Covering random graphs by monochromatic cycles

Rajko Nenadov

(joint with D. Korándi, F. Mousset, N. Škorić, and B. Sudakov)

Warmup

Theorem (Gerencsér, Gyárfás 1967)

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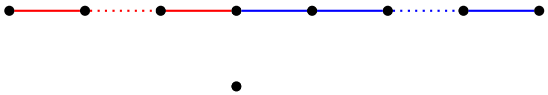

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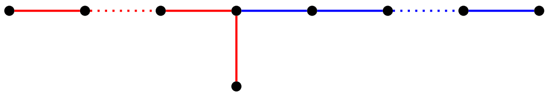

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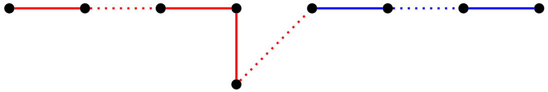

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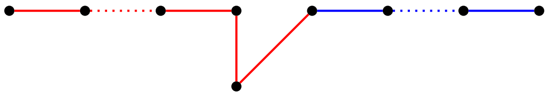

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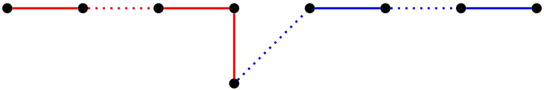

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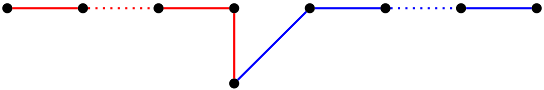

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subgraph that contains B. If Y is a point such that $Y \notin R$ and $Y \notin W$ then YV_i is yellow for $V_i \in V$. Let Q denote the maximal yellow-connected subgraph that contains Y. If there is no such Y, Q denotes the empty set. R, W, Q contain together all points of G. Namely any points $S \notin R$ is connected with a

[1] The weaker result $g(k, l) \leq k+l$ can be easily proved. Let us consider any vertex P and a pair of paths of G and $\overline{G}$ without common vertices except P. It can be proved that a pair of paths with maximal sum of lengths contains all points. (Maximality with respect to all P and all pairs.) From that the statement follows.

Covering and partitioning by monochromatic cycles

For an edge-coloured graph G, let

$\text{cp}(G) =$ minimum no. of vertex-disjoint monochromatic cycles covering $V(G)$

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The vertex set of any 2-edge-coloured complete graph K_n can be partitioned into a red and a blue cycle,

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- Gyárfás (1983) $\rightarrow$ cover by two cycles intersecting in at most one vertex;
- Łuczak, Rödl, Szemerédi (1998) $\rightarrow$ proof for large n;
- Allen (2008) $\rightarrow$ proof for smaller n;
- Bessy, Thomassé (2010) $\rightarrow$ proof for all n.

More colours

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- Pokrovskiy (2012) $\rightarrow$ the conjecture is wrong

What about non-complete graphs?

Similar results hold in

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These graphs are all very dense.

Tree partitioning of random graphs

Theorem (Kohayakawa, Mota, Schacht, 2017+)

If $p \gg (\log n/n)^{1/2}$ then whp every 2-colouring of $G_{n,p}$ contains a partition into two monochromatic trees,

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- Proved by Bal and DeBiasio (2016) for $p \gg (\log n/n)^{1/3}$.

Cycle covering of random graphs

Theorem (Korándi, Mousset, N., Škorić, Sudakov)

Given $r \geq 2$ *and* $\epsilon > 0$*, if* $p \gg n^{-1/r+\epsilon}$ *then whp*

$$\mathrm{cc}_r(G_{n,p}) \leq Cr^6 \log r.$$

- Note: this is covering, not partitioning.

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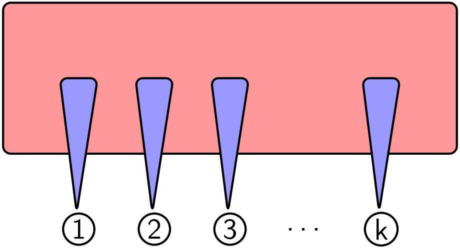
Construction for $r = 2$:



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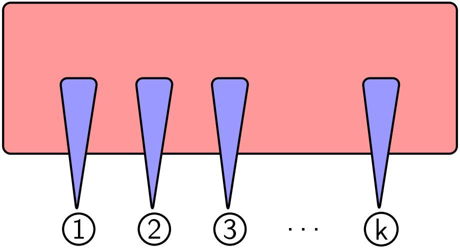

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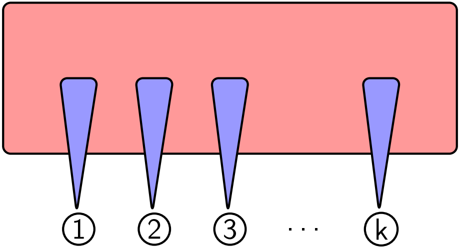

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For any constant k:

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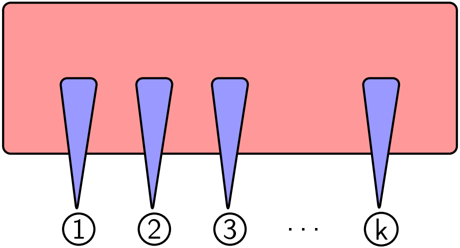

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A similar construction works for $r > 2$.

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Proof idea. Show that:

1. constantly many monochromatic cycles can cover all but $O(1/p)$ vertices;
2. every set of $O(1/p)$ can be covered by constantly many monochromatic cycles.

Covering all but $O(1/p)$ vertices

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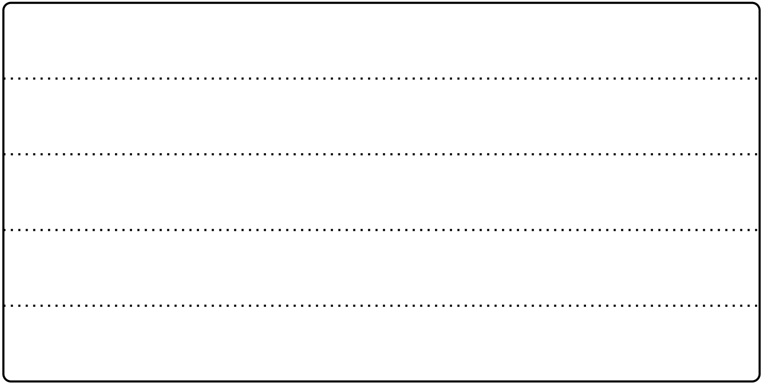
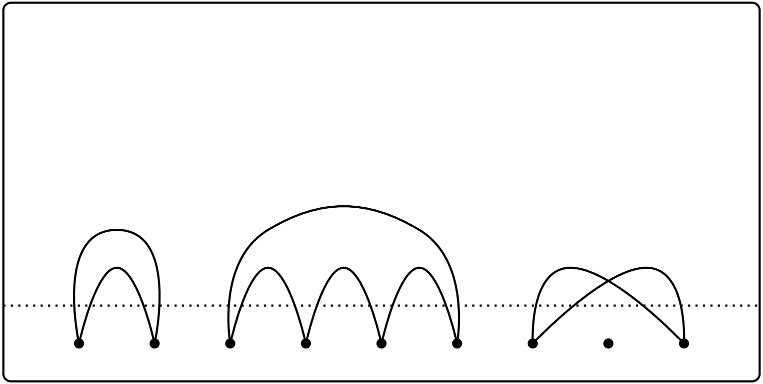


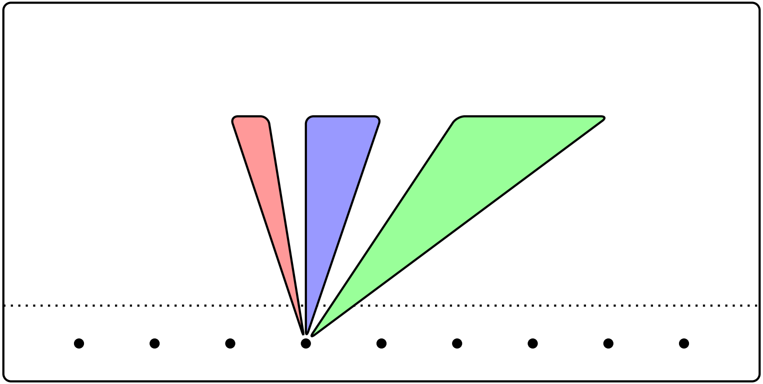
Split the vertices randomly into constantly many small parts.

Covering all but $O(1/p)$ vertices



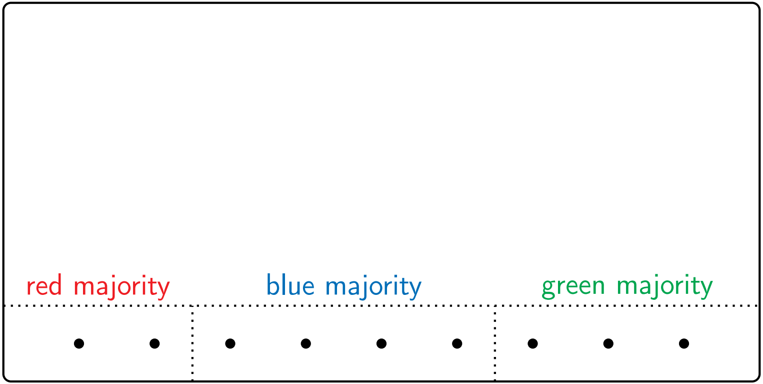
Goal: cover each part using vertices from other parts (except for $O(1/p)$ vertices).

Covering all but $O(1/p)$ vertices



Each vertex has a majortity colour to the top (at least np/r neighbours in that colour).

Covering all but $O(1/p)$ vertices



Classify the vertices according to the majority colour.

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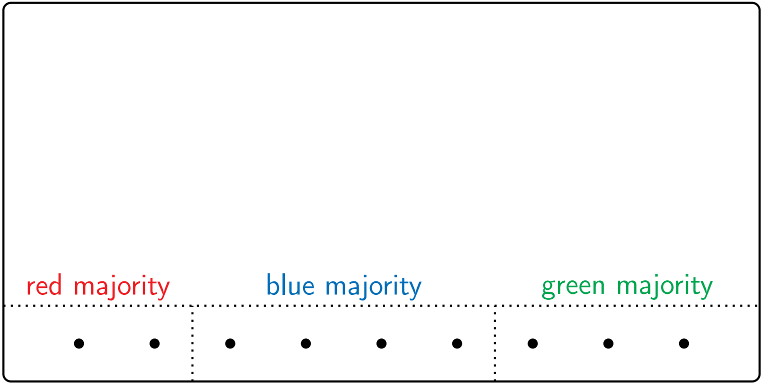
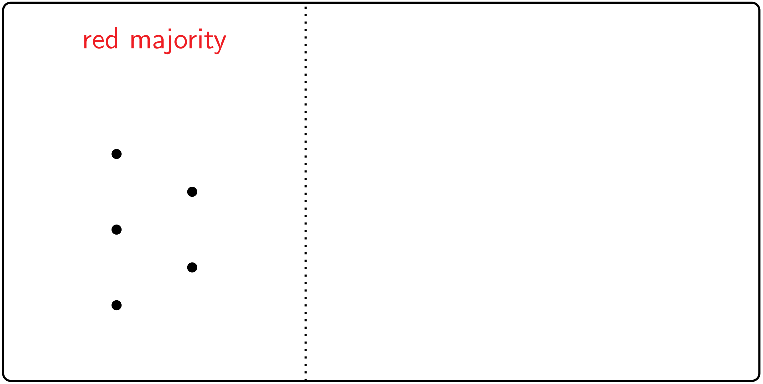


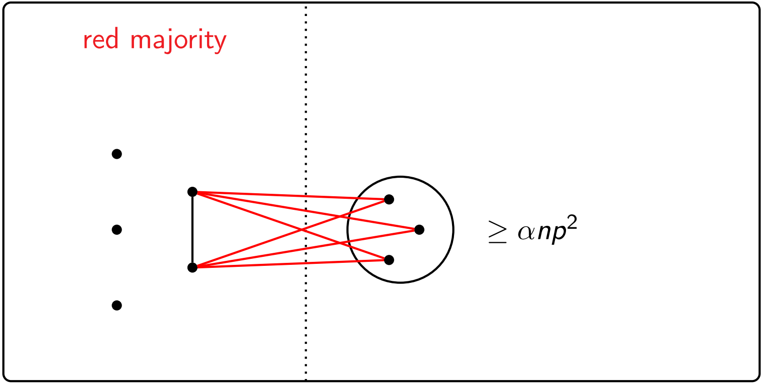
We handle each colour independently.

Covering all but $O(1/p)$ vertices



Each vertex has at least np/r red edges going to the right.

Covering all but $O(1/p)$ vertices



If two vertices have αnp^2 red common neighbours, place an auxiliary edge between them (here $\alpha > 0$ is a small constant).

In this way, we obtain an auxiliary graph on the red-majority vertices.

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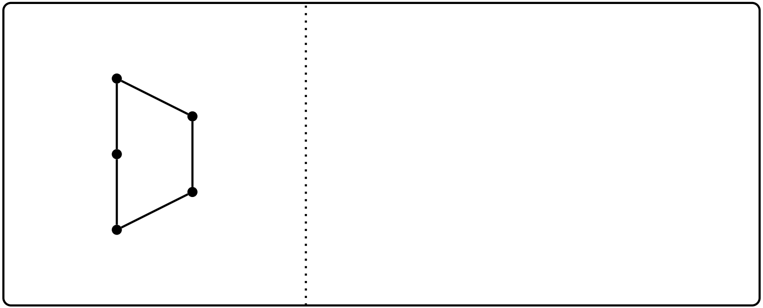

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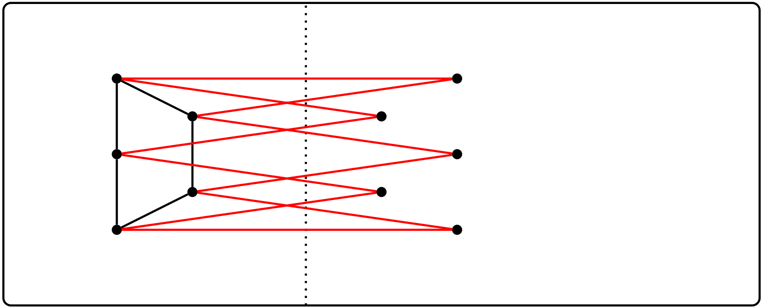

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Lemma (Structural lemma)

Let C be large enough and let $X_1, \ldots, X_{r+1}$ be disjoint subsets of C/p vertices in the auxiliary graph.

Then there are $i \neq j$ such that the auxiliary graph has an edge going from X_i to X_j.

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In other words: the complement of the auxiliary graph does not contain a complete $(r+1)$-partite graph with parts of size C/p.

Covering all but $O(1/p)$ vertices

The proof of the first step is thus completed by showing:

Lemma

Let G be a graph whose complement does not contain a complete k-partite graph with parts of size m. Then G contains k^2 vertex disjoint cycles covering all but k^2m vertices.

Covering C/p vertices

Next step: show that every subset of C/p vertices can be covered by a constant number of cycles.

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- Moreover, the auxiliary graph will have bounded independence number.
- Thus it can be partitioned into constantly many monochromatic cycles (Sárközy 2010).

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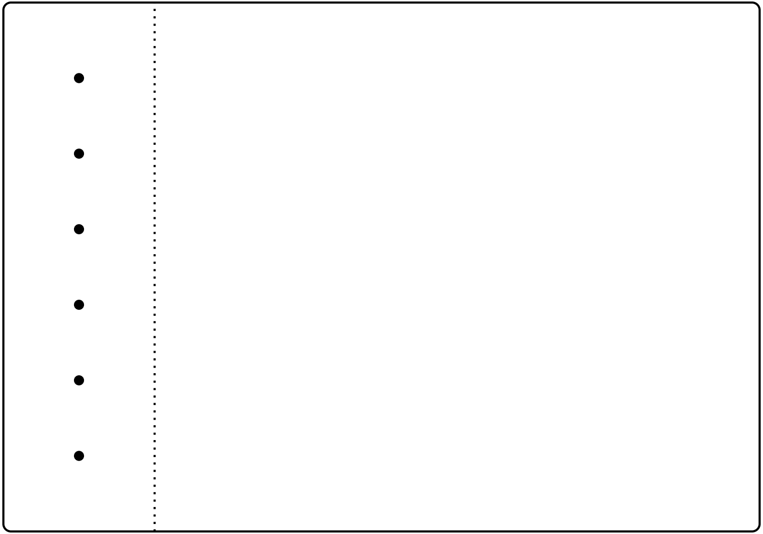

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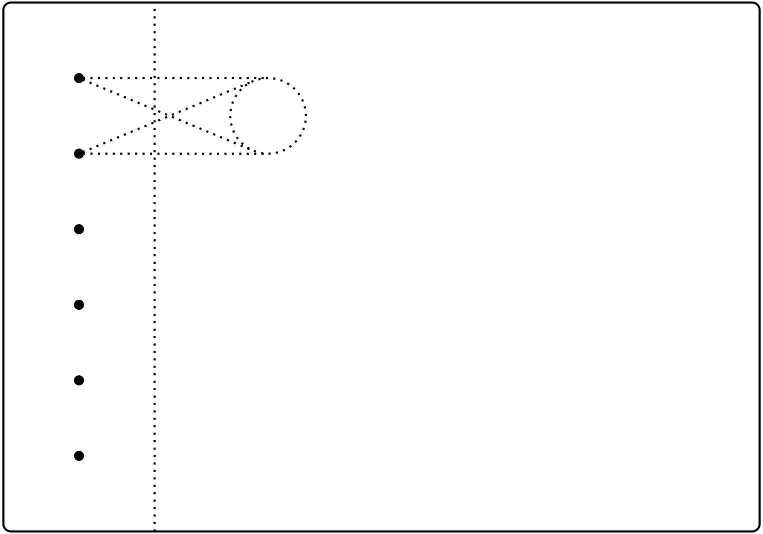

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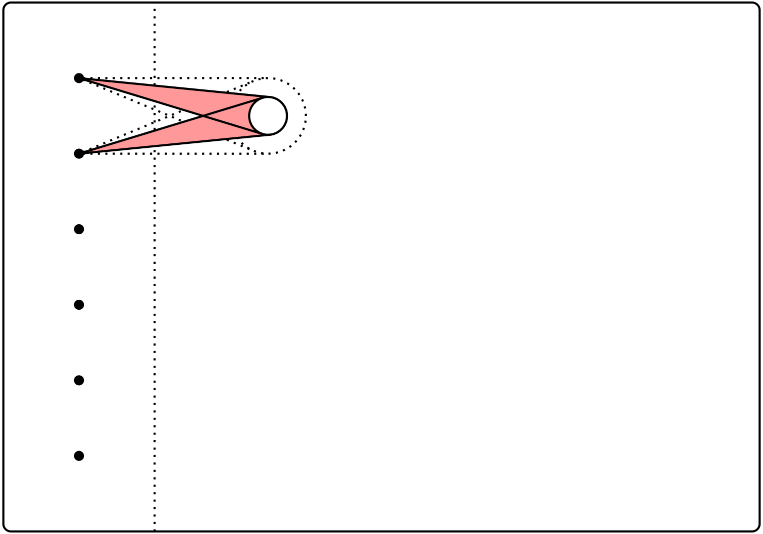

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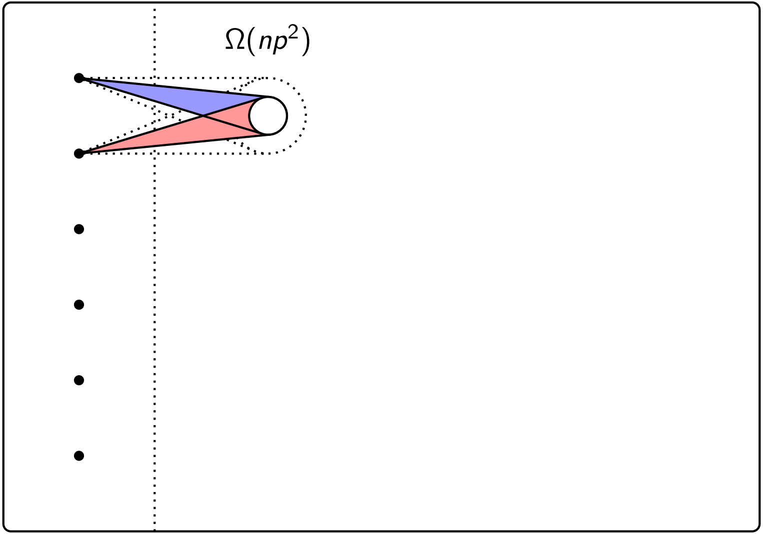

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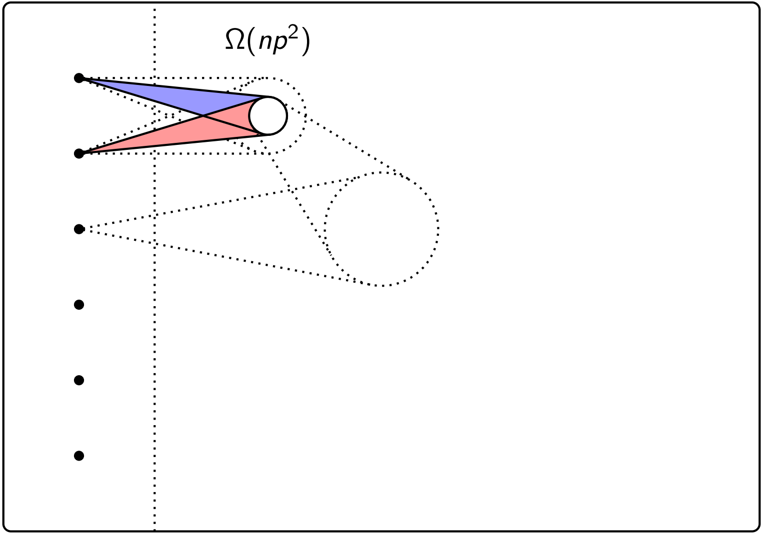

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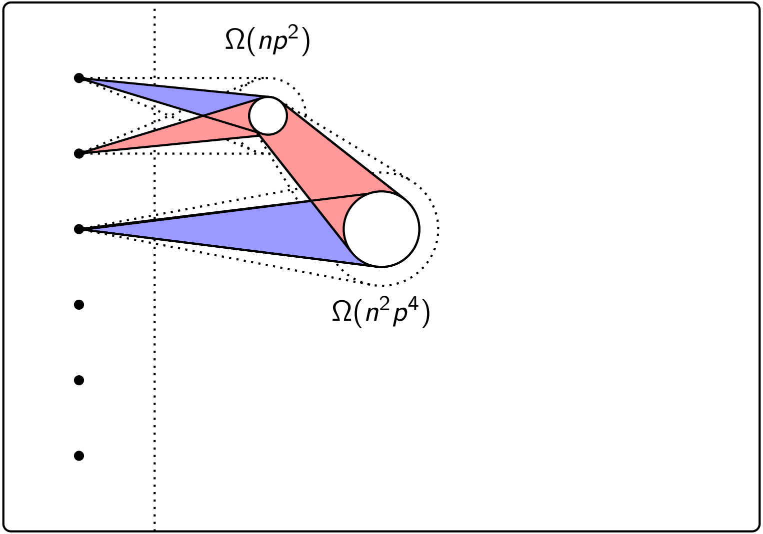

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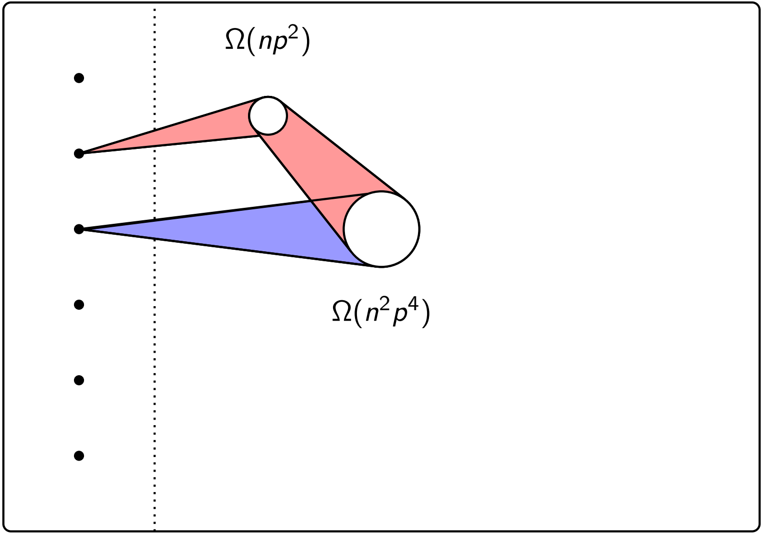

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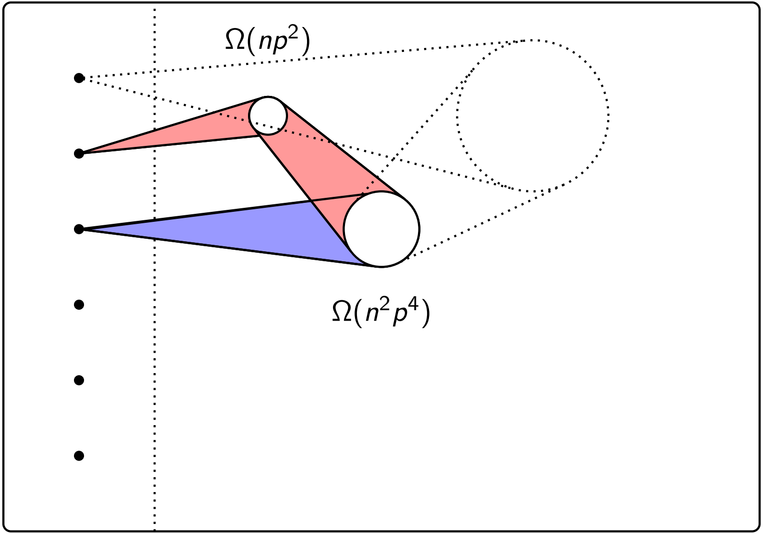

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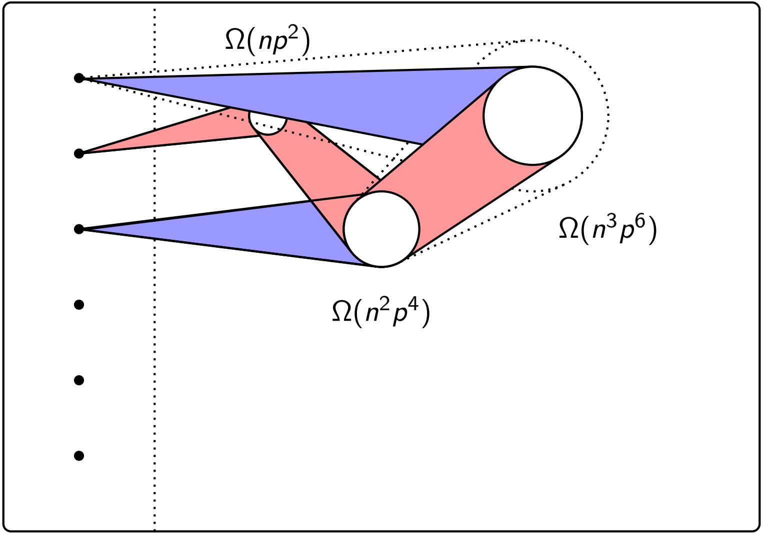

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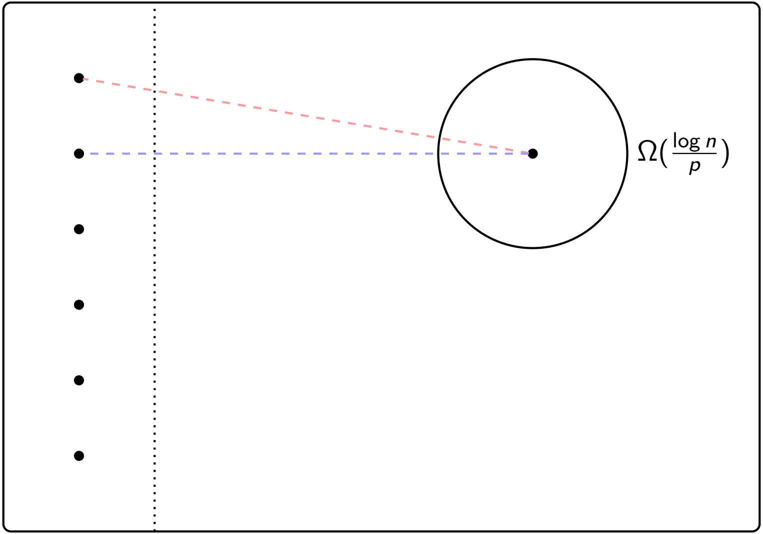

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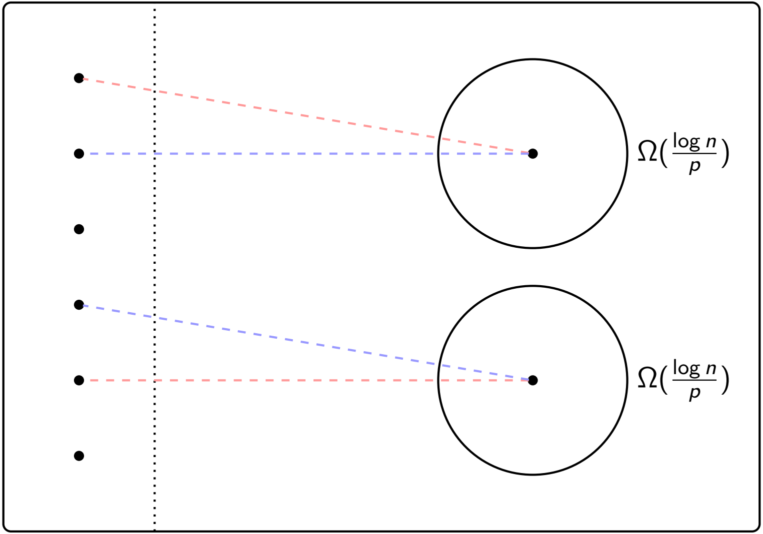

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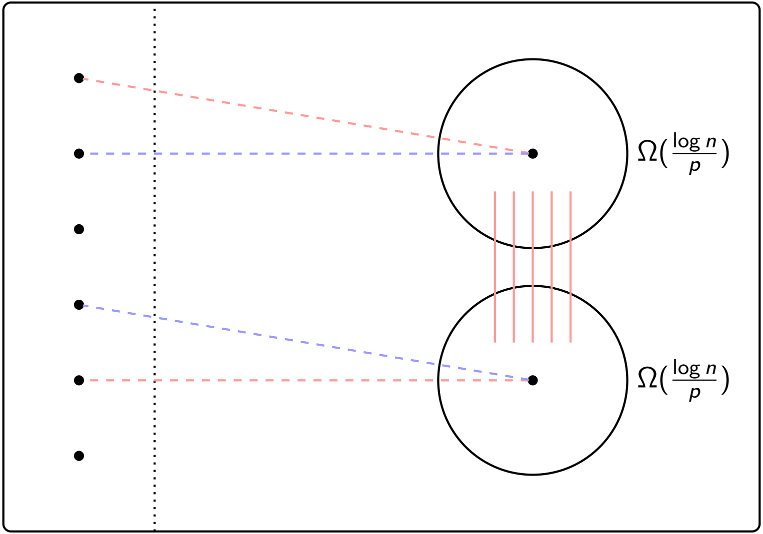

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