Bichromatic Lines in the Plane

Michael Payne

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Bichromatic Beck's Theorem

Theorem (Beck '83) Every set of n points in the plane with at most ℓ collinear determines at least $cn(n-\ell)$ lines for some $c > 0$.

- Payne-Wood: $c > \frac{1}{93}$

Theorem (Theran '11) Let P be a set of n red and n blue points with at most ℓ collinear. Then P determines at least $c'n(2n-\ell)$ bichromatic lines for some $c' > 0$.

▷ Theran's result is implied by:

Theorem (Pach–Pinchasi '00) Let P be a set of n red and n blue points, not all collinear. Then the number of bichromatic lines is at least $1/10$ times the total number of lines determined by P.

▷ Purdy–Smith ('10) improve this to $1/4$ of the total.

▷ Using our constant, one gets $c' > \frac{1}{186}$ in Theran's theorem.

Bivisibility Graphs

- A visibility graph of a point set has an edge between two points if there is no other point on the line segment connecting them.
- Big-line-big clique conjecture (Kára-Pór-Wood)
  "Every sufficiently large visibility graph has a k-clique or ℓ collinear points."
- Given a set of red and blue points, the bivisibility graph has an edge between each red-blue pair if there is no other point between them.

Theorem (Kővari-Sós-Turán '54) Fix an integer t, and let G be a bipartite graph with n vertices in each part. If G has no $K_{t,t}$ subgraph then the number of edges is $O(n^{2-1/t})$.

Corollary For all $t, \ell \geq 2$, there exists N s.t. if $n > N$, then every bivisibility graph on n red and n blue vertices has a $K_{t,t}$ subgraph or ℓ collinear points.

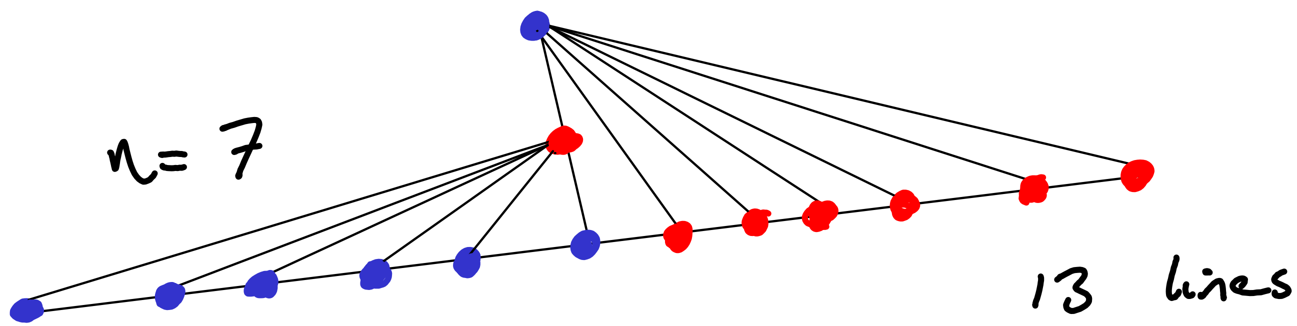
Proof: If G has no $K_{t,t}$, then by K-S-T G has $o(n^2)$ edges, but by Theran it has $\Omega(n^2)$ edges.

Kleitman-Pinchasi Conjecture

▷ We saw lower bounds on # bichromatic lines that depend on ℓ.

<u>Conjecture</u> (Kleitman-Pinchasi '03)
P a set of n red and n or $n-1$ blue points. If neither colour class is collinear, then P determines at least $|P|-1$ bichromatic lines.

Tight:



Motivation

Theorem (de Bruijn - Erdős '48)

$|S| = n$ and $S_i \subseteq S$ $i = 1, \ldots, k$ such that each pair in S is in exactly one S_i.
Then $S = S_i$ for some i or $k \geq n$.

- ▷ If $S \subset \mathbb{R}^2$ this follows from Sylvester-Gallai by induction.

Theorem (Motzkin '67) Every non-collinear set of red and blue points in $\mathbb{R}^2$ determines a monochromatic line.

- ▷ K-P Conj + Motzkin $\Rightarrow$ de B-E. in $\mathbb{R}^2$.

Theorem (Kleitman-Pinchasi '03)

P a set of n red and n or $n-1$ blue points. If neither colour class is collinear, then P determines at least $|P|-3$ bichromatic lines.

- ▷ Purdy and Smith ('10) proved the K-P Conjecture for $n \geq 79$.
- ▷ I improved this to $n \geq 10$, and also to give $|P|-2$ bichromatic lines for all n.
- ▷ Use Kleitman-Pinchasi proof method with improvements.

METHOD OF K-P PROOF.

- K-P use proof by induction. They establish an inductive step that works for Conj & Thm and for $n \geqslant 20$.
- The base cases $n < 19$ are checked by computer using linear programming.
- LP seeks to minimise # bichromatic lines under combinatorial & geometric constraints.
- I added more constraints and also eliminated a certain case by direct argument.

▷ Let $s_{i,j}$ be # lines with exactly i red and j blue points from P.

Combinatorial constraints: (n red, n blue case)

Red pairs $\sum \binom{i}{2} s_{i,j} = \binom{n}{2}$ Blue pairs $\sum \binom{j}{2} s_{i,j} = \binom{n}{2}$

Bichromatic Pairs $\sum ij\, s_{i,j} = n^2$

Geometric constraints:

▷ $\sum (i+j-3) s_{i,j} \leq -3$ Melchior's inequality ('41).

▷ If $i+j \geq n$ then $s_{i,j} = 0$. Lemma by K-P.

Also, $s_{i,j} \in \mathbb{N}_0$ (so really integer programming).

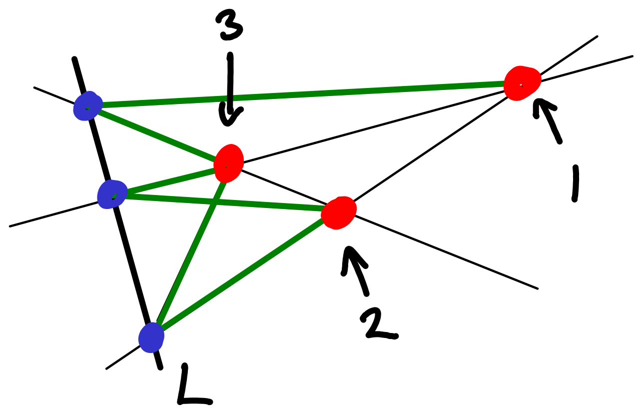
More Constraints (n red, n blue case)

- Observation: n blue points. Each red on $\leq \lfloor \frac{n}{2} \rfloor$ lines with 2 or more blues. Hence: $\sum_{j \geq 2} i \, s_{i,j} \leq n \lfloor n/2 \rfloor$

- Hirzebruch Inequality ('86):
  Monochromatic version: $s_2 + \frac{3}{4} s_3 \geq n + \sum_{i \geq 5} (2i-9) s_i$

- Lemma: If $i^2 + i + j^2 + j \geq 4n-2$ then $s_{i,j} = 0$.

Idea:



Integer Program Results.

Min found $-$ $(|P| - 1)$

n	(n, n)	$(n, n-1)$	n	(n, n)	$(n, n-1)$
4	1		12	3	3
5	0	-1	13	5	4
6	$\boxed{-2}$	-1	14	7	6
7	0	-1	15	9	8
8	0	-1	16	11	10
9	0	-1	17	13	12
→ 10	1	0	18	16	14
11	2	1	19	19	17

IMPLICATIONS

- K-P conjecture true for $n \geq 10$
- Only one case with possibly $|P|-3$:

$(6, 6)$ LP gives: $s_{2,2} = 9$ $s_{0,2} = 6$ $s_{2,0} = 6$.

Puzzle: Is this possible??

- If not, then $s_{2,2} \leq 8$. Adding this to the LP gives at least 10 bichrom. lines.

Hence $\geq |P|-2$ bichrom. lines for all n.

PUZZLES

$(6, 6)$: $s_{2,2} = 9$ $s_{0,2} = 6$ $s_{2,0} = 6$.

① Is it possible to arrange 6 blue and 6 red points so that each red is on 3 lines with 2 blue points?

② Is it possible to arrange 6 blue and 6 red points so that there are 9 lines with 2 of each colour??

Audience solutions?

Real Projective Plane

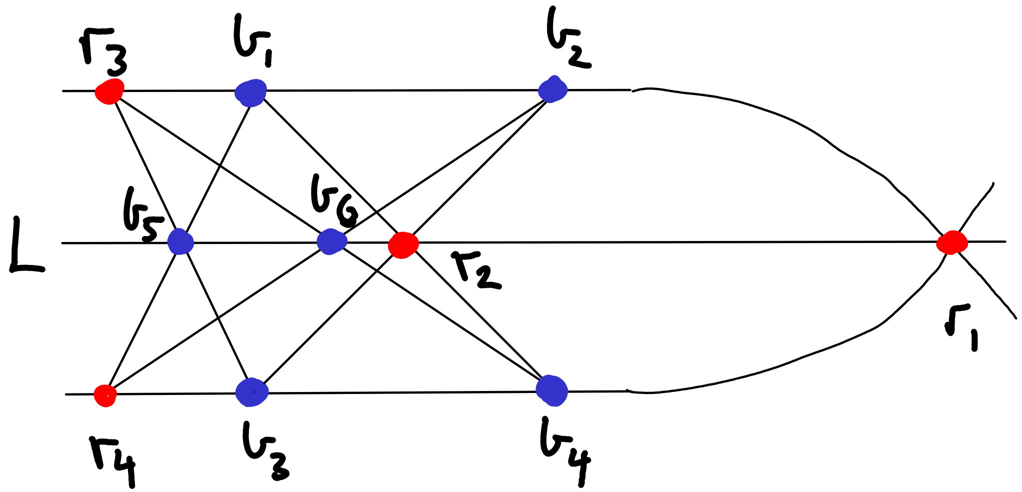
Fact: Let V, W be real projective planes. Given $v_1, \dots, v_4 \in V$ in general position, and $w_1, \dots, w_4 \in W$ in general position, there exists a collineation mapping $v_i \mapsto w_i$.

Proof idea:

- Each point is an equivalence class $[p]$ for some $p \in \mathbb{R}^3$. (say $v_i = [v_i]$ etc).
- Can map $v_i \mapsto c_i w_i$ for $i = 1, 2, 3$ and any choice of $c_i \neq 0$.
- Use these c_i to ensure $v_4 \mapsto c_4 w_4$ for some $c_4 \neq 0$.

$S_{2,2} = 9$ IMPOSSIBLE

- Suppose $s_{2,2} = 9$. Then every point on 3 of them.
- Suppose $L \supset \{r_1, r_2, b_5, b_6\}$. So $\{b_1, \dots, b_4\}$ gen pos.
- WLOG $b_1 = (-1,1)$ $b_2 = (1,1)$ $b_3 = (-1,-1)$ $b_4 = (1,-1)$
- WLOG $r_1 = \overline{b_1 b_2} \cap \overline{b_3 b_4}$ $r_2 = \overline{b_1 b_4} \cap \overline{b_2 b_3}$
- So L is $y = 0$.
- There is another red on $\overline{b_1 b_2}$, say r_3. And say $r_4 \in \overline{b_3 b_4}$.
- Position of r_3 or r_4 determines $\{b_5, b_6\}$.

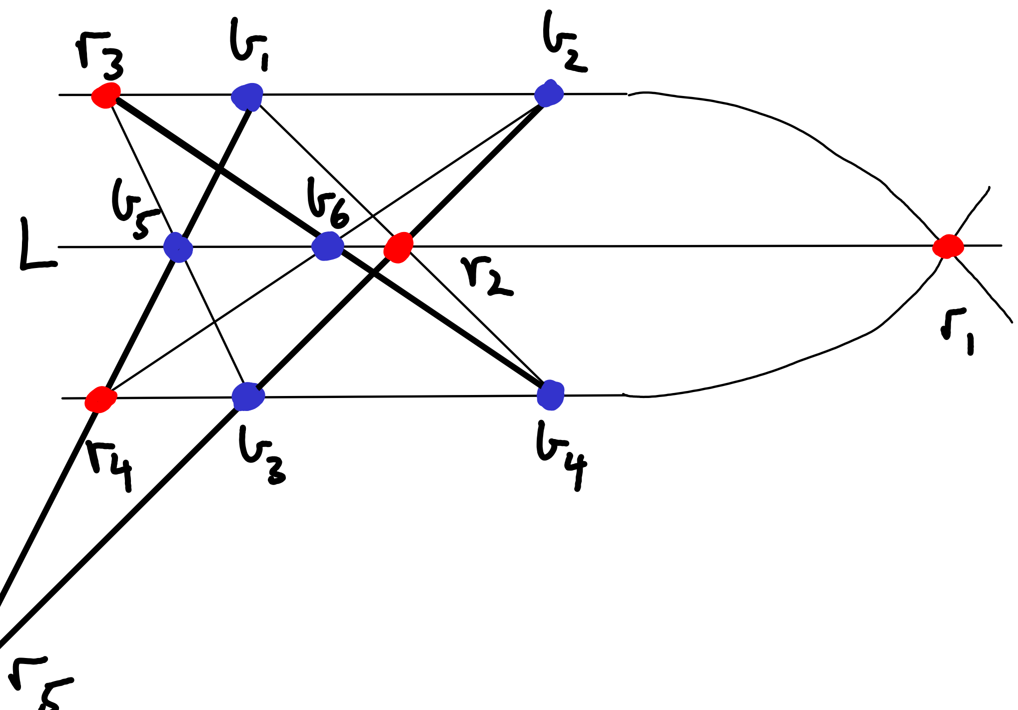


- Note if $r_3 = (a,1)$ then $r_4 = (a,-1)$

- 6 lines still need a red point:

$\overline{b_1 b_5}$ $\overline{b_5 b_3}$ $\overline{b_3 b_2}$

$\overline{b_2 b_6}$ $\overline{b_6 b_4}$ $\overline{b_4 b_1}$



- We may assume

$r_5 \in$ 1,5 3,2 6,4

- $r_5 = (c, c)$ (can check $c \neq \infty$)
- Since $r_5 \in \overline{r_4 b_1}$ $\quad (c,c) = \lambda(a,-1) + (1-\lambda)(-1,1)$

This gives $ac = a - 1 - 3c$

- Since $r_5 \in \overline{r_3 b_4}$ $\quad (c, c) = \gamma(a, 1) + (1-\gamma)(1,-1)$

This gives $ac = 3c - a - 1$

Solving gives $3c^2 = -1$ $\lightning$

PUZZLE ①