Covering random graphs by monochromatic cycles

Rajko Nenadov

(joint with D. Korándi, F. Mousset, N. Škorić, and B. Sudakov)
The vertex set of any 2-edge-coloured complete graph $K_n$ can be partitioned into a red and a blue path.
Theorem (Gerencsér, Gyárfás 1967)

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Take a maximal red-blue-path:
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Warmup

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Take a maximal red-blue-path:

![Diagram showing a red-blue path in a graph]
Warmup

**Theorem (Gerencsér, Gyárfás 1967)**

The vertex set of any 2-edge-coloured complete graph $K_n$ can be partitioned into a **red** and a **blue** path.

Take a maximal **red-blue**-path:

![Diagram of a red-blue path in a complete graph](image)
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![Diagram of a red-blue-path]

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Covering and partitioning by monochromatic cycles

For an edge-coloured graph $G$, let

- $\text{cp}(G) = \text{minimum no. of vertex-disjoint monochromatic cycles covering } V(G)$
- $\text{cc}(G) = \text{minimum no. of monochromatic cycles covering } V(G)$

$\text{cc}(G) \leq \text{cp}(G)$
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For a graph $G$, let

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Conjecture (Lehel 1979)

The vertex set of any 2-edge-coloured complete graph $K_n$ can be partitioned into a red and a blue cycle,

$$cp_2(K_n) = 2.$$
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- Gyárfás (1983) → cover by two cycles intersecting in at most one vertex;
- Łuczak, Rödl, Szemerédi (1998) → proof for large $n$;
- Allen (2008) → proof for smaller $n$;
- Bessy, Thomassé (2010) → proof for all $n$. 
Conjecture (Erdős, Gyárfás, Pyber 1991)

For every $r \geq 2$

$$cp_r(K_n) \leq r.$$
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- Erdős, Gyárfás, Pyber (1991) $\rightarrow$ $cp_r(K_n) = O(r^2 \log r)$
- Gyárfás, Ruszinkó, Sárközy, Szemerédi (2006) $\rightarrow$ $O(r \log r)$. 
More colours

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- Gyárfás, Ruszinkó, Sárközy, Szemerédi (2006) $\rightarrow O(r \log r)$.
- Pokrovskiy (2012) $\rightarrow$ the conjecture is wrong
What about non-complete graphs?

Similar results hold in
- complete bipartite graphs
- graphs with sufficiently large minimum degree
- graphs with bounded independence number
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Similar results hold in

- complete bipartite graphs
- graphs with sufficiently large minimum degree
- graphs with bounded independence number

These graphs are all very dense.
**Theorem (Kohayakawa, Mota, Schacht, 2017+)**

If \( p \gg (\log n / n)^{1/2} \) then whp every 2-colouring of \( G_{n,p} \) contains a partition into two monochromatic trees,

\[
\text{tp}_2(G_{n,p}) \leq 2.
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- Haxell, Kohayakawa (1996) \( \rightarrow \) \( \text{tp}_r(K_n) \leq r \)
Tree partitioning of random graphs

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- The statement is false if $p \ll (\log n/n)^{1/2}$.
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- Haxell, Kohayakawa (1996) \( \rightarrow \) \( tp_r(K_n) \leq r \)
- The statement is false if \( p \ll (\log n/n)^{1/2} \).
- Proved by Bal and DeBiasio (2016) for \( p \gg (\log n/n)^{1/3} \).
Cycle covering of random graphs

Theorem (Korándi, Mousset, N., Škorić, Sudakov)

Given $r \geq 2$ and $\epsilon > 0$, if $p \gg n^{-1/r+\epsilon}$ then whp

$$cc_r(G_{n,p}) \leq Cr^6 \log r.$$ 

- Note: this is covering, not partitioning.
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Construction for $r = 2$:

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\Pr[v \text{ has at least two neighbours in } \{1, \ldots, k\}] \leq \binom{k}{2} p^2
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For any constant \( k \):

\[ \Pr[\text{such } v \text{ exists}] \leq n \binom{k}{2} p^2 \rightarrow 0 \]
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A similar construction works for $r > 2$. 
Theorem

If \( p \gg n^{-1/r+\epsilon} \) then

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\text{cc}_r(G_{n,p}) \leq f(r).
\]
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\text{cc}_r(G_{n,p}) \leq f(r).
\]

Proof idea. Show that:

1. constantly many monochromatic cycles can cover all but \( O(1/p) \) vertices;
2. every set of \( O(1/p) \) can be covered by constantly many monochromatic cycles.
Covering all but $O(1/p)$ vertices
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Split the vertices randomly into constantly many small parts.
Goal: cover each part using vertices from other parts (except for $O(1/p)$ vertices).
Covering all but $O(1/p)$ vertices

Each vertex has a **majority colour** to the top (at least $np/r$ neighbours in that colour).
Covering all but $O(1/p)$ vertices

Classify the vertices according to the majority colour.
Covering all but $O(1/p)$ vertices

We handle each colour independently.
Covering all but $O(1/p)$ vertices

Each vertex has at least $np/r$ red edges going to the right.
Covering all but $O(1/p)$ vertices

If two vertices have $\alpha np^2$ red common neighbours, place an auxiliary edge between them (here $\alpha > 0$ is a small constant).
In this way, we obtain an auxiliary graph on the red-majority vertices.
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**Goal:** show that the auxiliary graph contains cycles covering all but $O(1/p)$ vertices.
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**Lemma (Structural lemma)**

Let \(C\) be large enough and let \(X_1, \ldots, X_{r+1}\) be disjoint subsets of \(C/p\) vertices in the auxiliary graph.

Then there are \(i \neq j\) such that the auxiliary graph has an edge going from \(X_i\) to \(X_j\).
**Goal:** show that the auxiliary graph contains cycles covering all but $O(1/p)$ vertices.

**Lemma (Structural lemma)**

Let $C$ be large enough and let $X_1, \ldots, X_{r+1}$ be disjoint subsets of $C/p$ vertices in the auxiliary graph.

Then there are $i \neq j$ such that the auxiliary graph has an edge going from $X_i$ to $X_j$.

In other words: the complement of the auxiliary graph does not contain a complete $(r + 1)$-partite graph with parts of size $C/p$. 
Covering all but $O(1/p)$ vertices

The proof of the first step is thus completed by showing:

**Lemma**

Let $G$ be a graph whose complement does not contain a complete $k$-partite graph with parts of size $m$. Then $G$ contains $k^2$ vertex disjoint cycles covering all but $k^2 m$ vertices.
Covering $C/p$ vertices

Next step: show that every subset of $C/p$ vertices can be covered by a constant number of cycles.
Covering $C/p$ vertices

Suppose $|X| \leq C/p$. Here’s the strategy:

- We again define an auxiliary graph on $X$, but this time, an **edge-coloured** one.
Covering $C/p$ vertices

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- We again define an auxiliary graph on $X$, but this time, an edge-coloured one.
- Place a red auxiliary edge between $u$ and $v$ if $G_{n,p}$ contains “many” short red paths from $u$ to $v$. (Same for other colours.)
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- Place a red auxiliary edge between $u$ and $v$ if $G_{n,p}$ contains “many” short red paths from $u$ to $v$. (Same for other colours.)
- A monochromatic cycle in the auxiliary graph will correspond to a monochromatic cycle in $G_{n,p}$.
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- A monochromatic cycle in the auxiliary graph will correspond to a monochromatic cycle in $G_{n,p}$.
- Moreover, the auxiliary graph will have bounded independence number.
Covering $C/p$ vertices

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- We again define an auxiliary graph on $X$, but this time, an edge-coloured one.
- Place a red auxiliary edge between $u$ and $v$ if $G_{n,p}$ contains “many” short red paths from $u$ to $v$. (Same for other colours.)
- A monochromatic cycle in the auxiliary graph will correspond to a monochromatic cycle in $G_{n,p}$.
- Moreover, the auxiliary graph will have bounded independence number.
- Thus it can be partitioned into constantly many monochromatic cycles (Sárközy 2010).
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\[ \Omega(n^2 p^4) \]

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\[ \Omega\left(\frac{\log n}{p}\right) \]
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Open problems

Cycles:

- Partitioning instead of covering
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- Better $p$ (get rid of the $\epsilon$)
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- Partitioning instead of covering
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Trees:
- if $p \gg (\log n/n)^{1/r}$ then whp $t_{p_r}(G_{n,p}) \leq r$