Universality properties of random graphs

Rajko Nenadov

joint work with David Conlon, Asaf Ferber and Nemanja Škorić

Embedding – definition

Given graphs G and H, an injective function $f : V(H) \to V(G)$ is an embedding of H into G if

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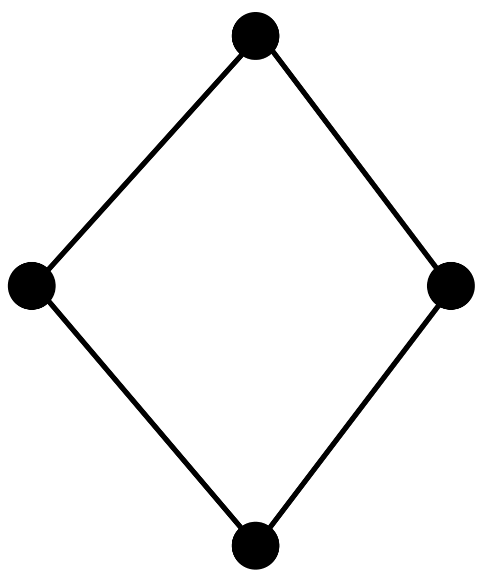
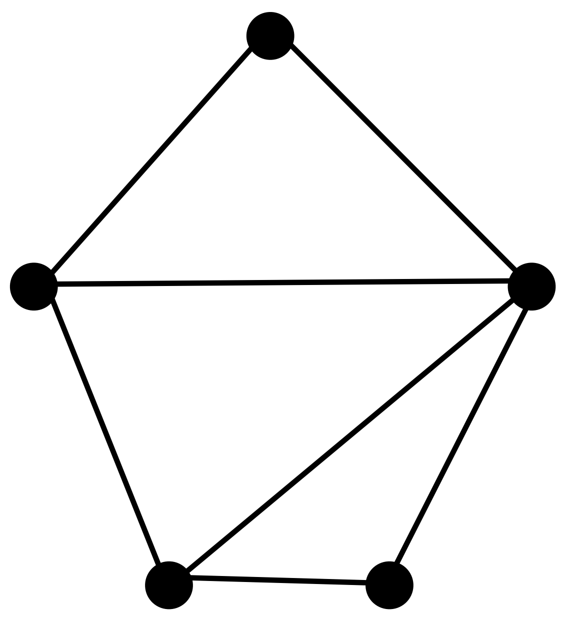


H



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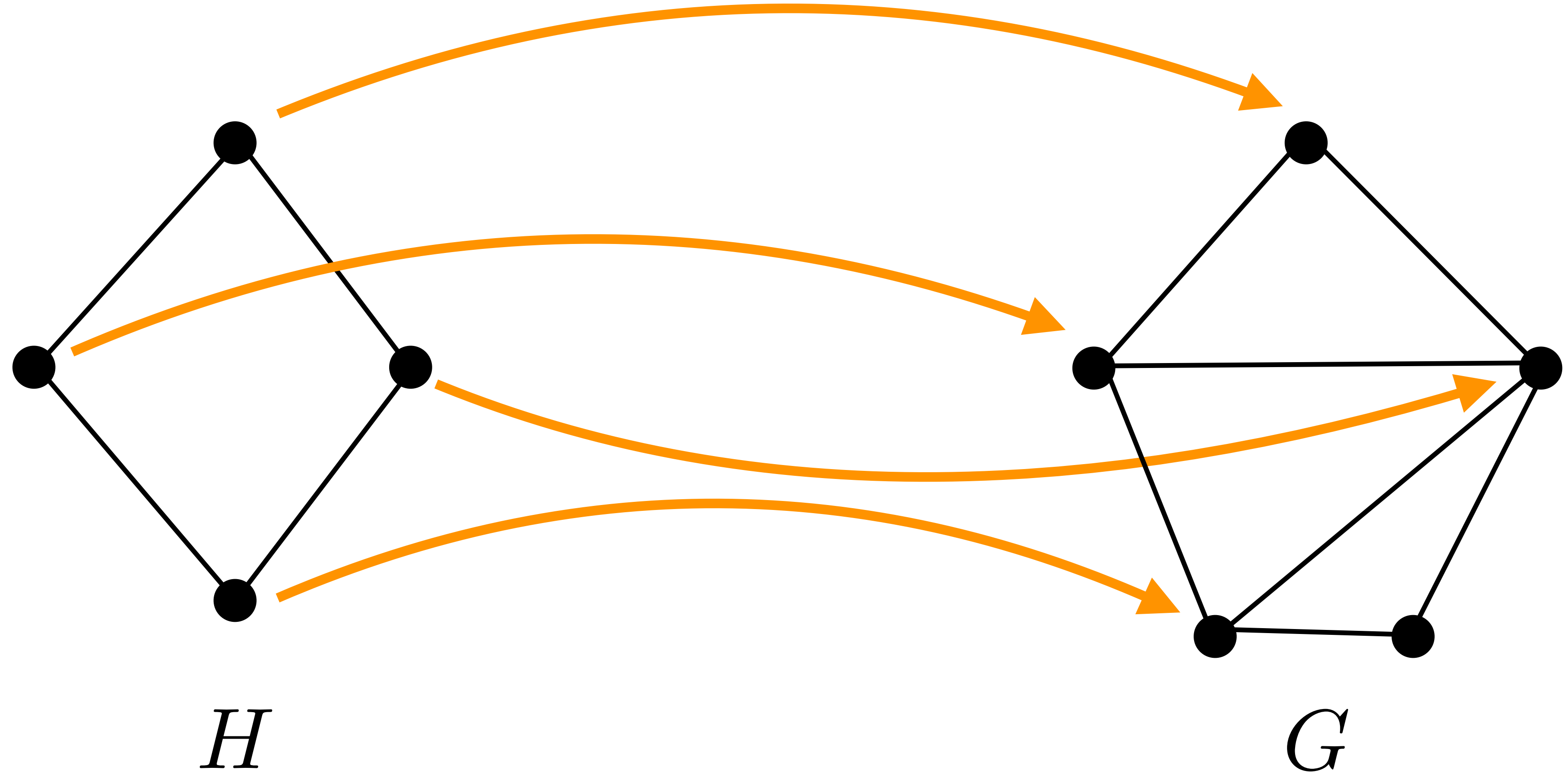

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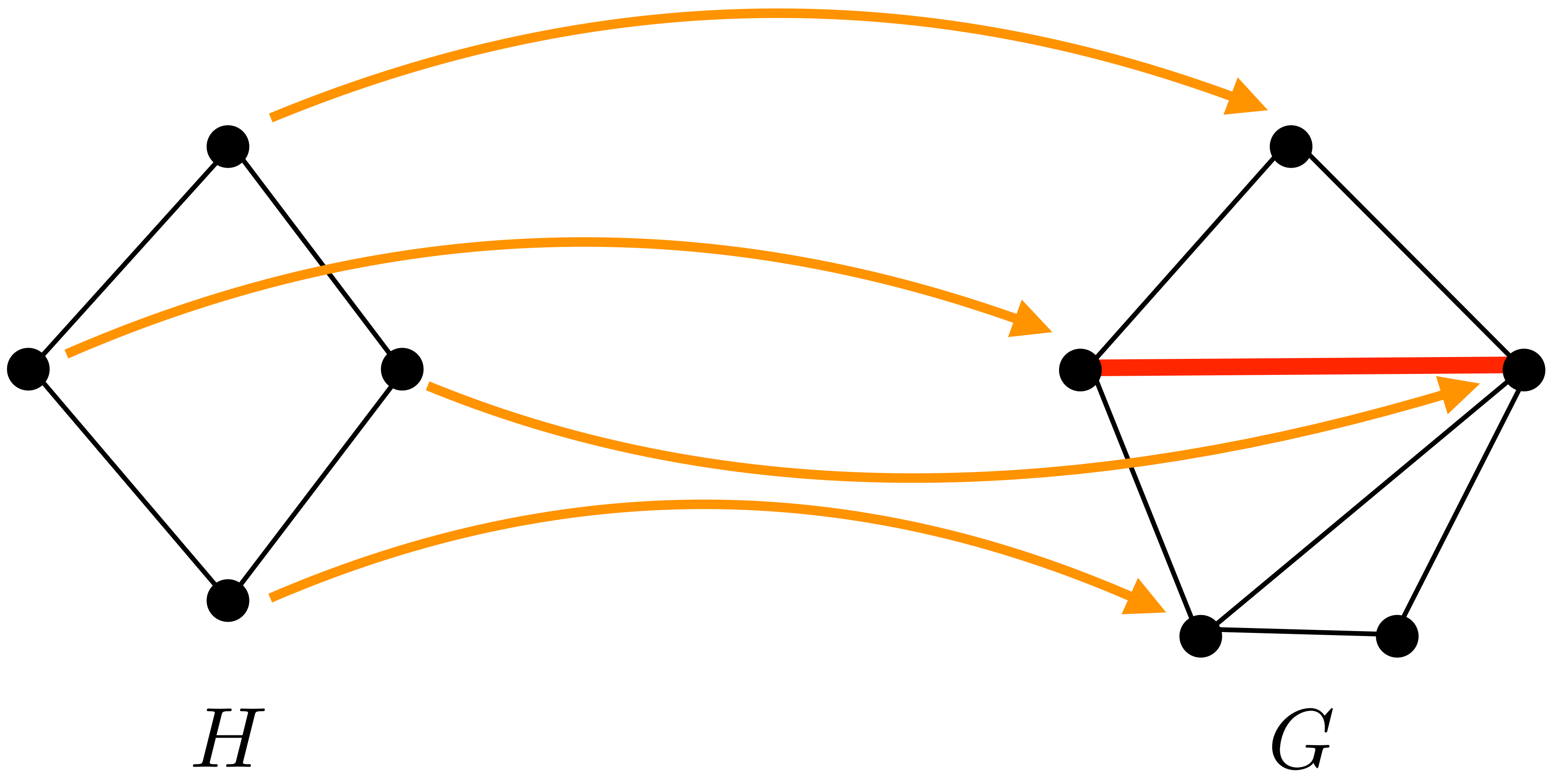

Not necessarily induced!

Random graphs

Binomial random graph $G(n, p)$

- graph on n vertices
- each edge present with probability p (independently)

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Theorem (Bollobás, Thomason '87) – threshold functions

For every monotone graph property $\mathcal{P}$ (connected, Hamiltonian, etc.) there exists $p_0 = p_0(n)$ such that

$$\lim_{n\to\infty} \Pr[G(n,p) \in \mathcal{P}] = \begin{cases} 1, & p \gg p_0(n) \\ 0, & p \ll p_0(n). \end{cases}$$

Embeddings in random graphs

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In this talk

we are interested in the case when H_n satisfies the following:

(i) $v(H_n) \leq (1 - \varepsilon)n$ ("almost-spanning")

(ii) $\Delta(H_n) \leq \Delta$ ("bounded-degree")

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Theorem (Alon, Füredi '91) – constructive proof

If H_n has maximum degree at most Δ, then

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Better bounds obtained by Riordan using the second-moment method; non-constructive!

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For which p does $G(n,p)$ simultaneously contain every $H_n \in \mathcal{H}_n$?

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$$\begin{aligned}\mathcal{H}_n(\varepsilon,\Delta) &= \{ \text{ all almost-spanning bounded-degree graphs } \} \\ &= \{H_n \ : \ v(H_n) \le (1-\varepsilon)n \text{ and } \Delta(H_n) \le \Delta\}\end{aligned}$$

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$$\not\Longrightarrow$$

$\lim_{n\to\infty}[G(n,p) \text{ is } \mathcal{H}_n\text{-universal}] = 1$

$$\Pr[G(n,p) \text{ is not } \mathcal{H}_n\text{-universal}] \le \overbrace{\sum_{H\in\mathcal{H}_n} \Pr[H_n \not\subset G(n,p)]}^{\text{useless if } \mathcal{H} \text{ is large}}$$

Universality in random graphs

Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi '00:

Theorem

For any constant $\Delta \in \mathbb{N}$ and $\varepsilon > 0$, if

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Remark: improved to $\varepsilon = 0$ (spanning) by Dellamonica, Kohayakawa, Rödl and Ruciński ('12) and Kim and Lee ('15)

A story about $\left(\frac{\log n}{n}\right)^{1/\Delta}$

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Fact

If $p \gg \left(\frac{\log n}{n}\right)^{1/\Delta}$ then $G(n,p)$ a.a.s. has the property that every set of $k \leq \Delta$ vertices has a common neighborhood of size $\approx np^k$.

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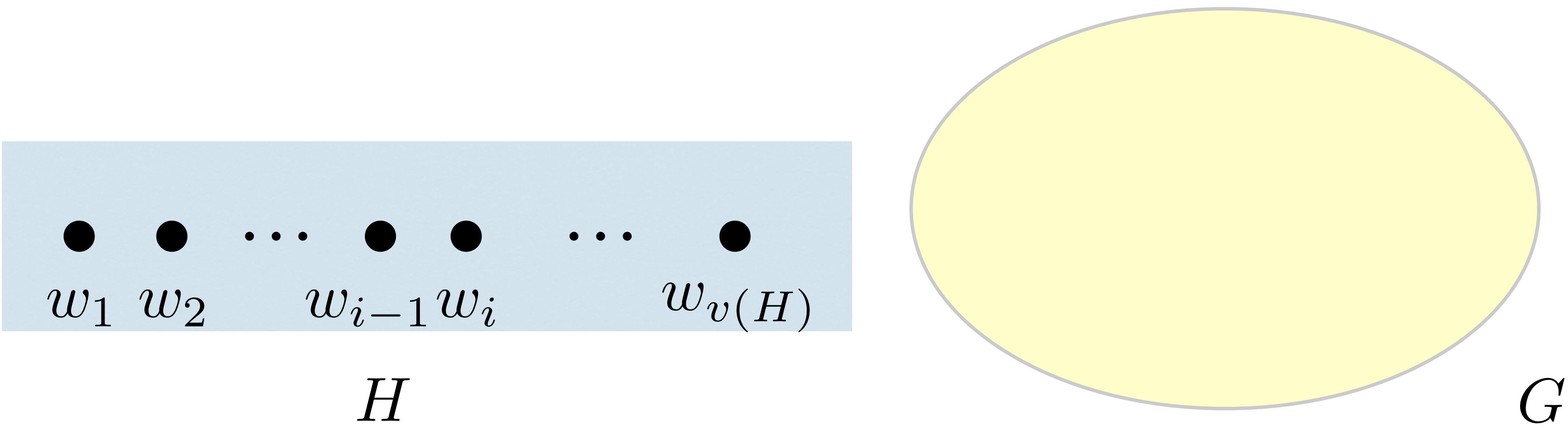

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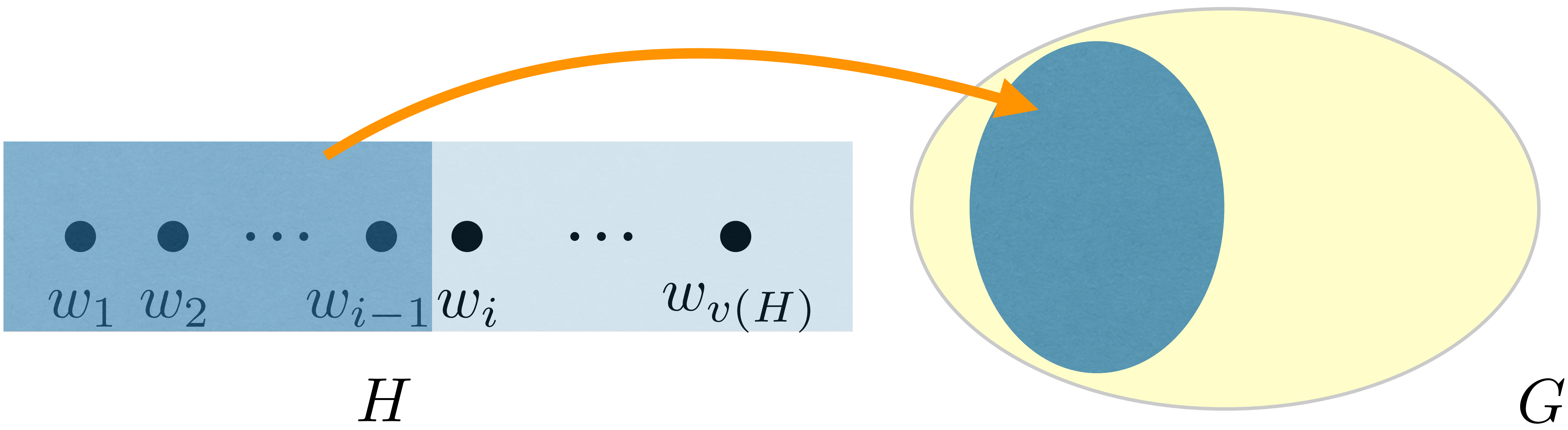

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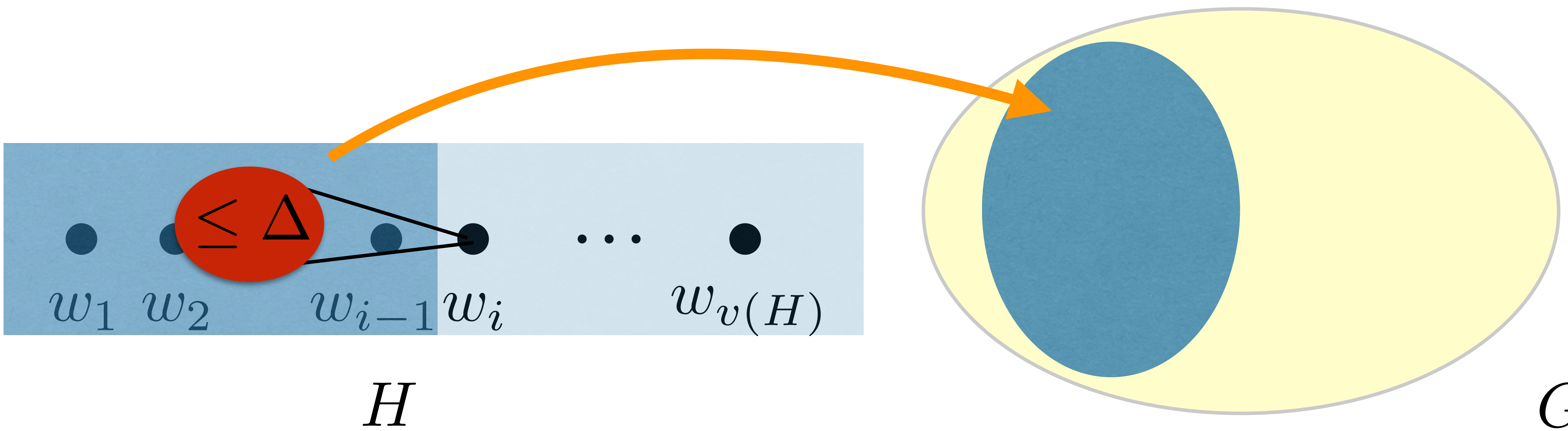

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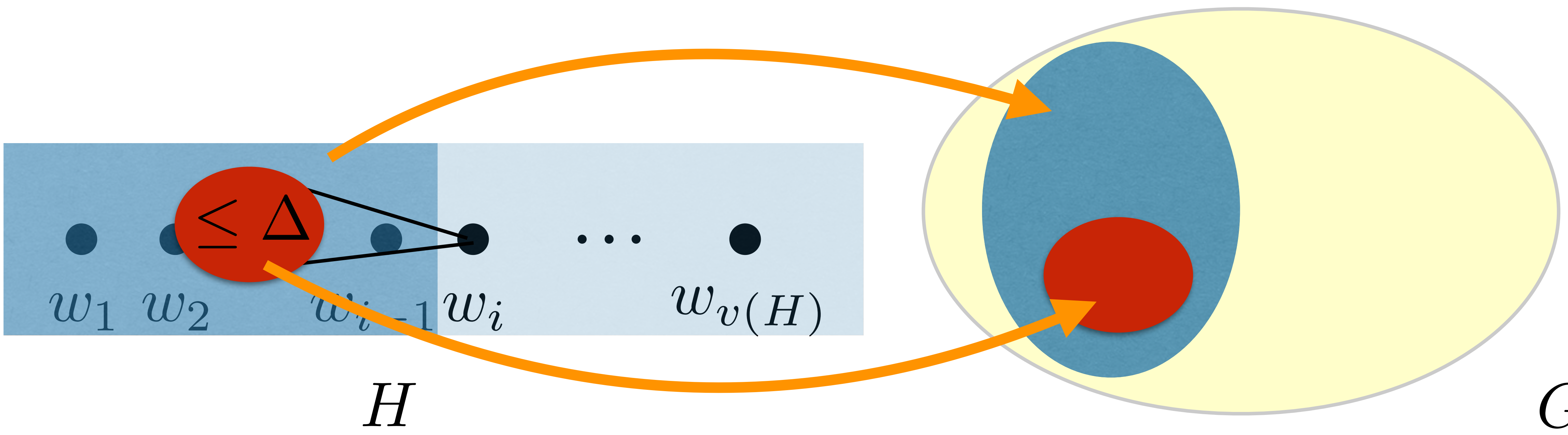

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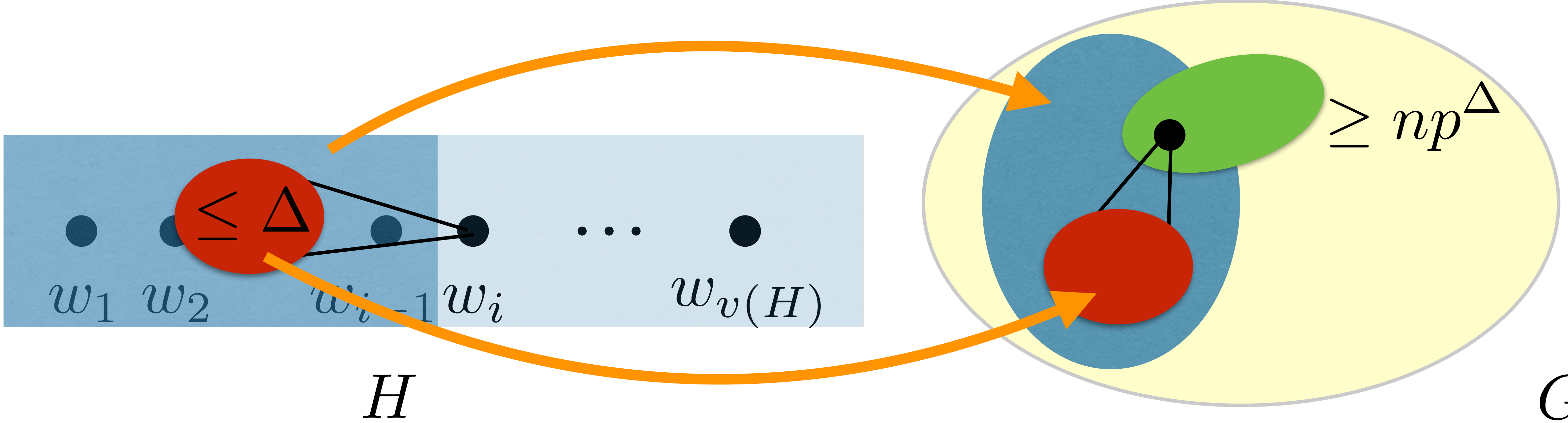

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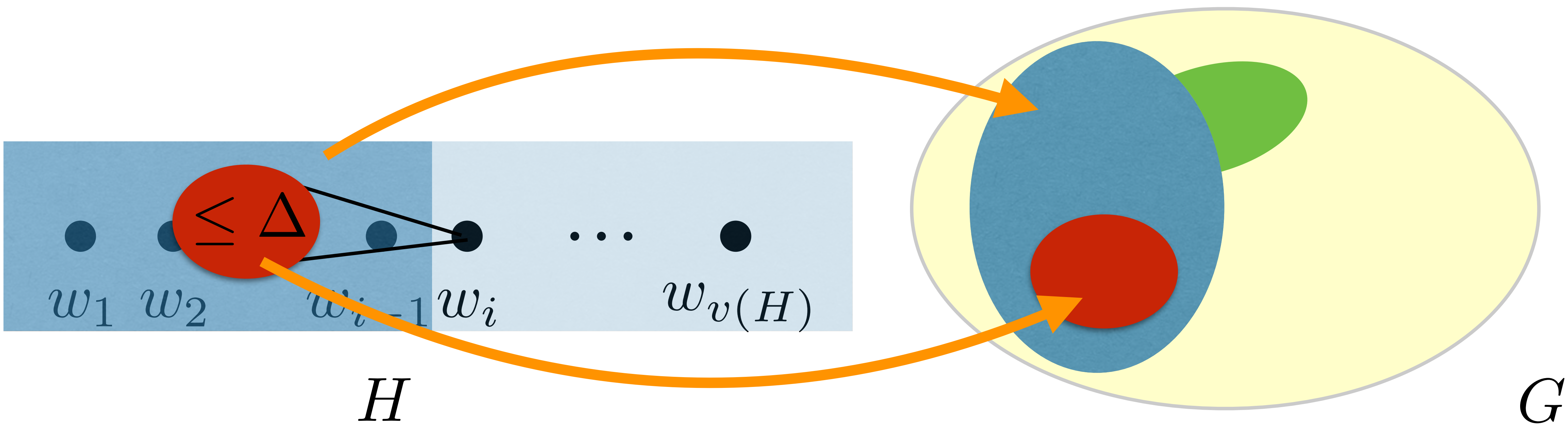

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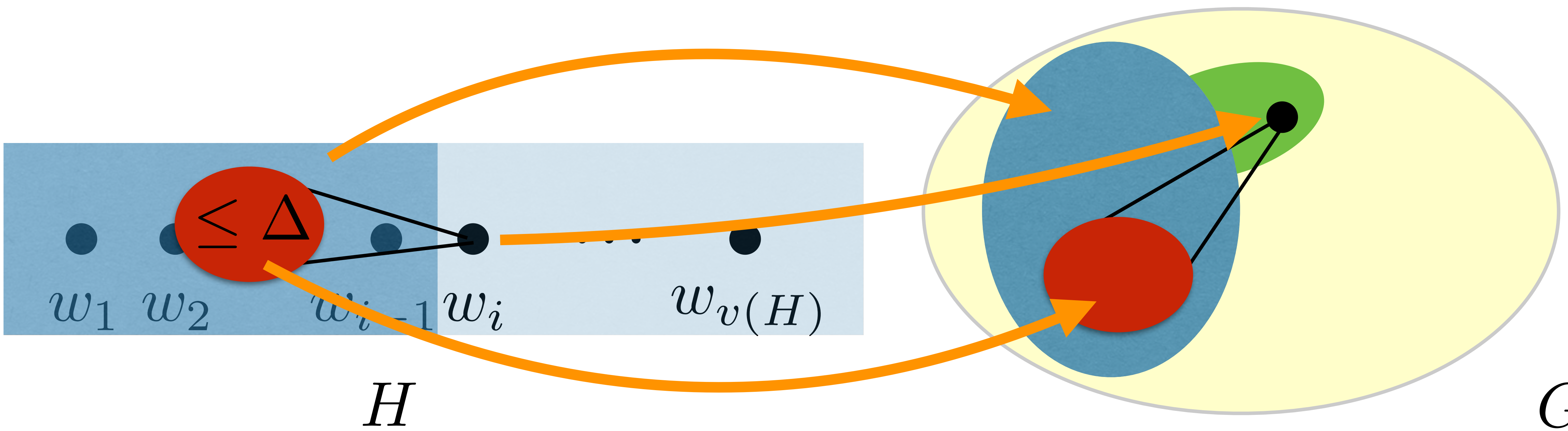

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All previous results in some way implement this approach.

Our result

Theorem [ACKRRSz '00]

For any constant $\Delta \in \mathbb{N}$ and $\varepsilon > 0$, if

$$p \gg \left(\frac{\log n}{n}\right)^{\frac{1}{\Delta}}$$

then $G(n,p)$ is a.a.s. $\mathcal{H}_n(\varepsilon, \Delta)$-universal.

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For any constant $\Delta \in \mathbb{N}$ ($\Delta \geq 3$) and $\varepsilon > 0$, if

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Remark

This is optimal (up to the logarithmic factor) for $\Delta = 3$:

- consider a disjoint union of $\frac{(1-\varepsilon)n}{4}$ copies of K_4

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Theorem [ACKRRSz '00]

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Theorem [Conlon, Ferber, N., Škorić '16]

If

$$p \gg \left(\frac{\log^3 n}{n}\right)^{\frac{1}{2 - 1/2}}$$

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Proof sketch

Embedding vertex-by-vertex – revisited

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If $p \gg \left(\frac{\log n}{n}\right)^{1/\Delta}$ then $G(n,p)$ a.a.s. has the property that every set of $k \le \Delta$ vertices has a common neighborhood of size $\approx np^k$.

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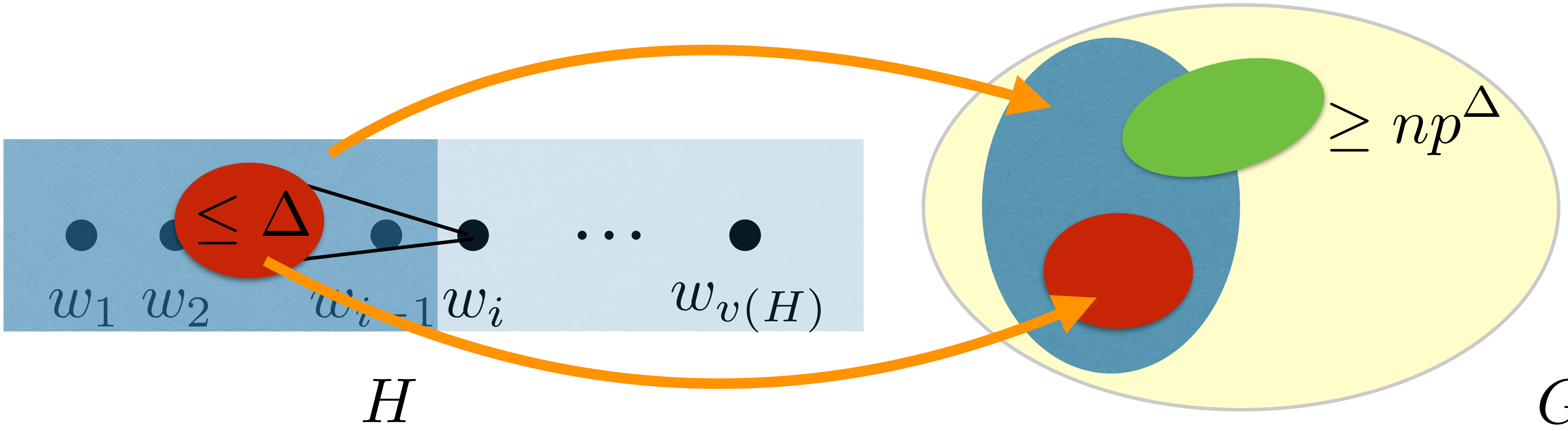

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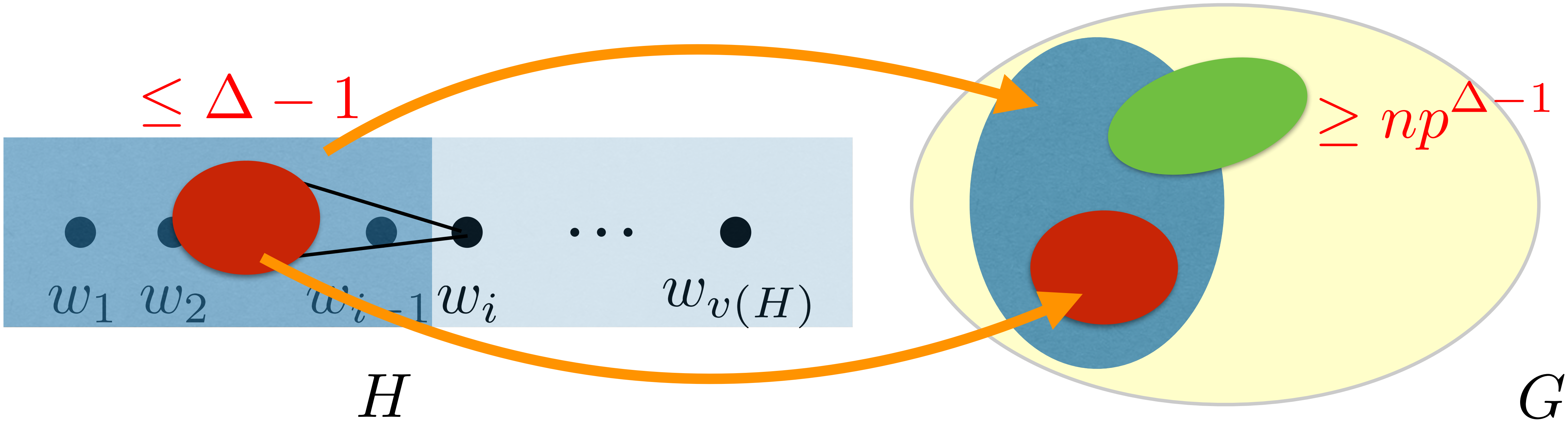
Assume: we can order the vertices of H such that each vertex has $\leq \Delta - 1$ left neighbors (i.e. it is $(\Delta - 1)$-degenerate)



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If $p \gg \left(\frac{\log n}{n}\right)^{1/(\Delta-1)}$ then $G(n,p)$ a.a.s. has the property that every set of $k \leq \Delta-1$ vertices has a common neighborhood $\approx np^k$.

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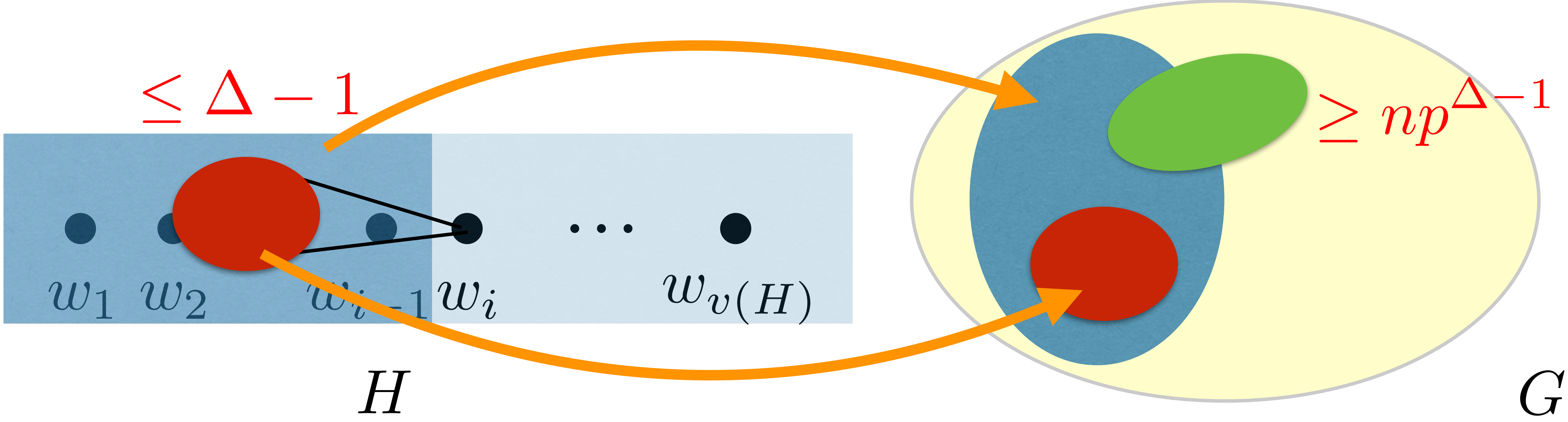

Universality for d-degenerate graphs

This intuition can be turned into a proof!

Theorem

For any constants $d, \Delta \in \mathbb{N}$ and $\varepsilon > 0$, if

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The case $d = 1$ (trees) was considered by Alon, Krivelevich and Sudakov ('07) and independently by Balogh, Csaba, Pei and Samotij ('10)

Strategy

Preparation: split $G \sim G(n,p)$ into two parts such that

(a) $|V_1| = (1-\varepsilon/2)n$ and $|V_2| = \varepsilon n/2$

(b) $G[V_1]$ is $D_n(\Delta - 1)$-universal

(c) to be discussed

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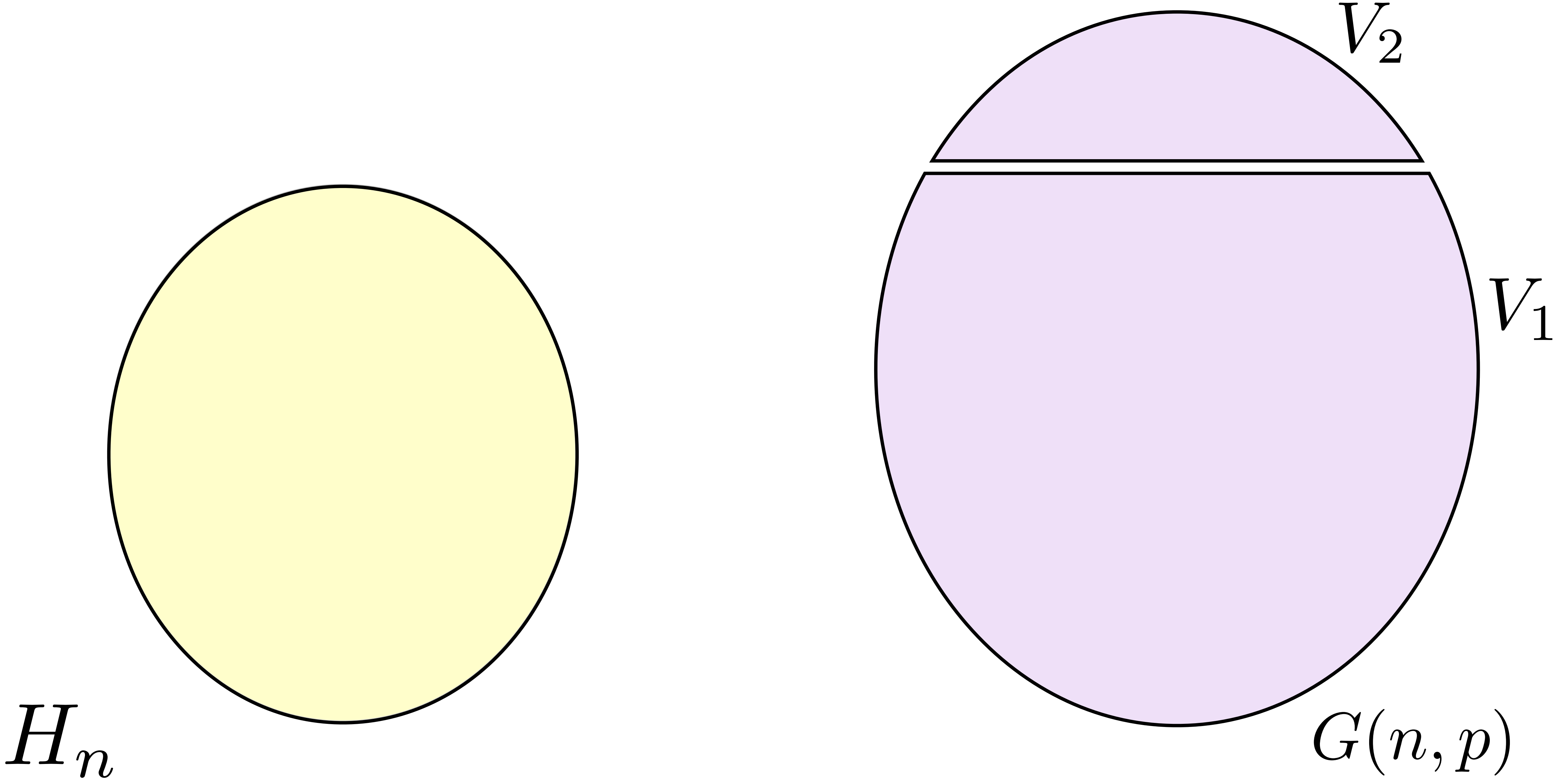
  - (b) S has a "nice" structure
- (ii) embed $H_n - S$ into $G[V_1]$
- (iii) somehow embed the vertices from S into V_2 (not vertex-by-vertex!!)

Strategy (assume H_n is connected)



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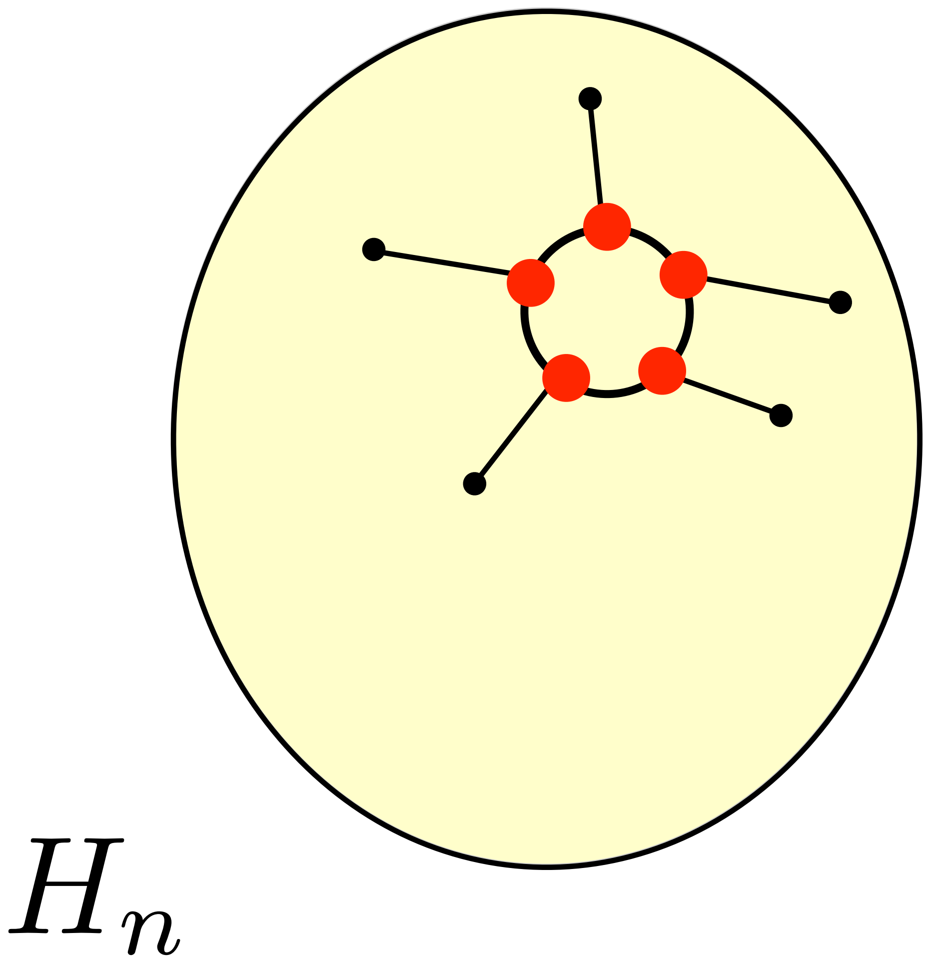

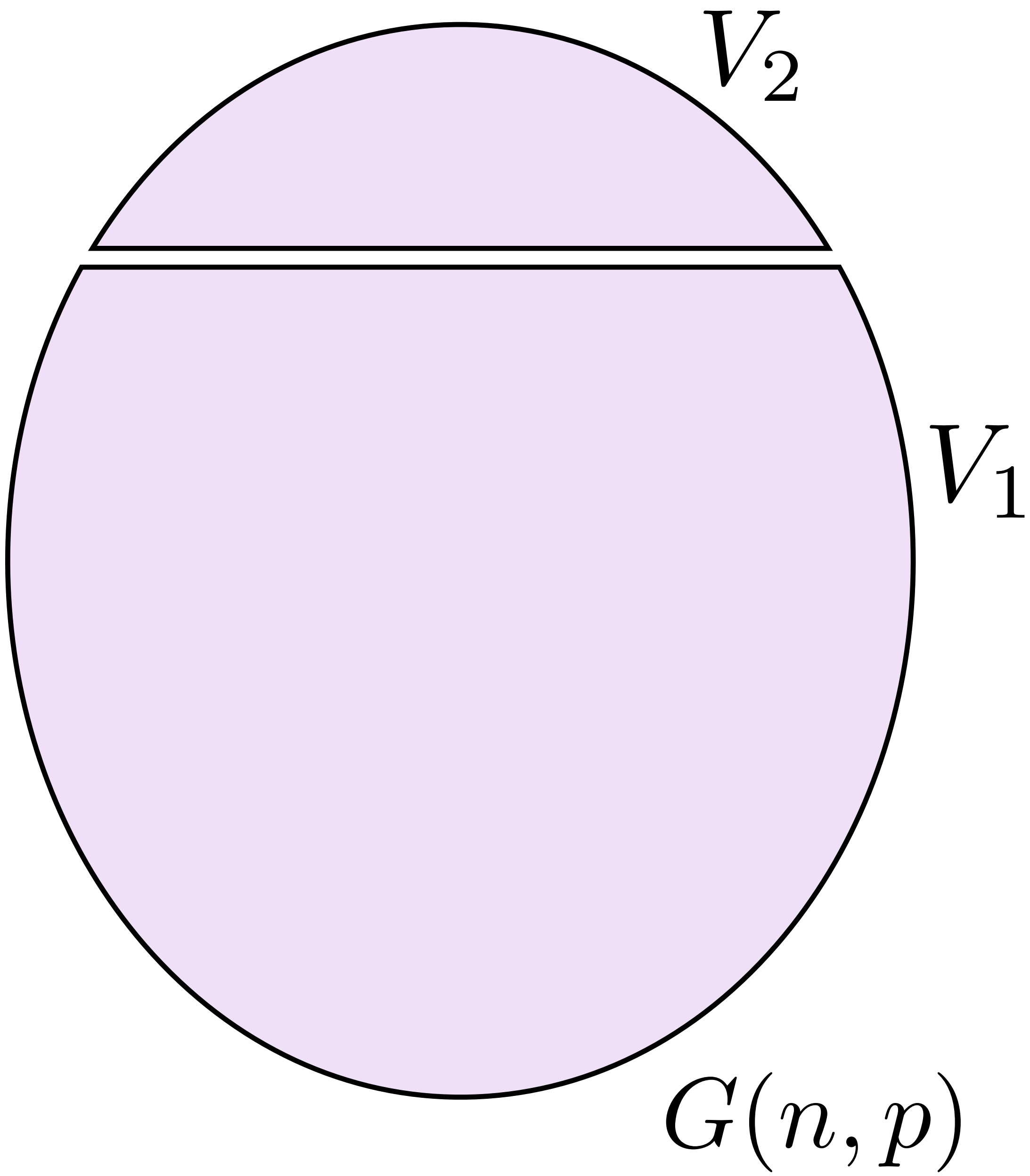

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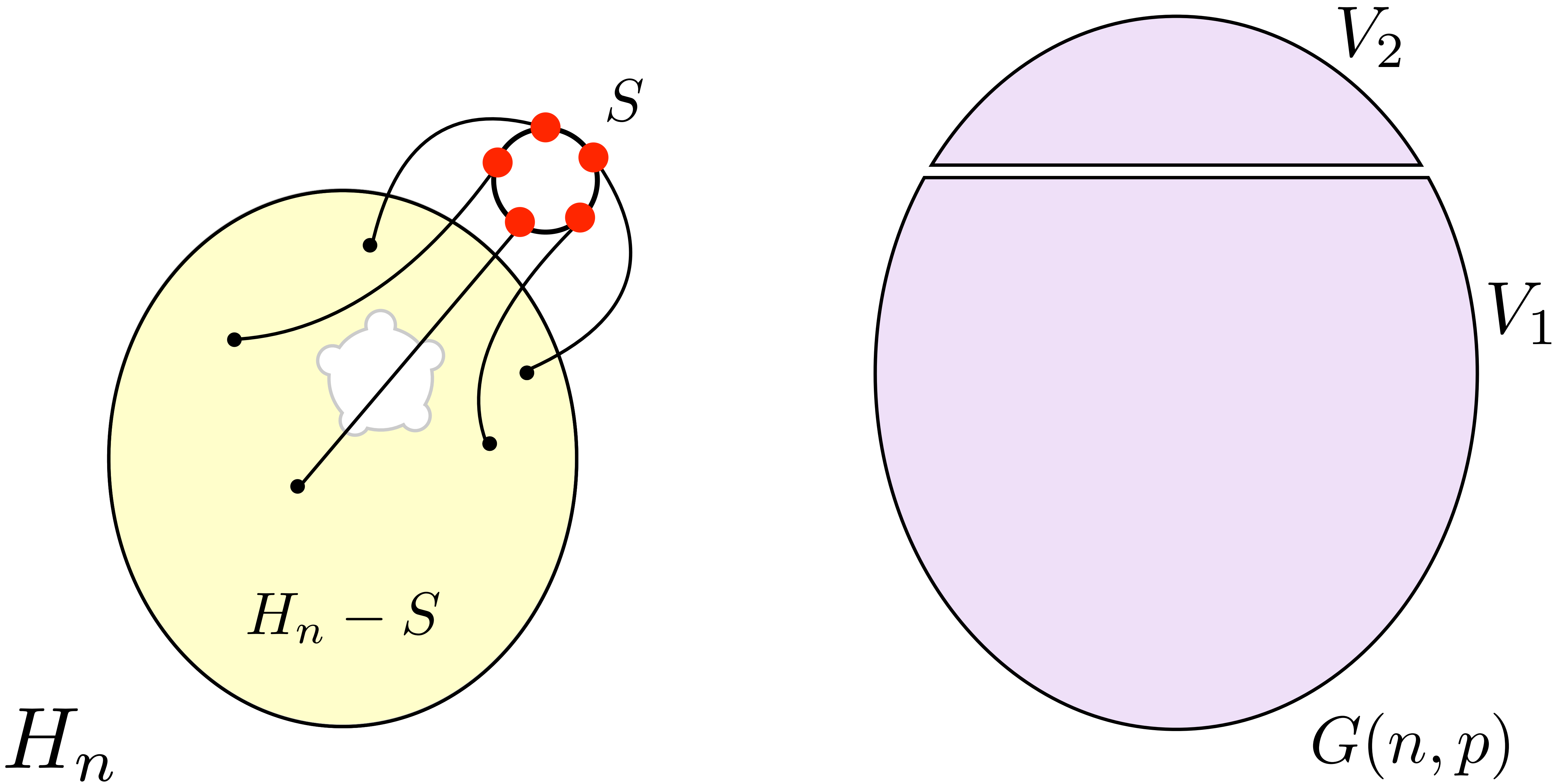

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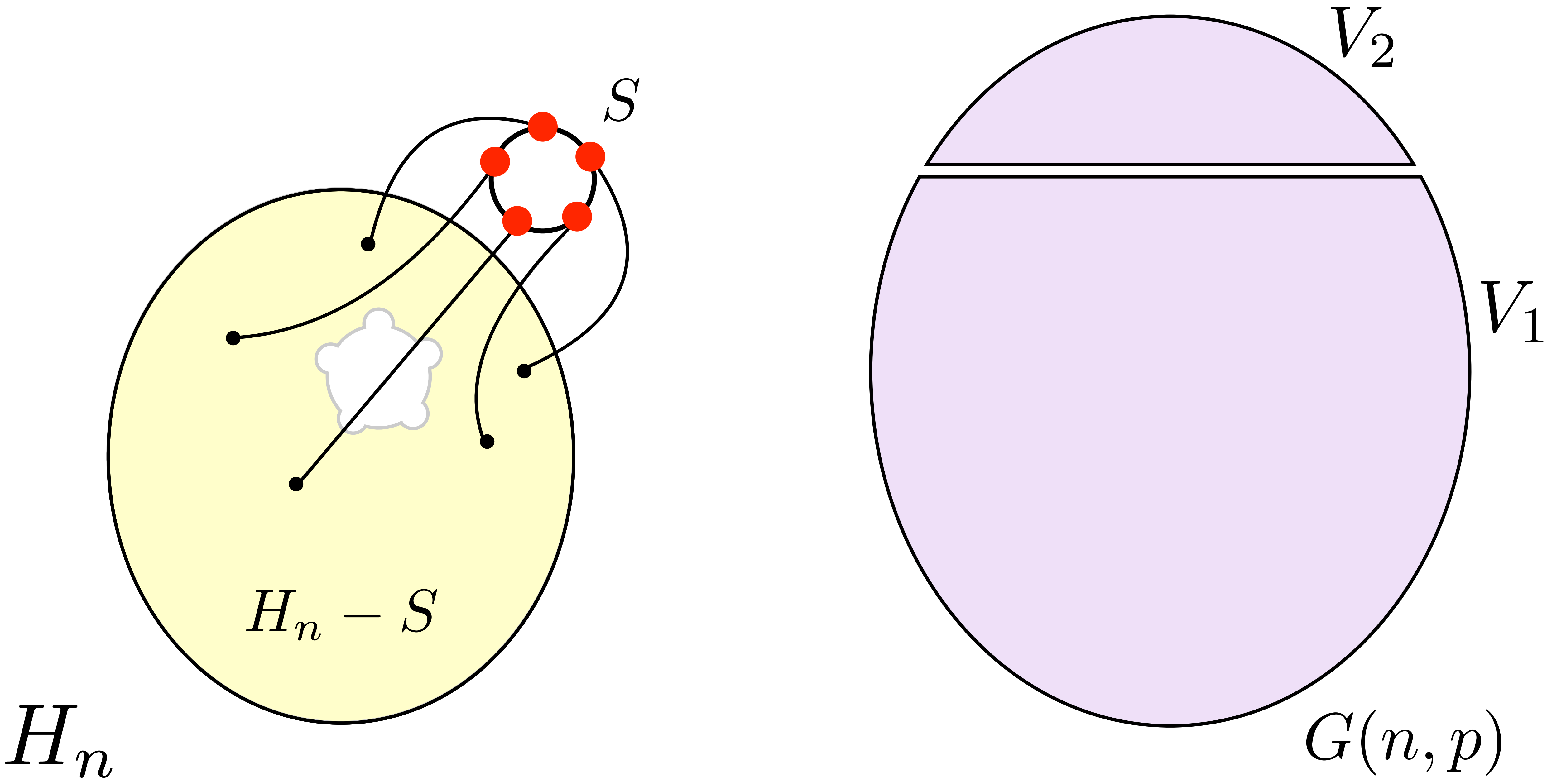

Step (ii): embed $H_n - S$ into $G[V_1]$

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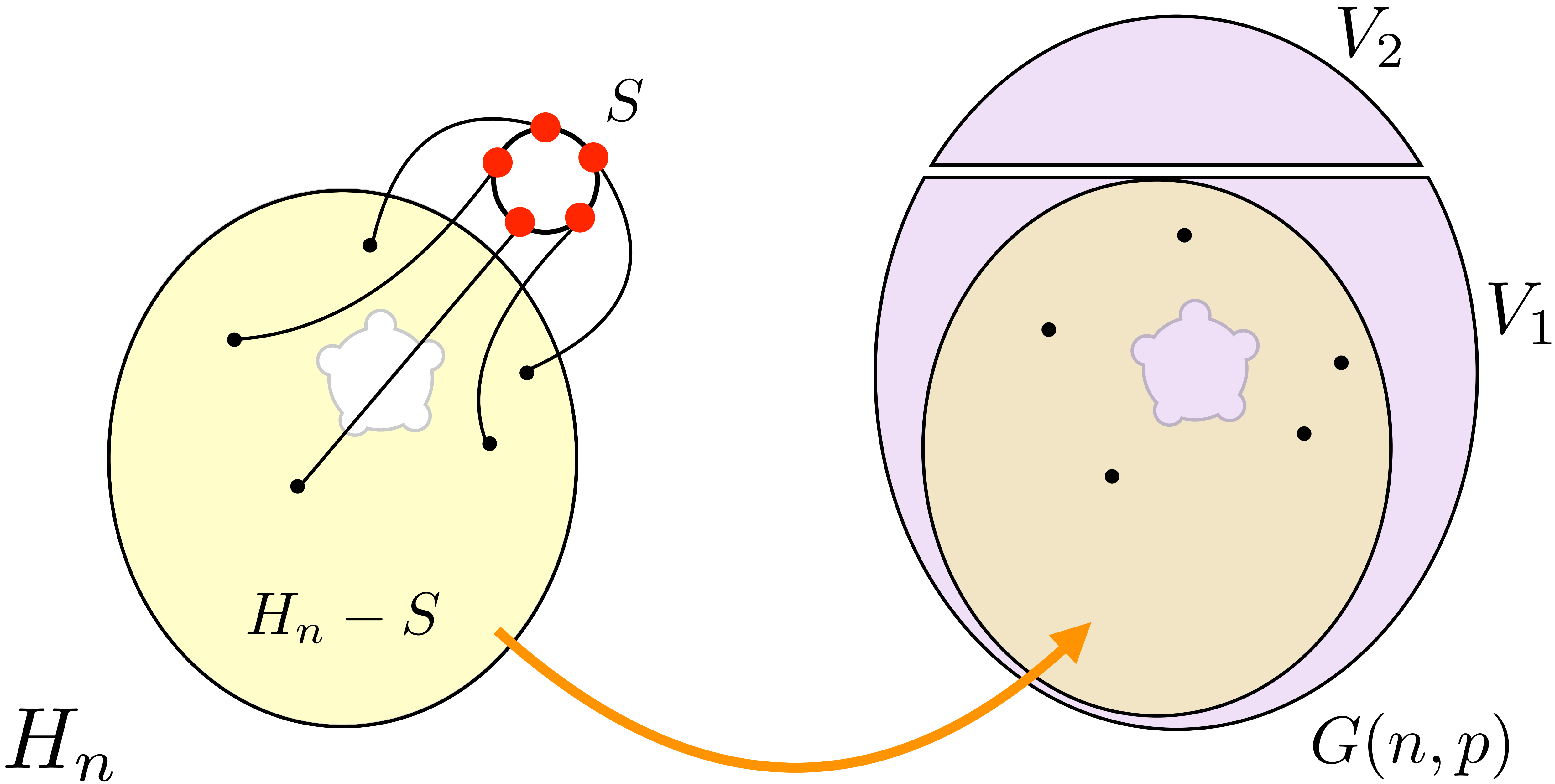

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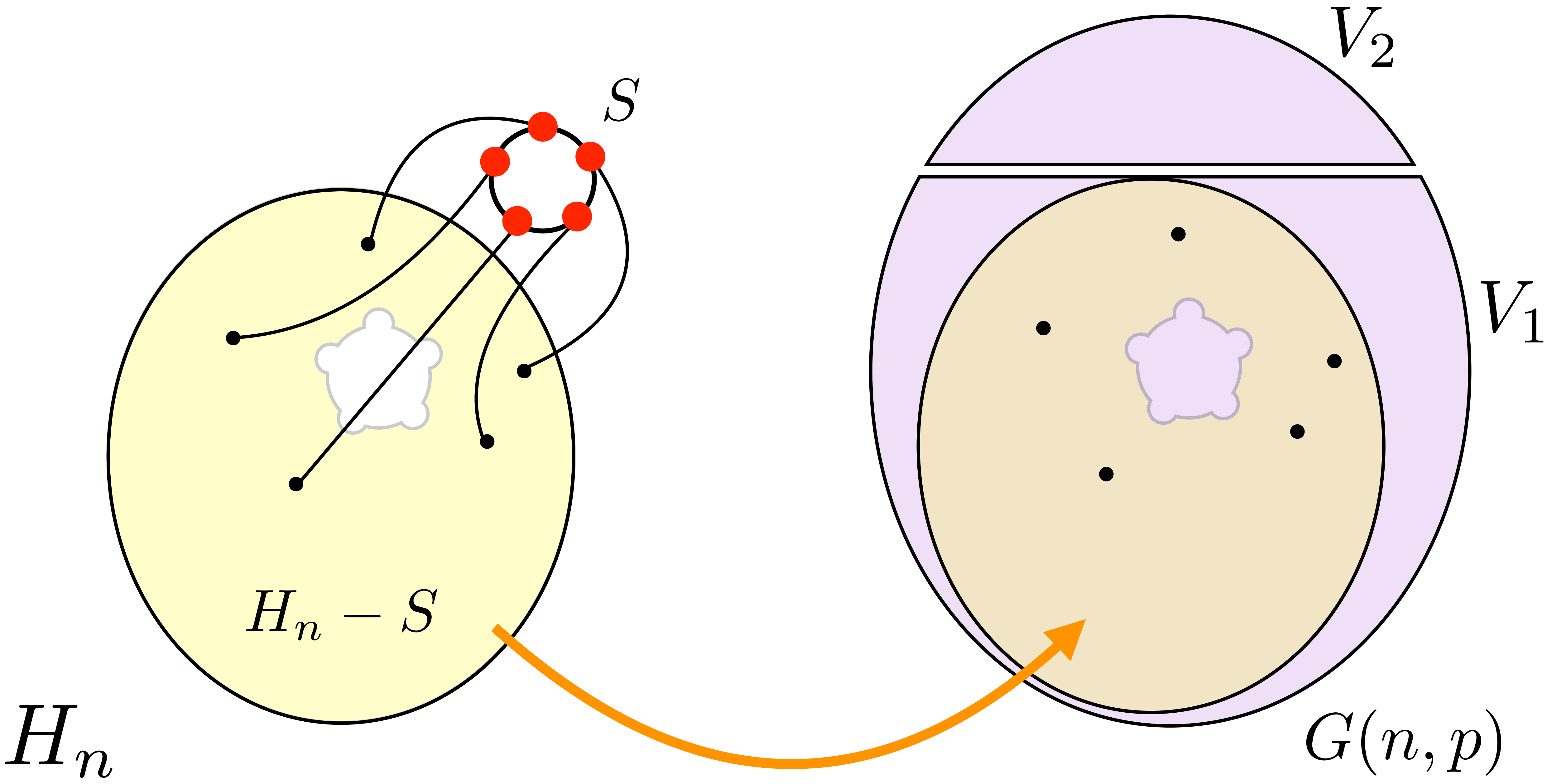

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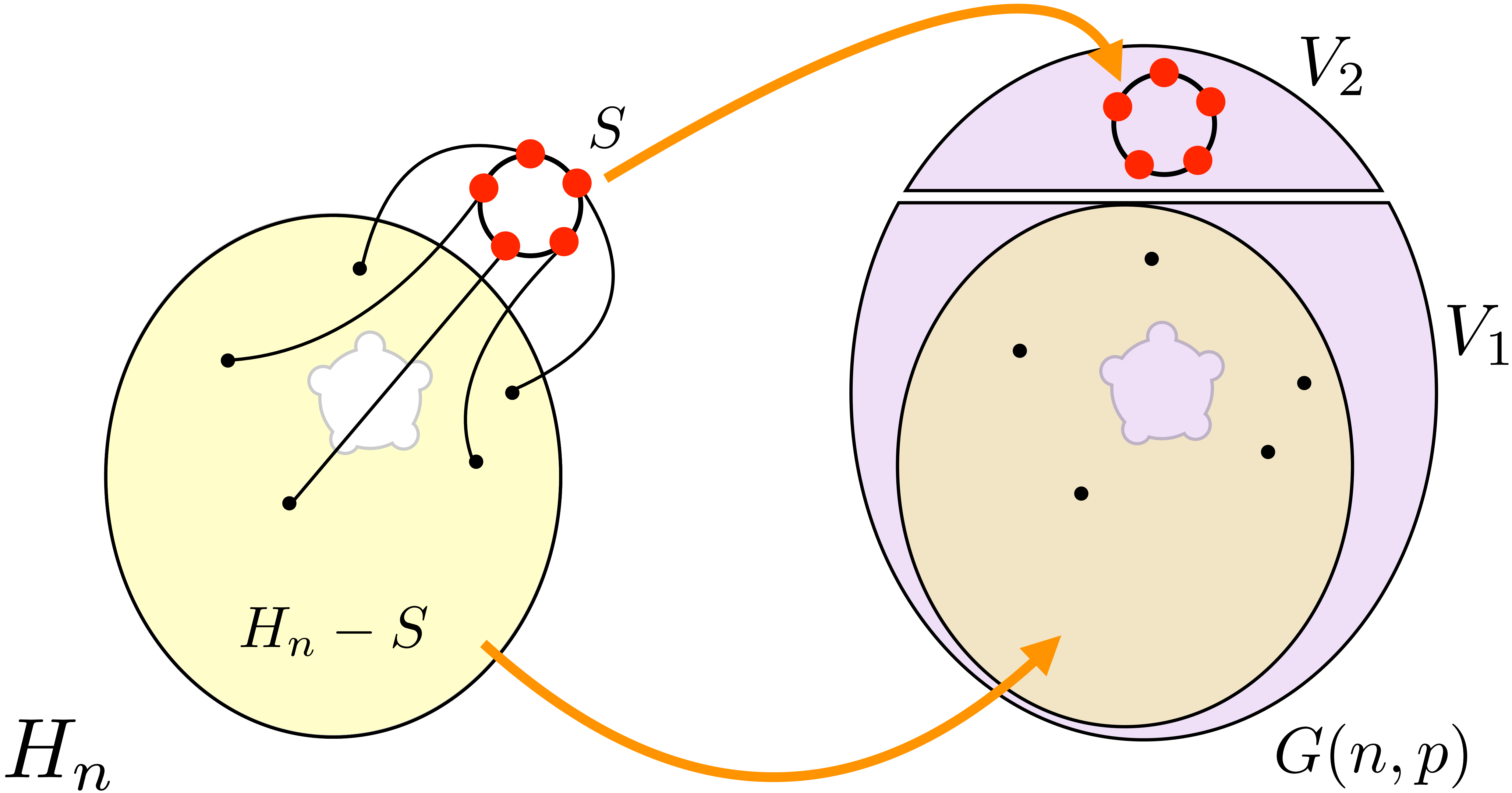

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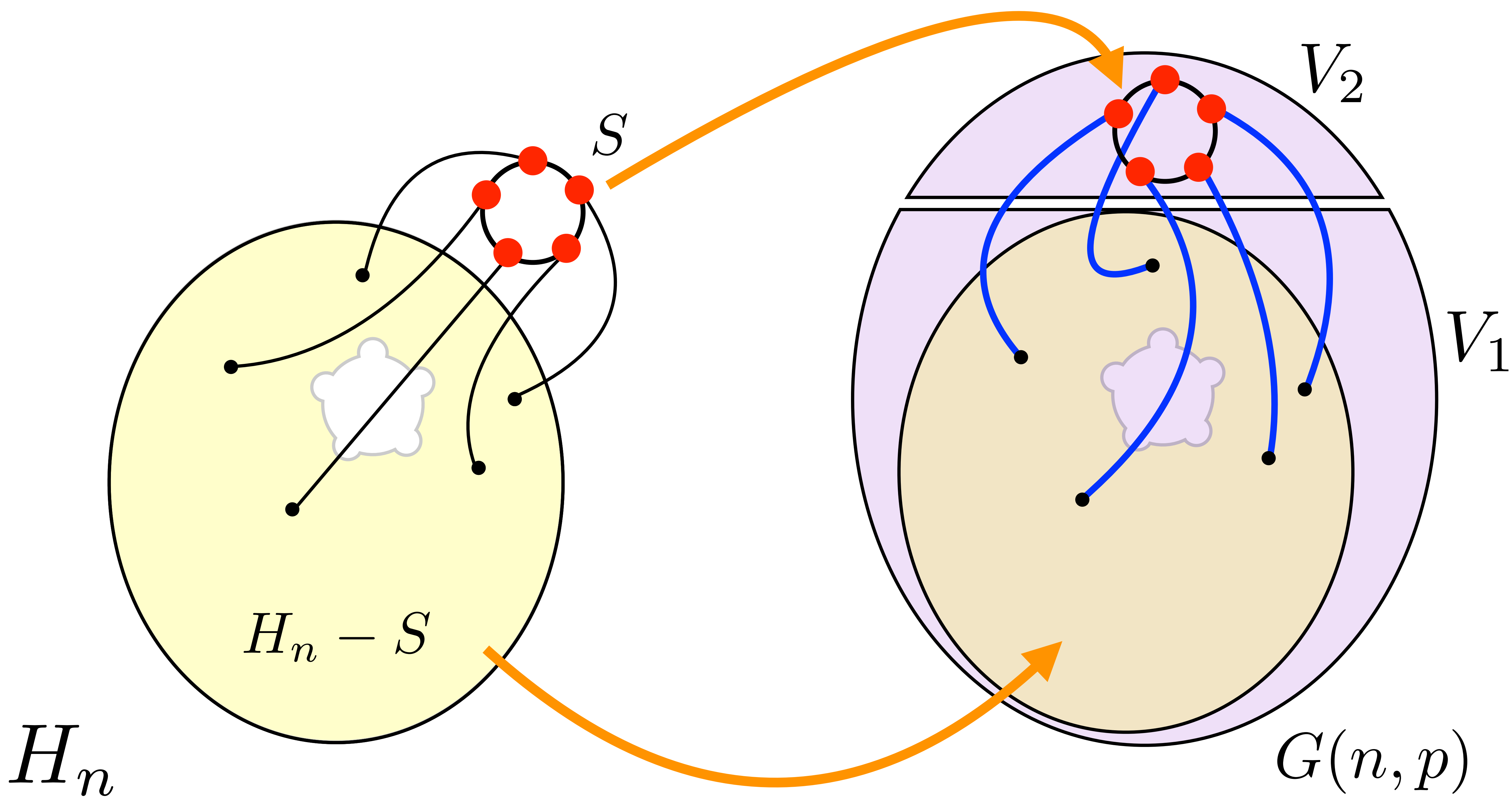

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Conclusion

Theorem [Conlon, Ferber, N., Škorić '16]

For any constant $\Delta \in \mathbb{N}$ ($\Delta \geq 3$) and $\varepsilon > 0$, if

$$p \gg \left(\frac{\log^3 n}{n}\right)^{\frac{1}{\Delta-1}}$$

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Applications

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1. Existence of 'sparse' universal graphs

- $G(n, p)$ has roughly $n^2 p$ edges
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Theorem (Alon, Capalbo ’07)

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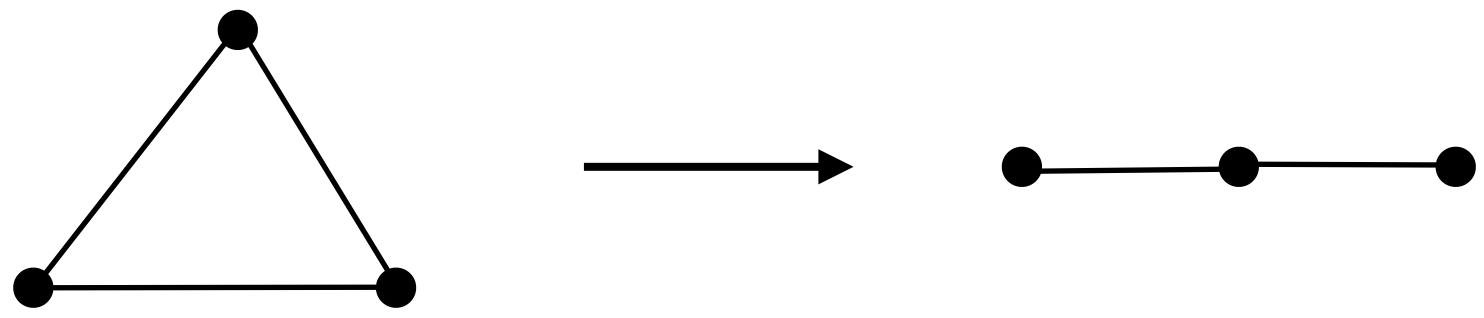

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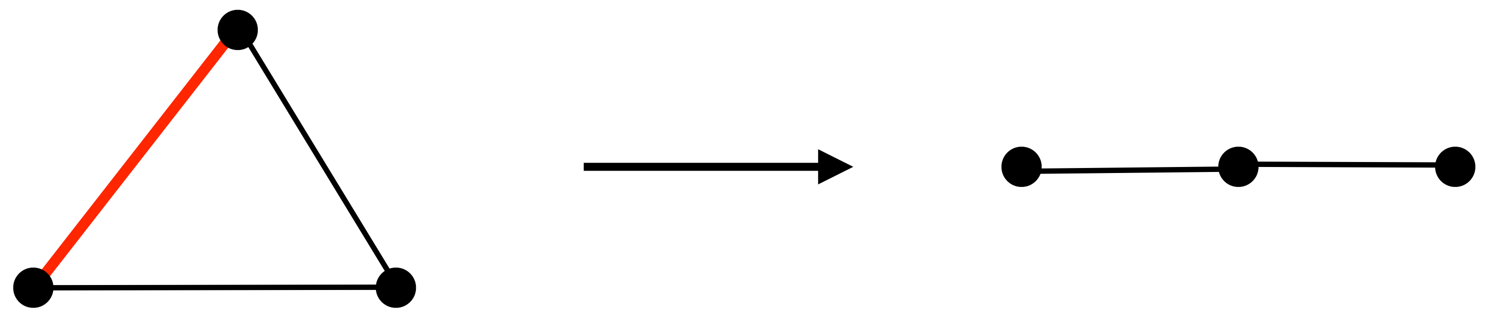

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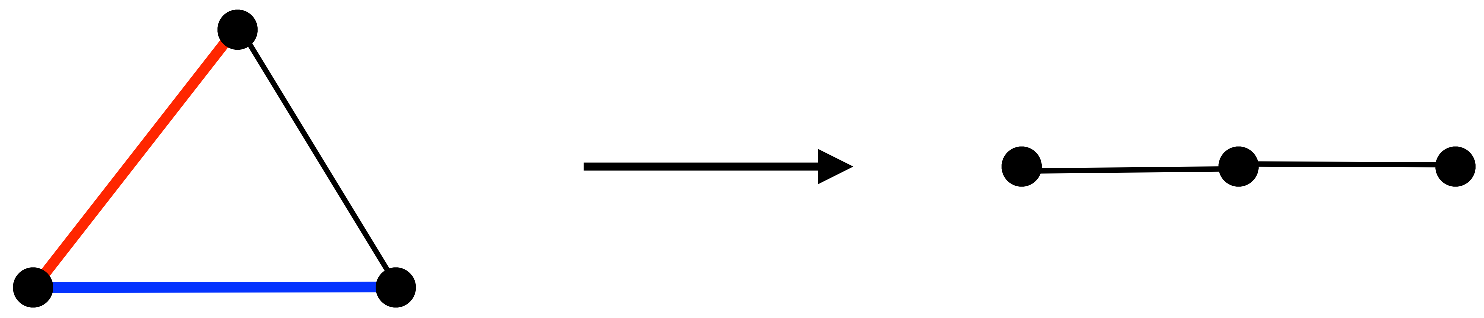

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$$r(H) = \min\{N \in \mathbb{N} \ : \ K_N \to H\}$$
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For every $\Delta \in \mathbb{N}$ there exists C_Δ such that if H is a graph with n vertices and maximum degree Δ then

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Theorem (Kohayakawa, Rödl, Schacht and Szemerédi '11)

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Theorem (Conlon, N. '16+)

If H is additionally triangle-free then

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$\mathcal{T}_n(\Delta)$ = family of all triangle-free graphs with n vertices and maximum degree at most Δ

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Proof implements previously described strategy in a more difficult setting of sparse regularity.

Thank you!