z}_4[x]}{(2x, x^2-2)}, \frac{\mathbb{Z}_4[x]}{(x,2)^2}, \frac{\mathbb{Z}_4[x,y]}{(x,y)^2}$.

(2) F is a field with $|F| > 2$, $\frac{\mathbb{F}_4[x]}{(x^2)}, \frac{\mathbb{Z}_4[x]}{(1+x+x^2)}, \mathbb{Z}_9, \frac{\mathbb{Z}_3[x]}{(x^2)}$.

(3) $\mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_3, \mathbb{Z}_2 \times \mathbb{F}_4, \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_3 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{F}_4$.

Used papers for this talk

[1] A. R. Naghipour, M. Rezagholibeigi. A Refinement of the Unit and Unitary Cayley Graphs of a Finite Ring. Bull. Korean Math. Soc. 53 (2016), No. 4, pp. 1197–1211.

[2] T. Tamizh Chelvam and S. Anukumar Kathirve, On generalized unit and unitary Cayley graphs of finite rings, Journal of Algebra and its Applications. Accepted, 2018.

[3] A. R. Naghipour, M. Rezagholibeigi. A Refinement of the Unit and Unitary Cayley Graphs of a Noncommutative Ring, in preparation.

Acknowlegement

Thank you for your attention