Ramsey properties of random graphs

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29th August 2016

A graph G is **Ramsey** for H

$$G \to H$$

if every red/blue colouring of the edges of G contains a **monochromatic** copy of H

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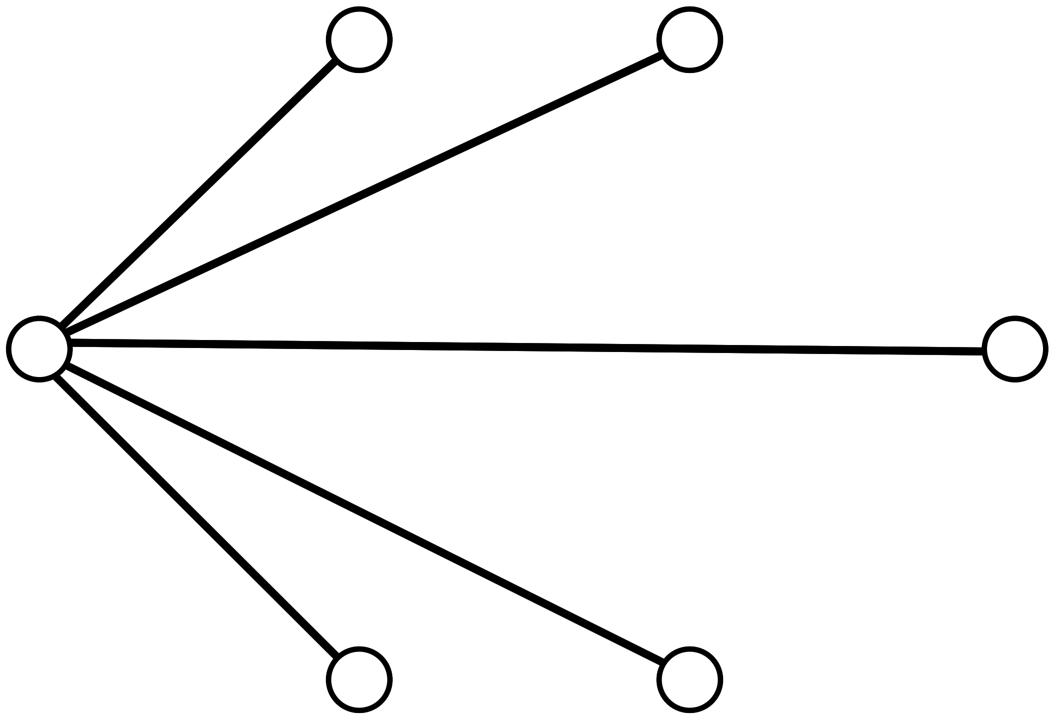

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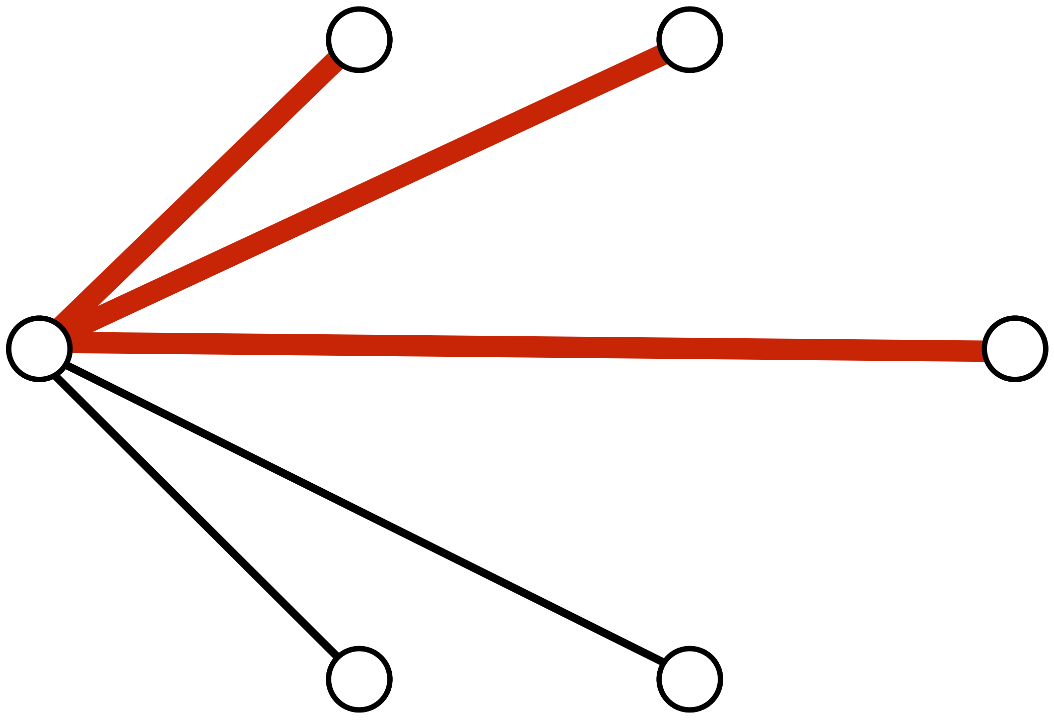

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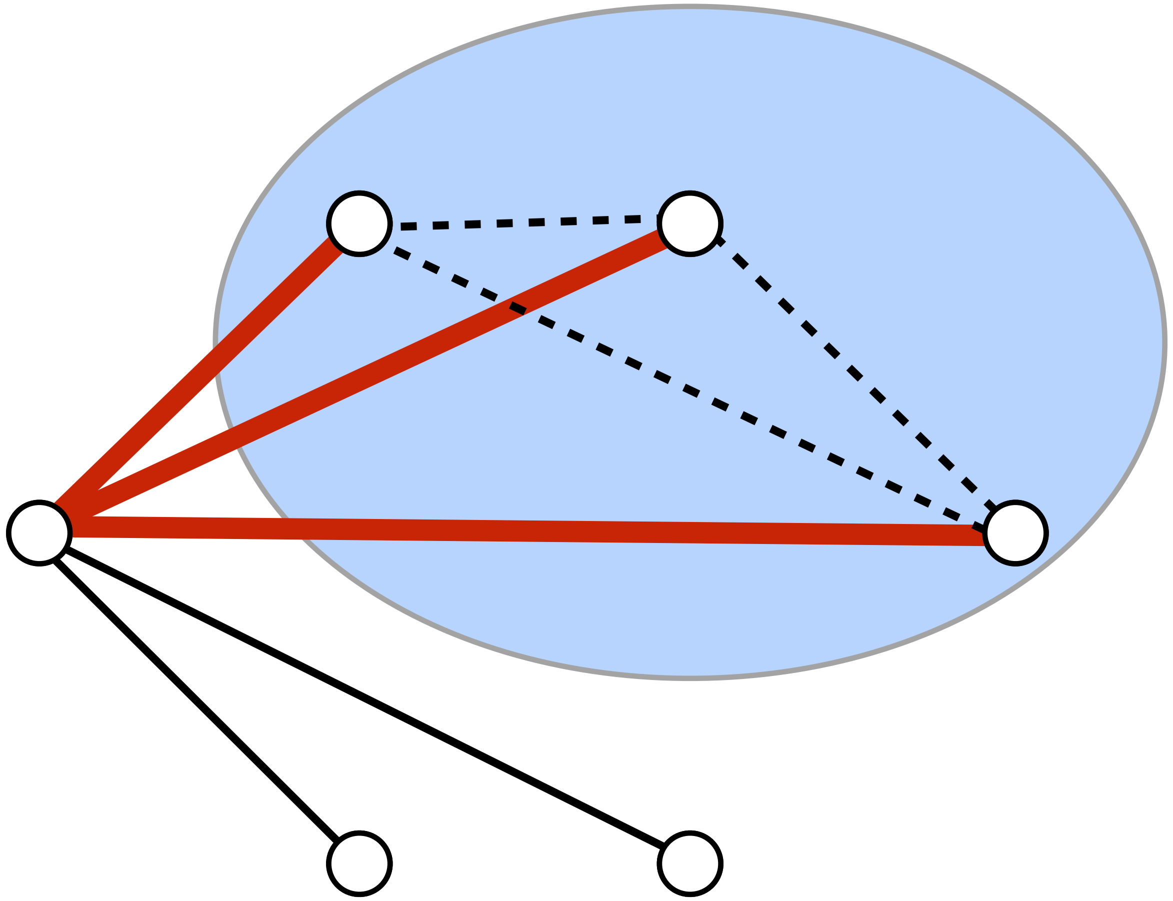

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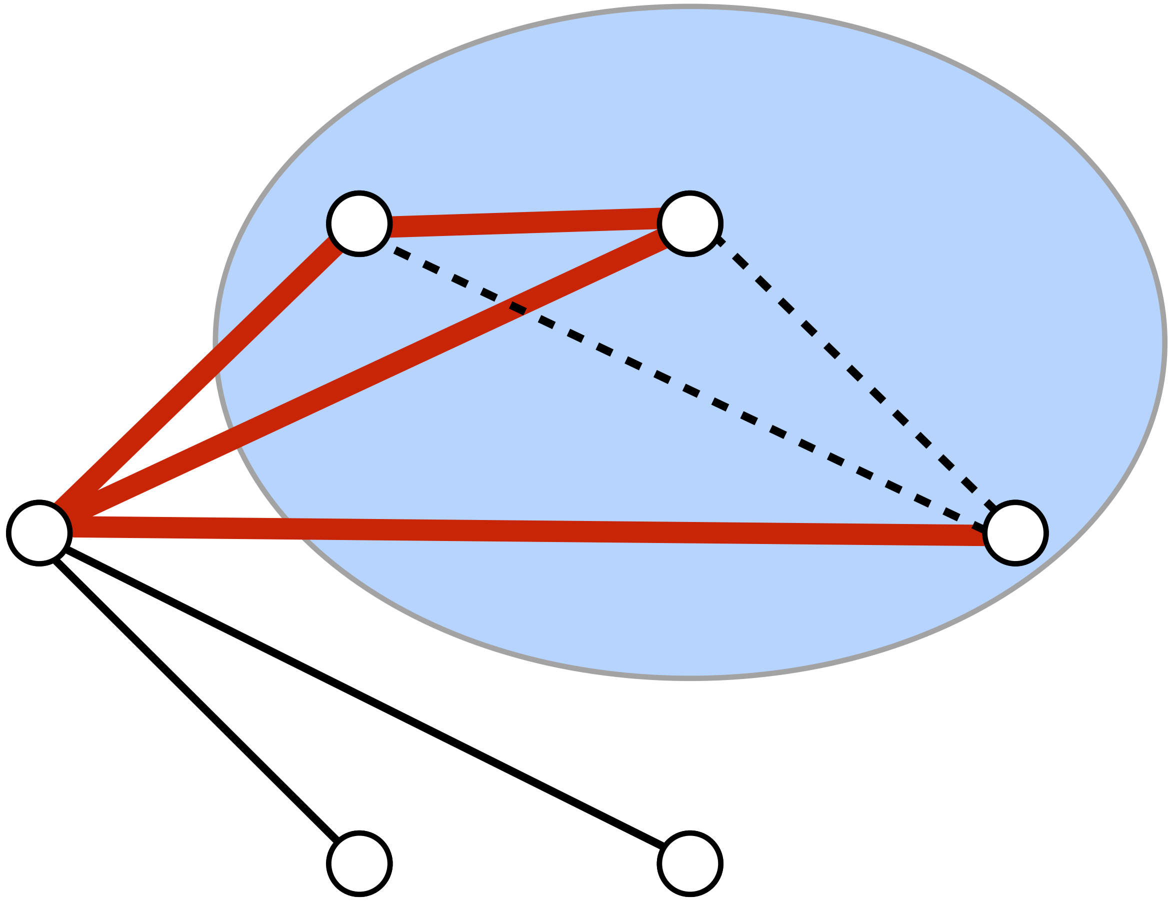

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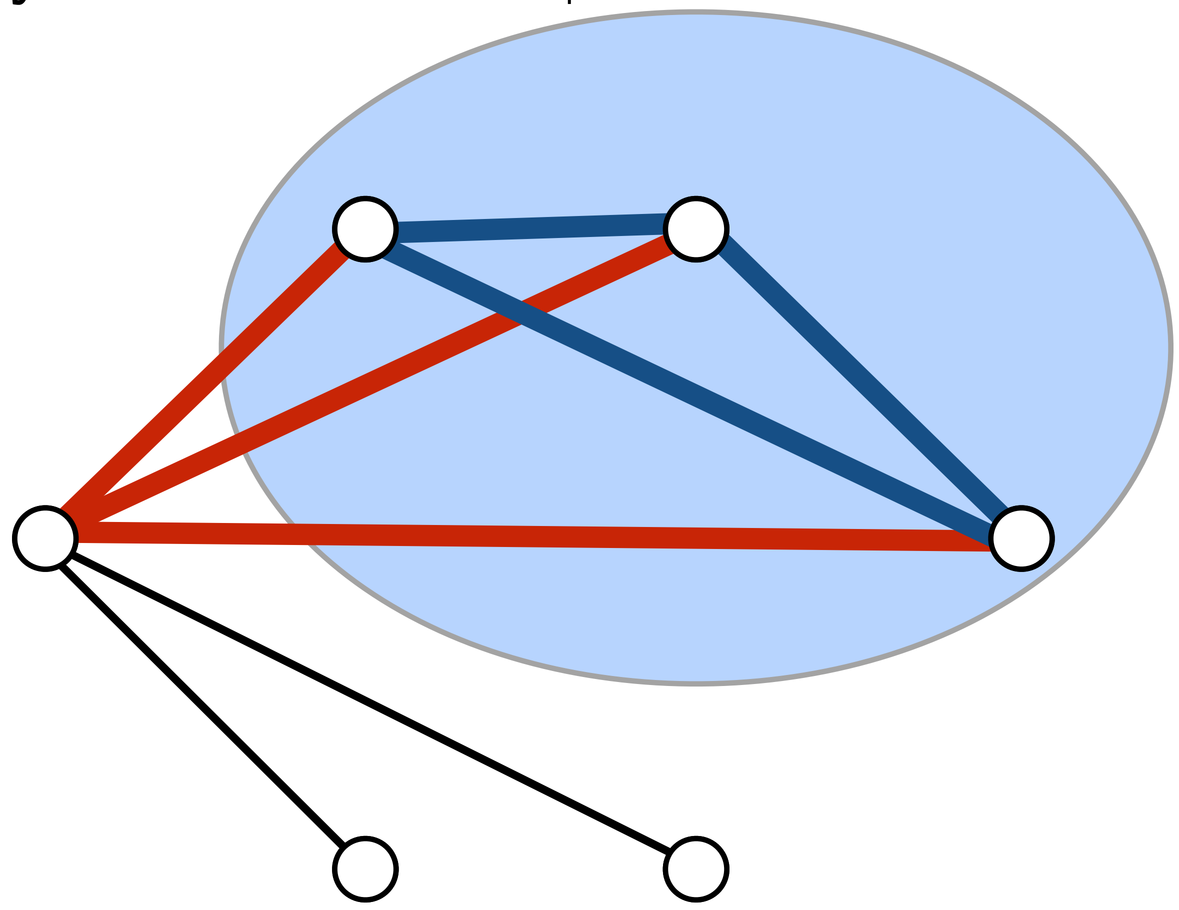

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Ramsey (1930)

For every graph H there exists (sufficiently large) $n \in \mathbb{N}$ such that

$$K_n \to H$$

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Binomial **random graph** $G(n, p)$

- n vertices
- each edge present with probability p

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- n vertices
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Given a graph $\mathbf{H}$ and $\mathbf{p = p(n) \in [0,1]}$, determine

$$\Pr[G(n,p) \to H]$$

Behaviour of $\Pr[G(n,p) \to H]$

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- "Being Ramsey for H" is a monotone property (preserved under edge addition)
- Bollobás-Thomason ('87): every non-trivial monotone property $\mathcal{P}$ has a threshold function $p^*(\mathcal{P})$

$$\lim_{n\to\infty} \Pr[G(n,p) \in \mathcal{P}] = \begin{cases} 0, & \text{if } p/p^*(\mathcal{P}) \to 0 \\ 1, & \text{if } p/p^*(\mathcal{P}) \to \infty \end{cases}$$

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Goal: find a threshold p_H for the property "being Ramsey for H"

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Warm-up

Given a graph $\mathbf{H}$, determine $p_H(n)$ such that

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Explanation: $G(n,p)$ contains K_6 with high probability

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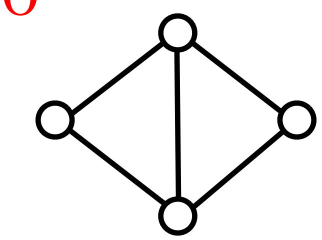
Explanation: $G(n,p)$ does not contain 

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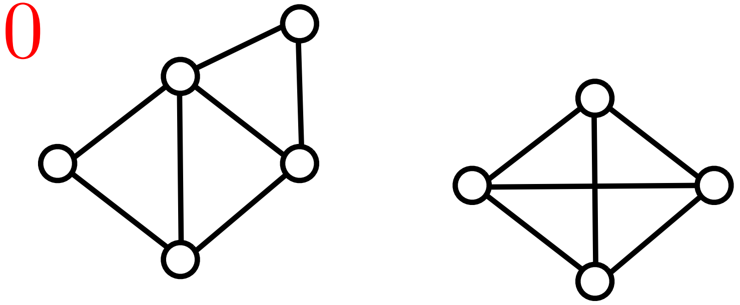

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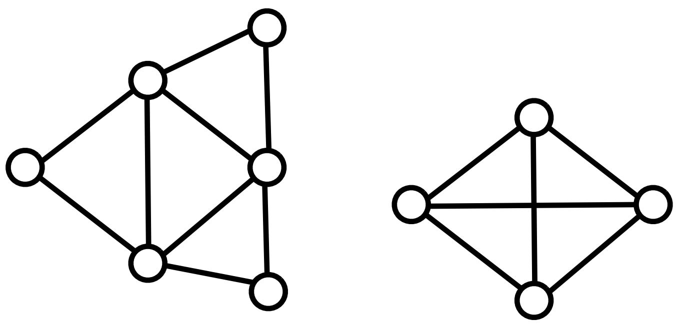

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Threshold for $G(n,p) \to K_3$

Frankl-Rödl ('86), Łuczak-Ruciński-Voigt ('92')

There exist constants $c, C > 0$ such that

$$\lim_{n\to\infty} \Pr[G(n,p) \to K_3] = \begin{cases} 0, & \text{if } p \le cn^{-1/2} \\ 1, & \text{if } p \ge Cn^{-1/2} \end{cases}$$

Threshold for $G(n,p) \to H$

Rödl-Ruciński ('93–'95)

For every graph H (which contains a cycle) there exist constants $c, C > 0$ such that

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Intuition: $\beta(H)$ is chosen such that

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N.-Steger (2015) – a 'short' proof

$$\lim_{n\to\infty} \Pr[G(n,p) \to K_3] = 1 \quad \text{for} \quad p \geq Cn^{-1/2}$$

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or

'A short introduction to hypergraph containers'

Hypergraph containers

Balogh–Morris–Samotij and Saxton-Thomason (2015)

$\forall \delta > 0 \; \exists K > 0$: for every $n \in \mathbb{N}$ there exists a collection $\mathcal{C}$ of graphs on n vertices and a function $f\colon 2^{E(K_n)} \to \mathcal{C}$ such that

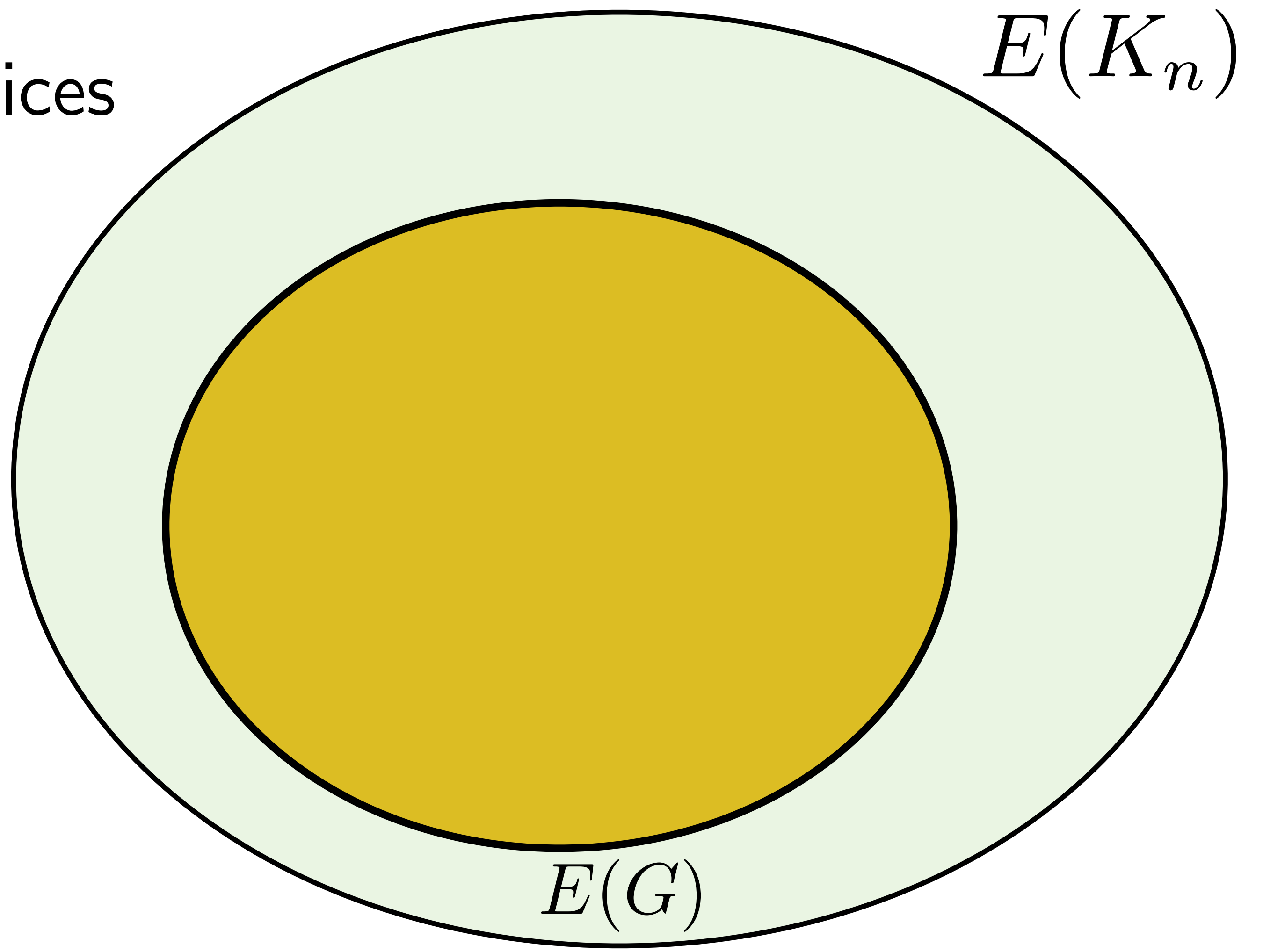
(a) each $C \in \mathcal{C}$ contains at most δn^3 triangles,

(b) for every K_3-free graph H there exists $S \subseteq E(K_n)$ such that

$$e(S) \leq K n^{3/2} \quad \text{and} \quad S \subseteq H \subseteq f(S)$$

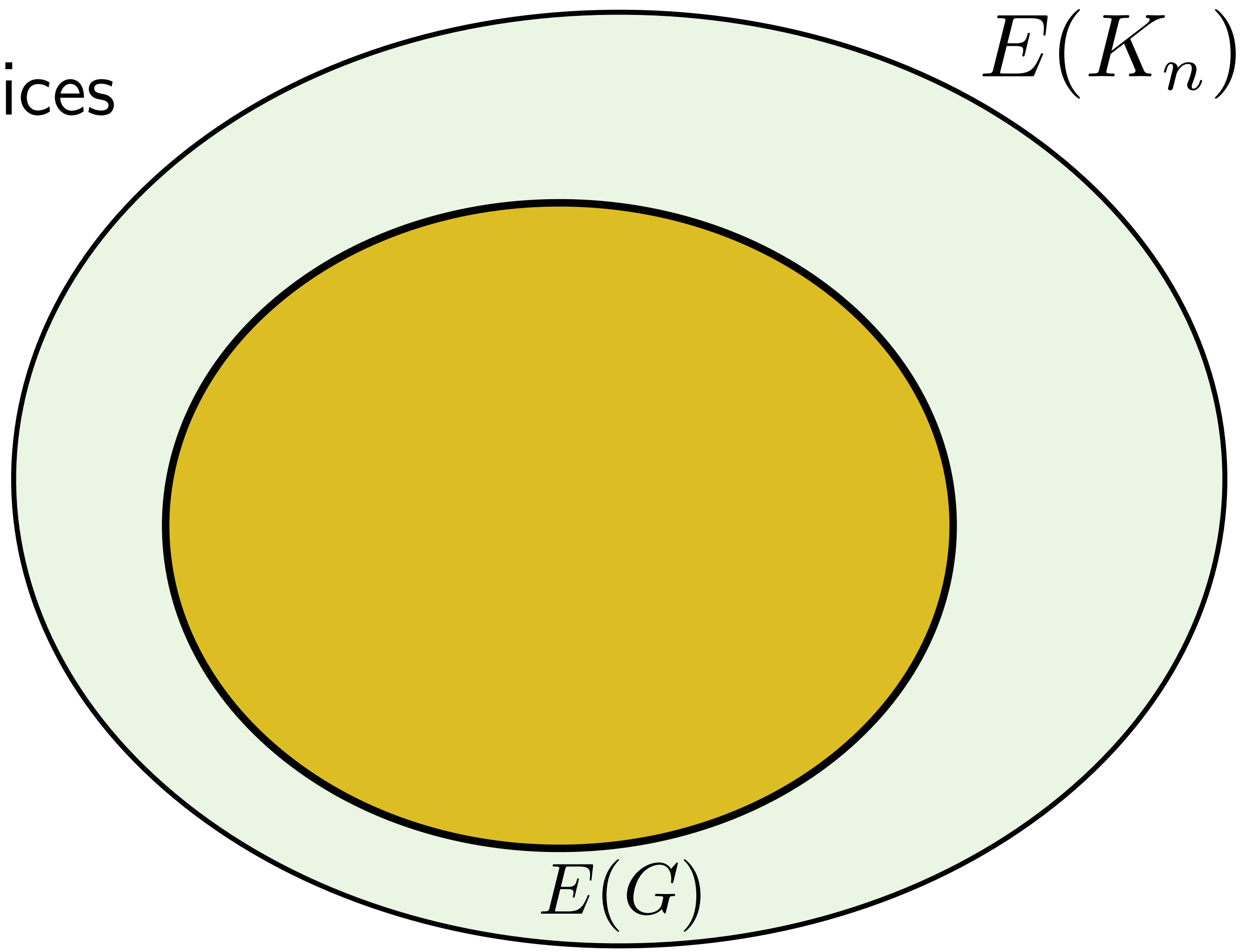
- $\mathcal{C}$ contains all triangle-free graphs (containers)
- container of a graph H is generated by its small subgraph

Let $G \subseteq K_n$ be a graph on n vertices



Let $G \subseteq K_n$ be a graph on n vertices

- Suppose there exists a colouring $E(G) = R \cup B$ with no monochromatic triangle



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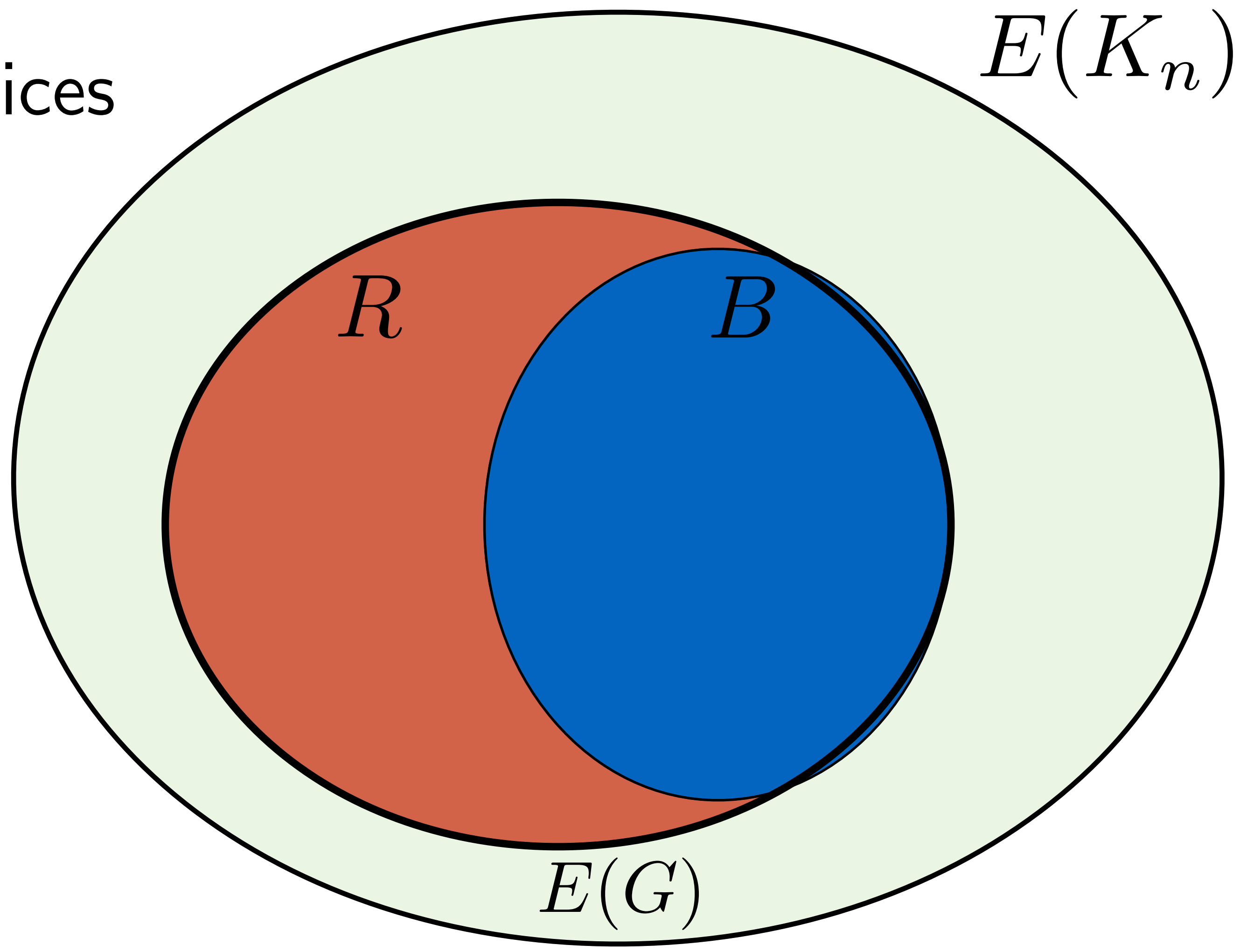

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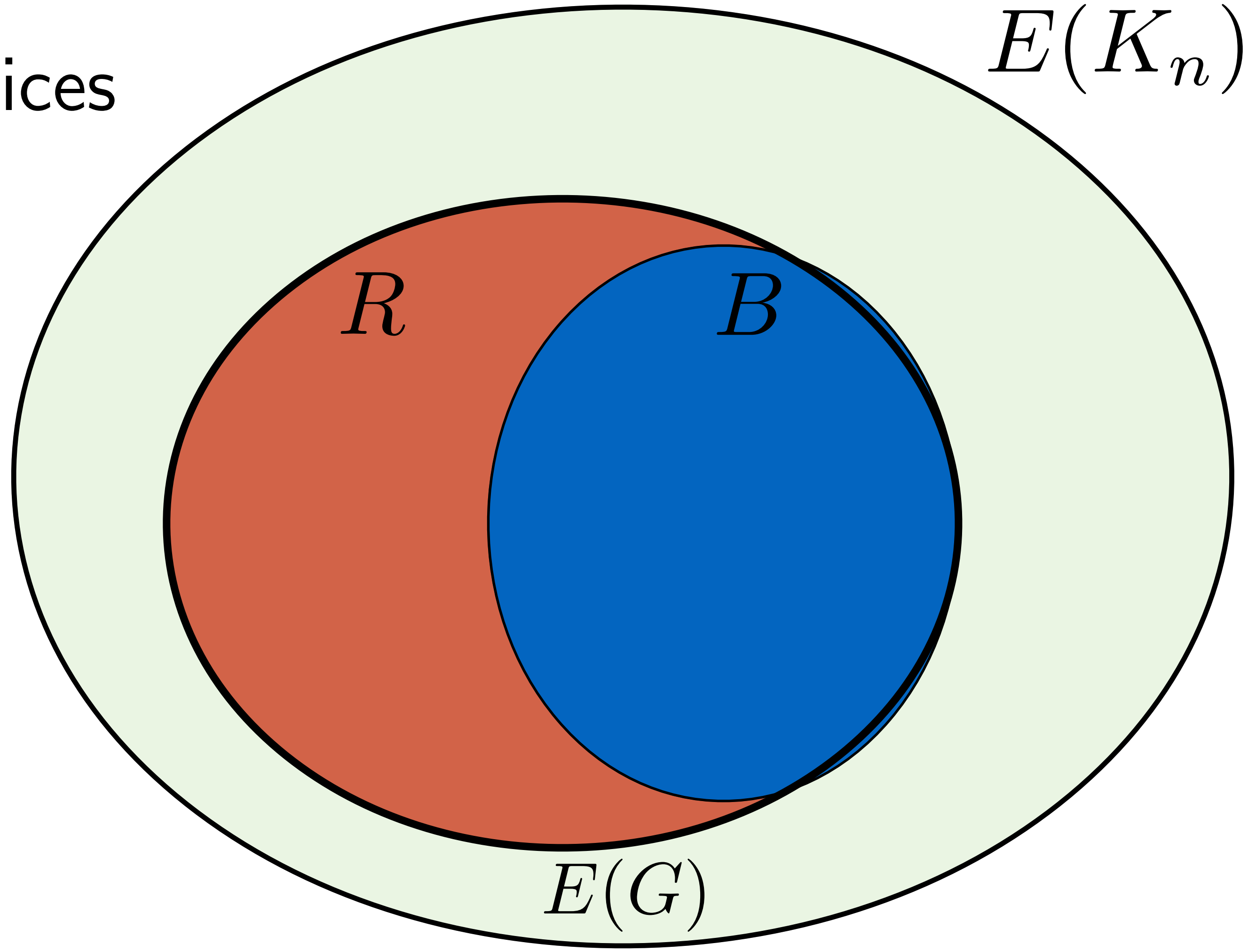

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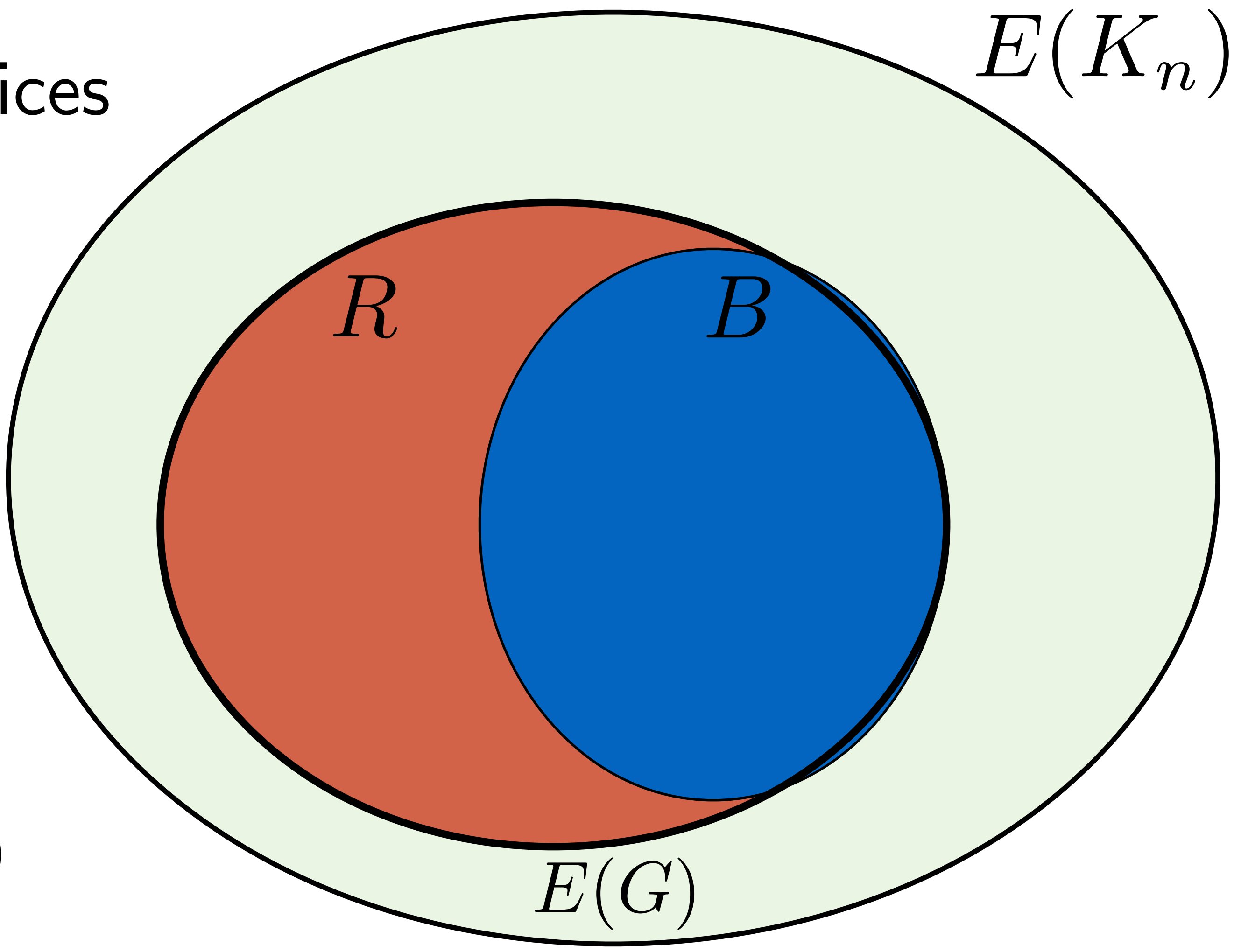

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- Container theorem – there exist small subgraphs $S_R, S_B \subseteq E(K_n)$ such that

$$S_R \subseteq R \subseteq f(S_R)$$
$$S_B \subseteq B \subseteq f(S_B)$$

Recall: $e(S_R) < Kn^{3/2}$, $f(S_R)$ contains at most δn^3 triangles

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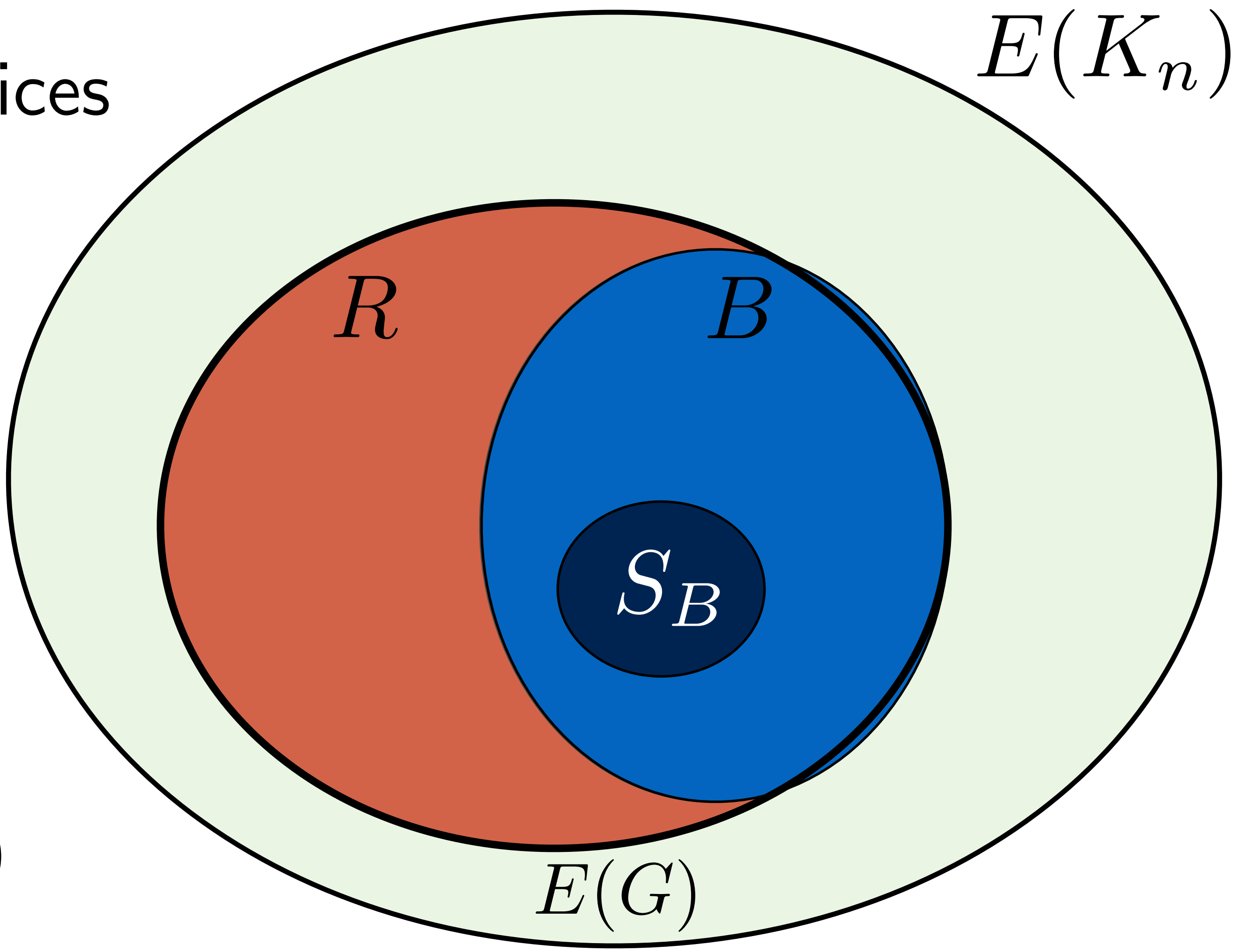

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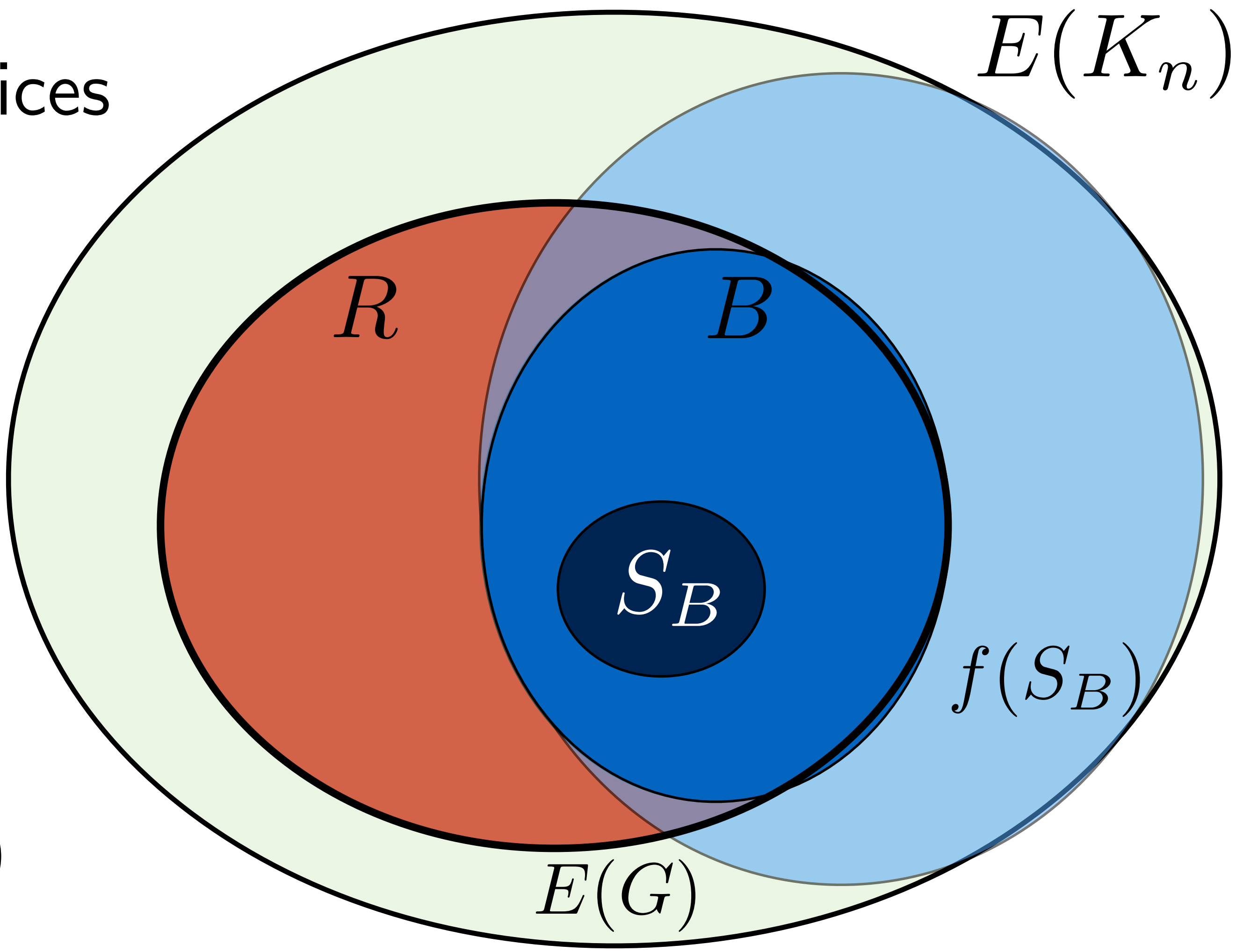

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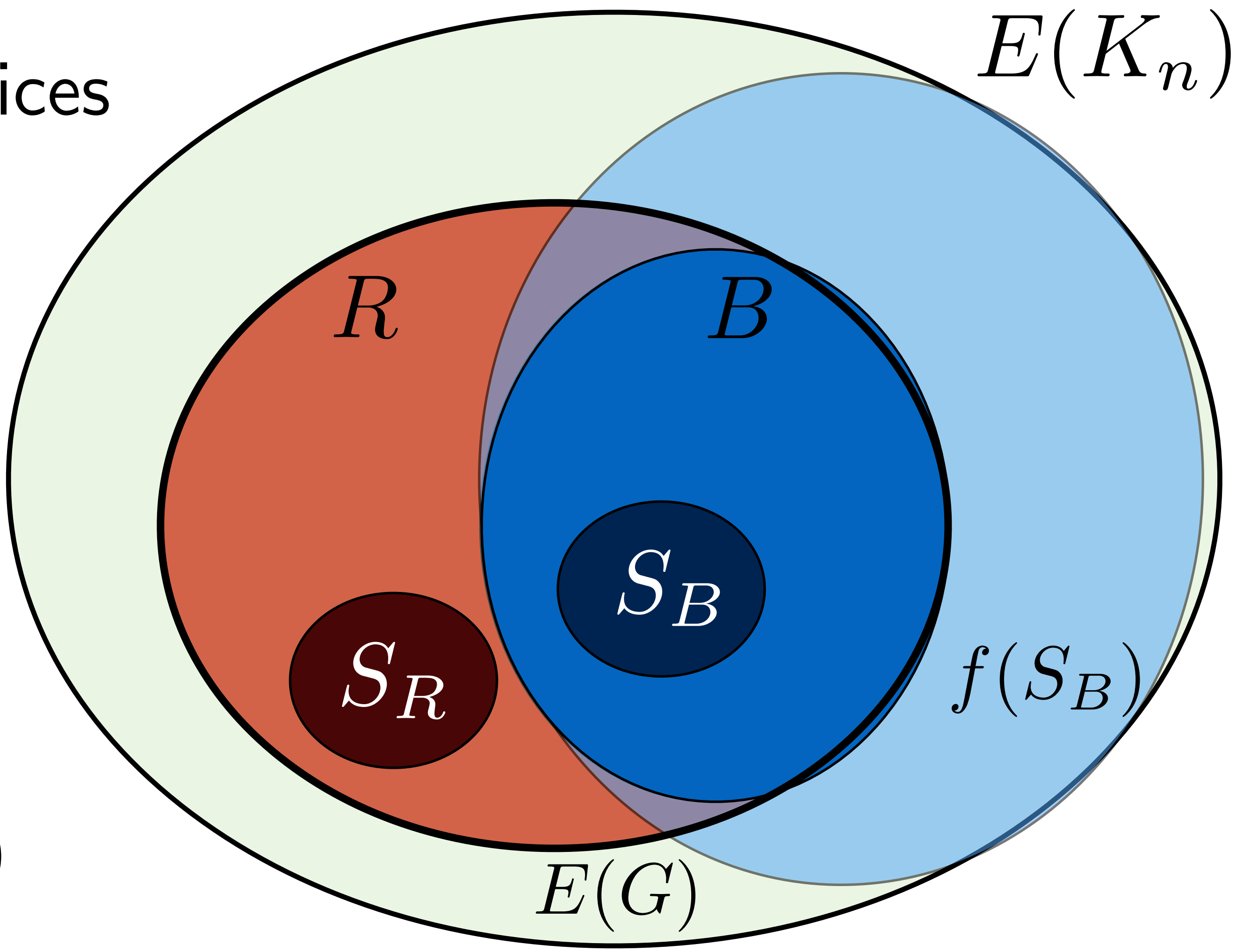

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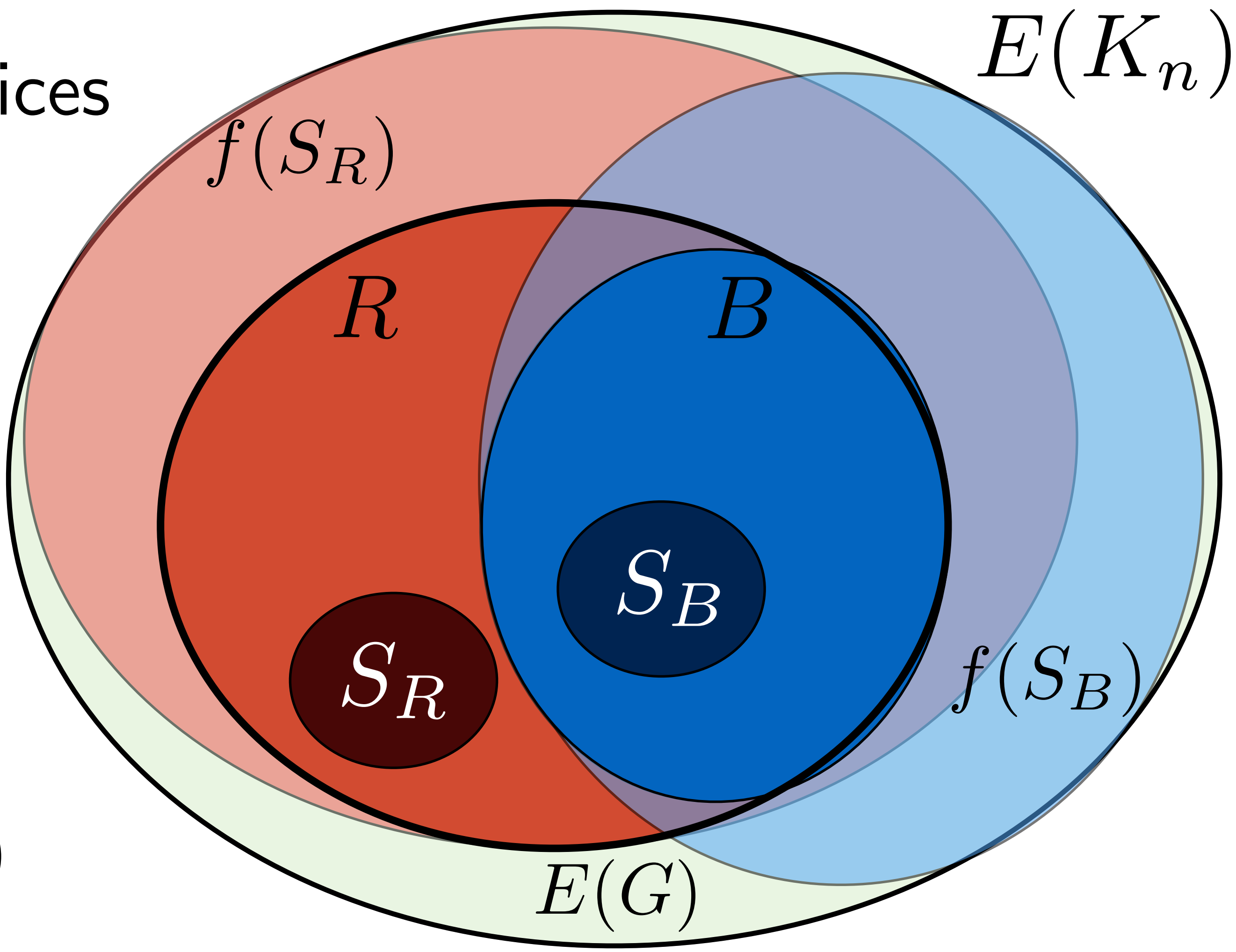

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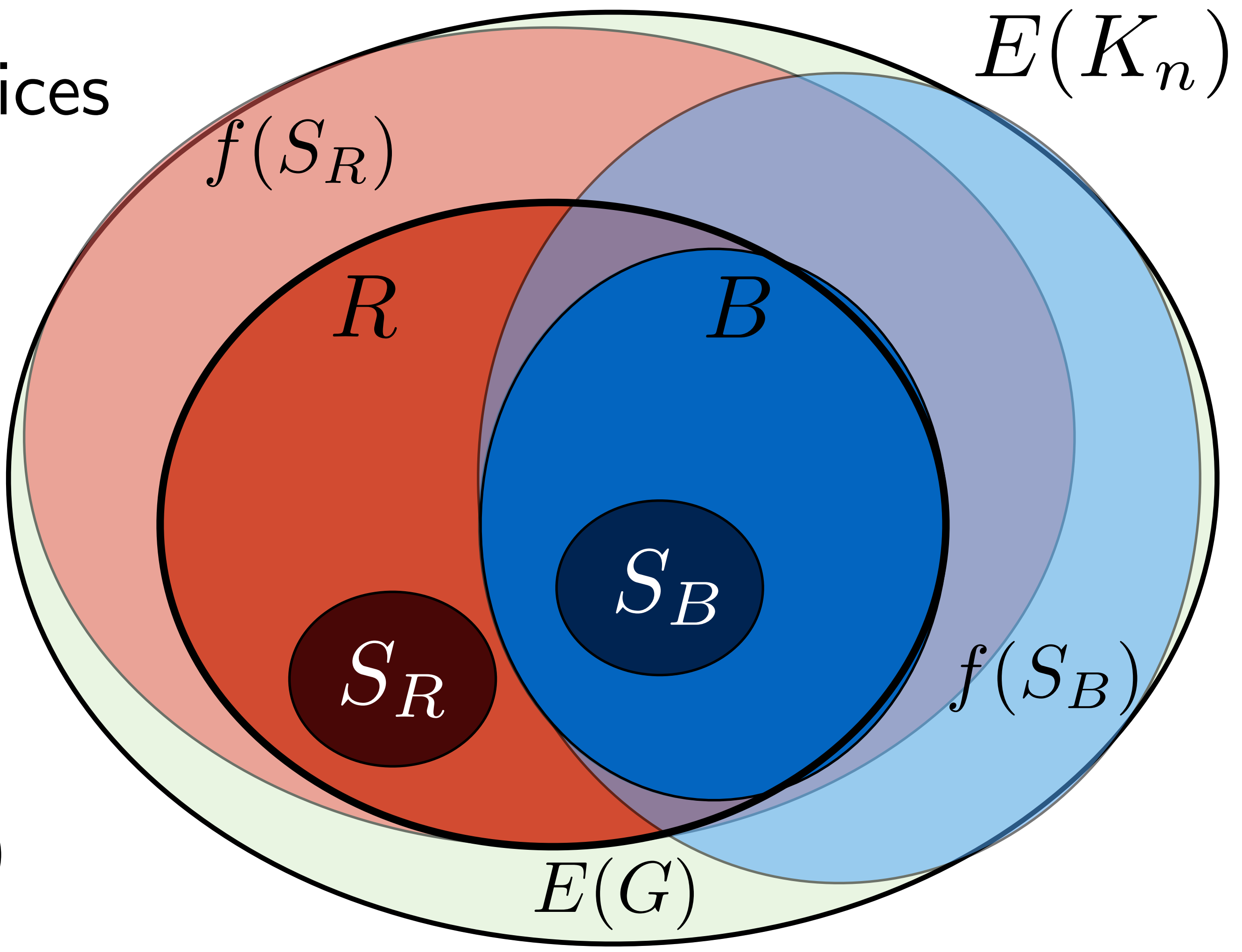

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- $L = K_n \setminus (f(S_R) \cup f(S_B))$

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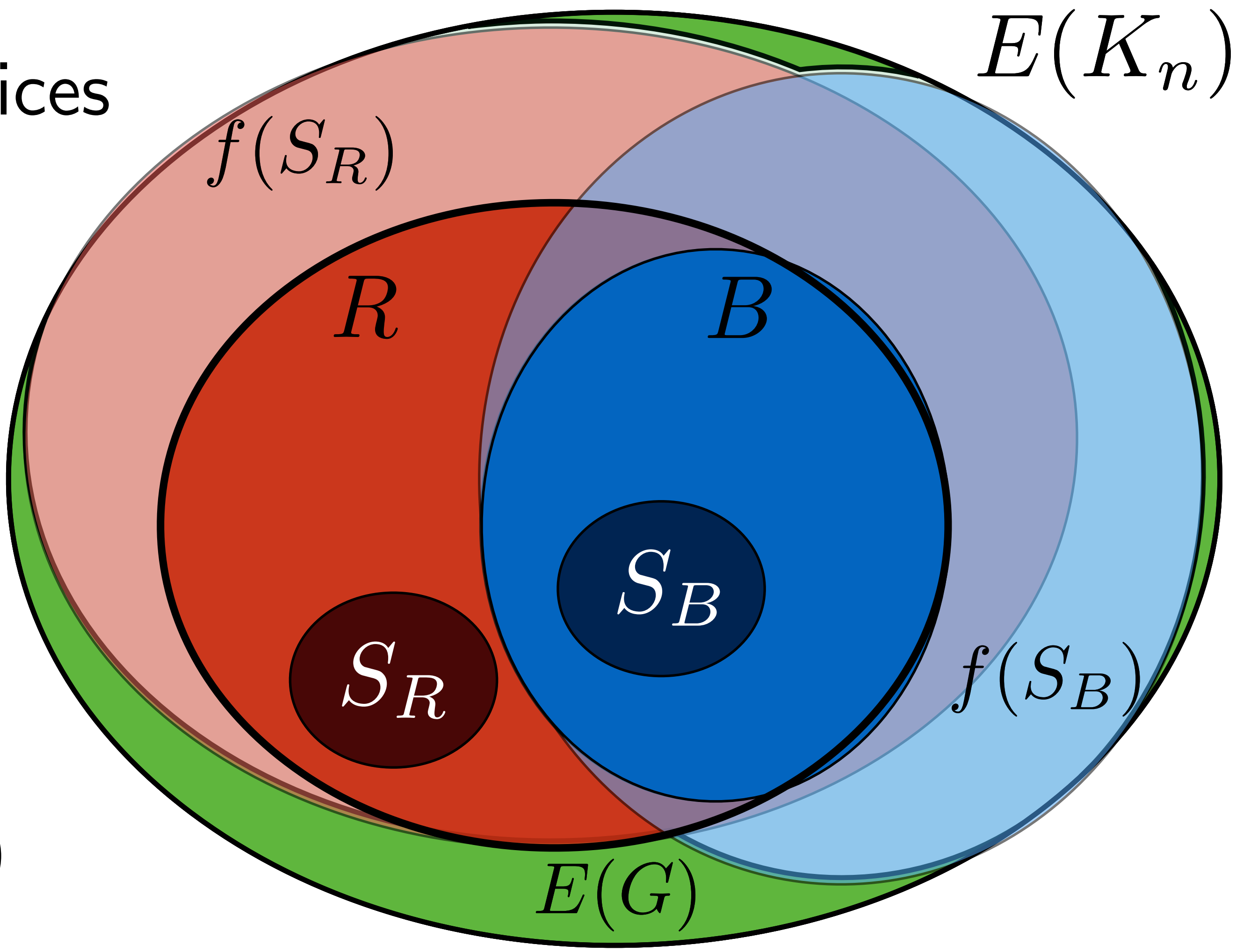

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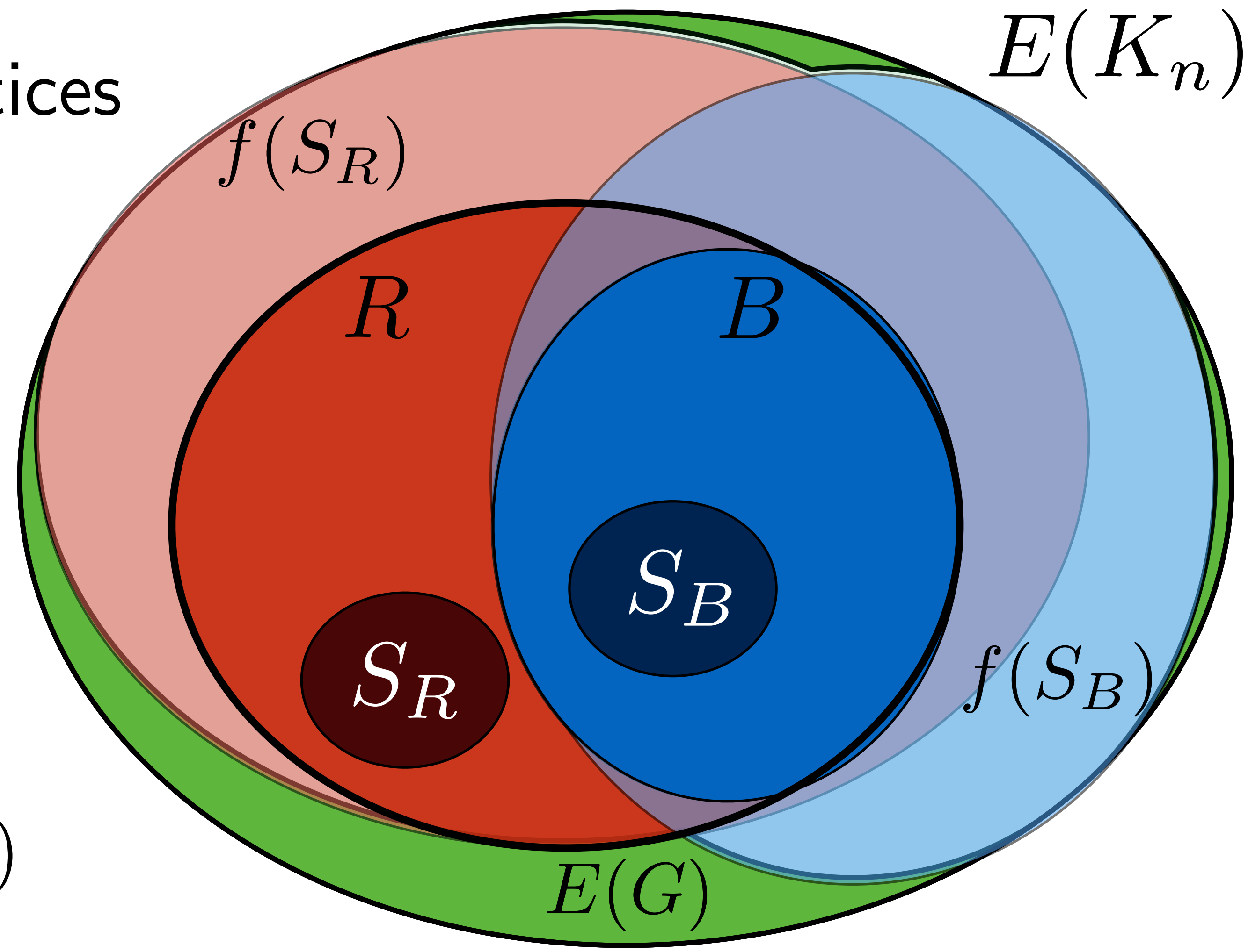

- Suppose there exists a colouring $E(G) = R \cup B$ with no monochromatic triangle (both R and B are triangle-free)

- Container theorem – there exist small subgraphs $S_R, S_B \subseteq E(K_n)$ such that

$$S_R \subseteq R \subseteq f(S_R)$$
$$S_B \subseteq B \subseteq f(S_B)$$

Recall: $e(S_R) < Kn^{3/2}$, $f(S_R)$ contains at most δn^3 triangles

- $L = K_n \setminus (f(S_R) \cup f(S_B))$

- Crucial observations:

$$L \cap G = \emptyset, \quad e(L) \geq \alpha n^2$$

Threshold for $G(n,p) \to H$

Rödl-Ruciński ('93–'95)

For every graph H (which contains a cycle) there exist constants $c, C > 0$ such that

$$\lim_{n\to\infty} \Pr[G(n,p) \to H] = \begin{cases} 0, & \text{if } p \le cn^{-\beta(H)} \\ 1, & \text{if } p \ge Cn^{-\beta(H)} \end{cases}$$

Other approaches and generalisations

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- Basic idea:
  - $G(n,p) = G(n,p_1) \cup G(n,p_2)$ where $p1 = \alpha p_2$
  - Show that any colouring of $G(n,p_1)$ either contains a mono. H or heaps of well-distributed mono. $H - e$

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  - Show that any colouring of $G(n,p_1)$ either contains a mono. H or heaps of well-distributed mono. $H - e$
  - There are $2^{n^2 p_1}$ colourings
  - Show that with probability $e^{-\Omega(n^2 p_2)}$ any extension to a colouring to $G(n,p_2)$ gives a mono. H

Other approaches and generalisations

Proof (1-statement):

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- Many partial results until finally settled
  - Balogh-Morris-Samotij and Saxton-Thomason (2015, containers)
  - Conlon, Gowers, Samotij, Schacht (2014, weaker in one sense/stronger in the other)

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- Hypergraphs

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Friedgut-Rödl-Schacht ('10) and Conlon-Gowers ('16)

For every k-hypergraph H there exists $C > 0$ such that if $p \geq Cn^{-\beta(H)}$ then

$$\lim_{n \to \infty} \Pr[G^{(k)}(n, p) \to H] = 1$$

$\beta(H)$ is chosen such that

- $p \leq cn^{-\beta(H)}$ (for some small constant $c > 0$)
  $\rightarrow$ most of the hyperedges do not belong to a copy of H
- $p \geq Cn^{-\beta(H)}$
  $\rightarrow$ each hyperedge belongs to many copies of H

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- Connection to asymmetric Ramsey properties
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  - Multiple exposure: Kohayakawa-Schacht-Spöhel ('14)
  - Containers: N.-Person-Steger-Škorić ('16+) (gives the hypergraph version)

Other approaches and generalisations

Proof (1-statement):

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Further applications – Folkman numbers

Given $r, k \in \mathbb{N}$, what is the smallest number $f(r,k)$ for which there exists a graph G on $f(r,k)$ vertices such that

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- Rödl-Ruciński-Schacht ('16+) – $f(k,r) \leq 2^{O(k^4 \log k + k^3 r \log r)}$ (similar to the presented proof)

Thank you!