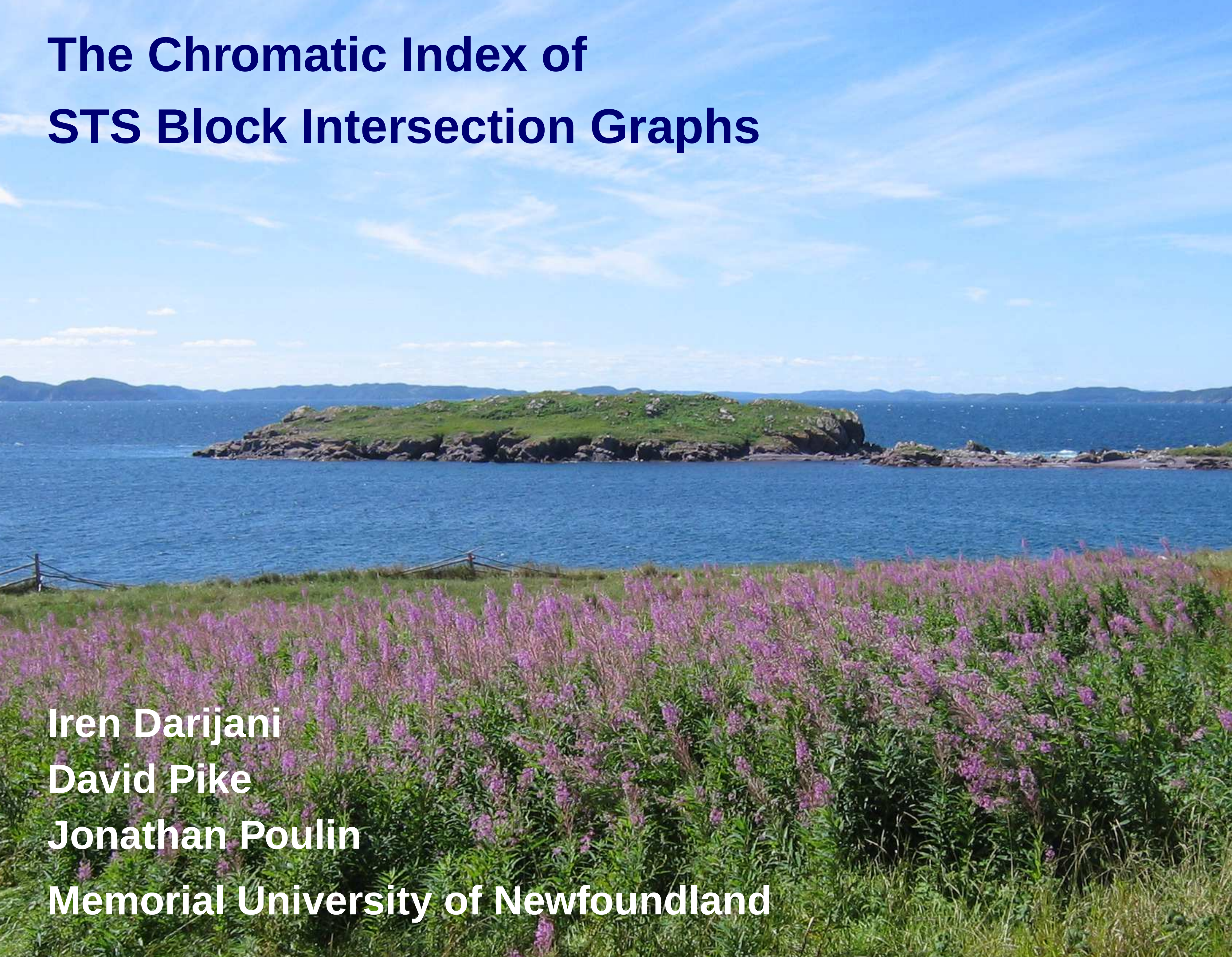
The Chromatic Index of STS Block Intersection Graphs



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Definition:

A combinatorial design $\mathcal{D}$ consists of

- a set V of elements (called points), together with
- a collection $\mathcal{B}$ of subsets (called blocks) of V.

A balanced incomplete block design, BIBD(v, k, λ), is a design in which:

- $|V| = v$,
- for each block $B \in \mathcal{B}$, $|B| = k$, and
- each 2-subset of V occurs in precisely λ blocks of $\mathcal{B}$.

A BIBD$(v, 3, 1)$ is a Steiner triple system, STS(v).

Example: a BIBD(13,3,1) ... *i.e.*, a STS(13):

$v = 13 \qquad k = 3 \qquad \lambda = 1$

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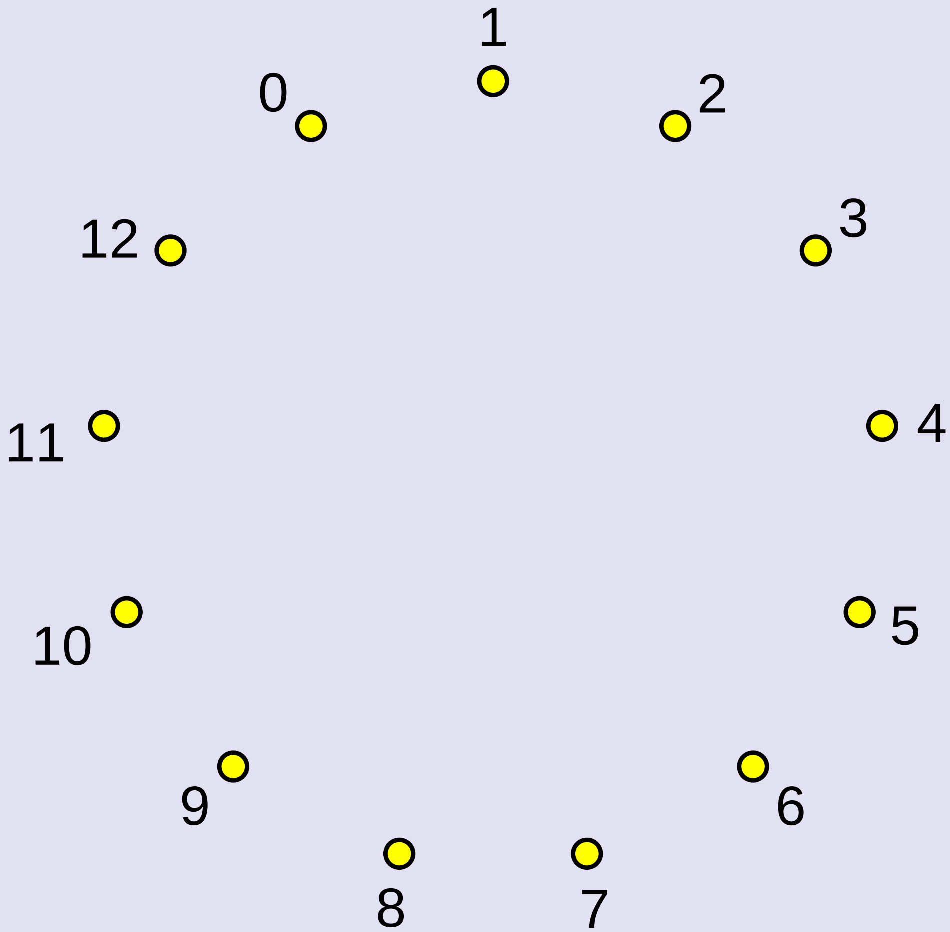

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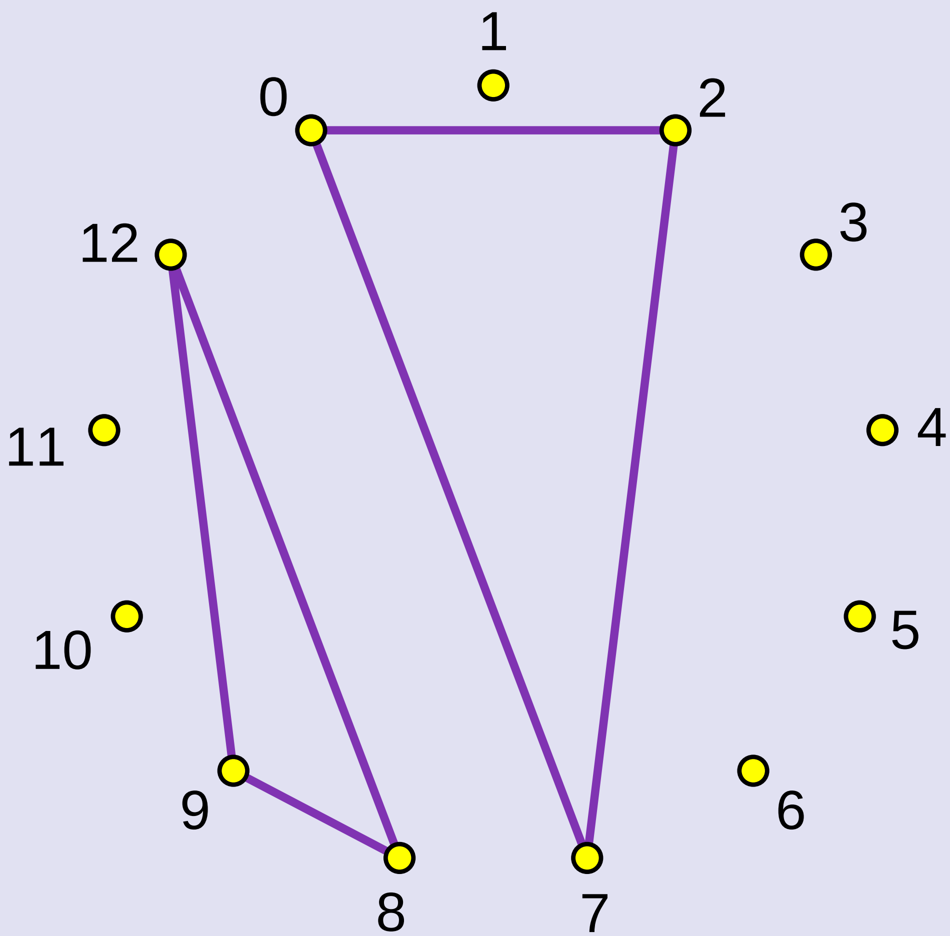

{0,2,7} {8,9,12}

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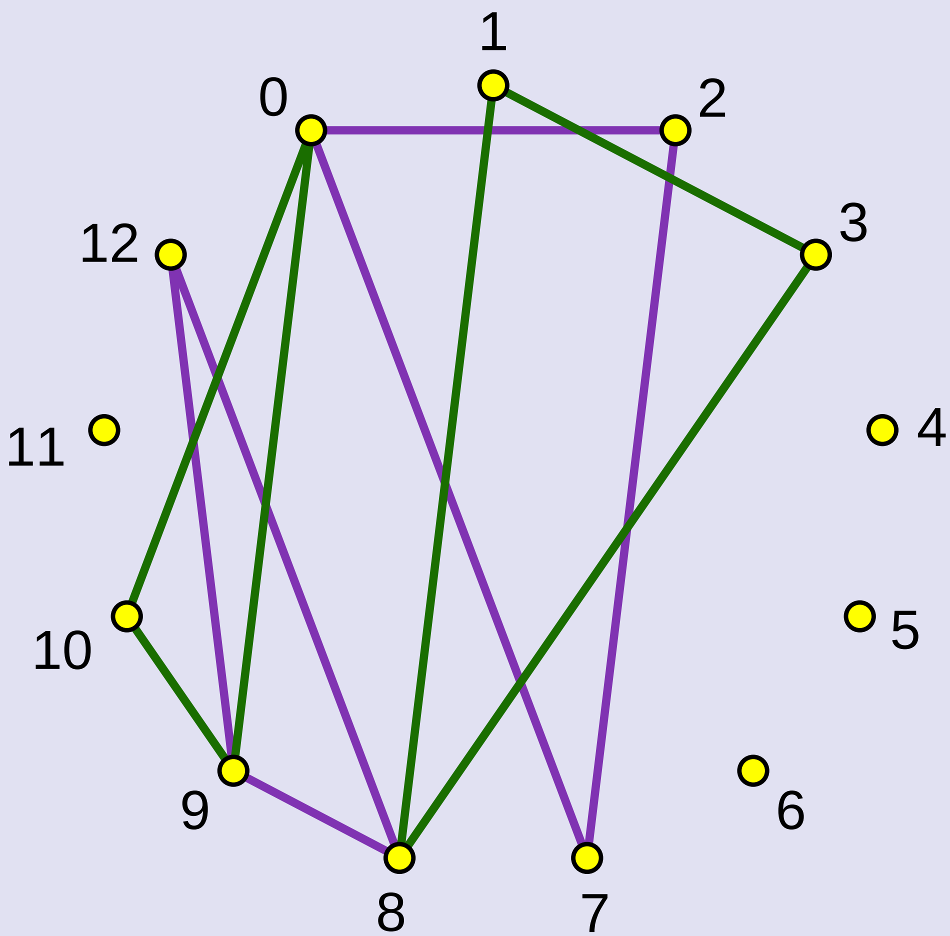

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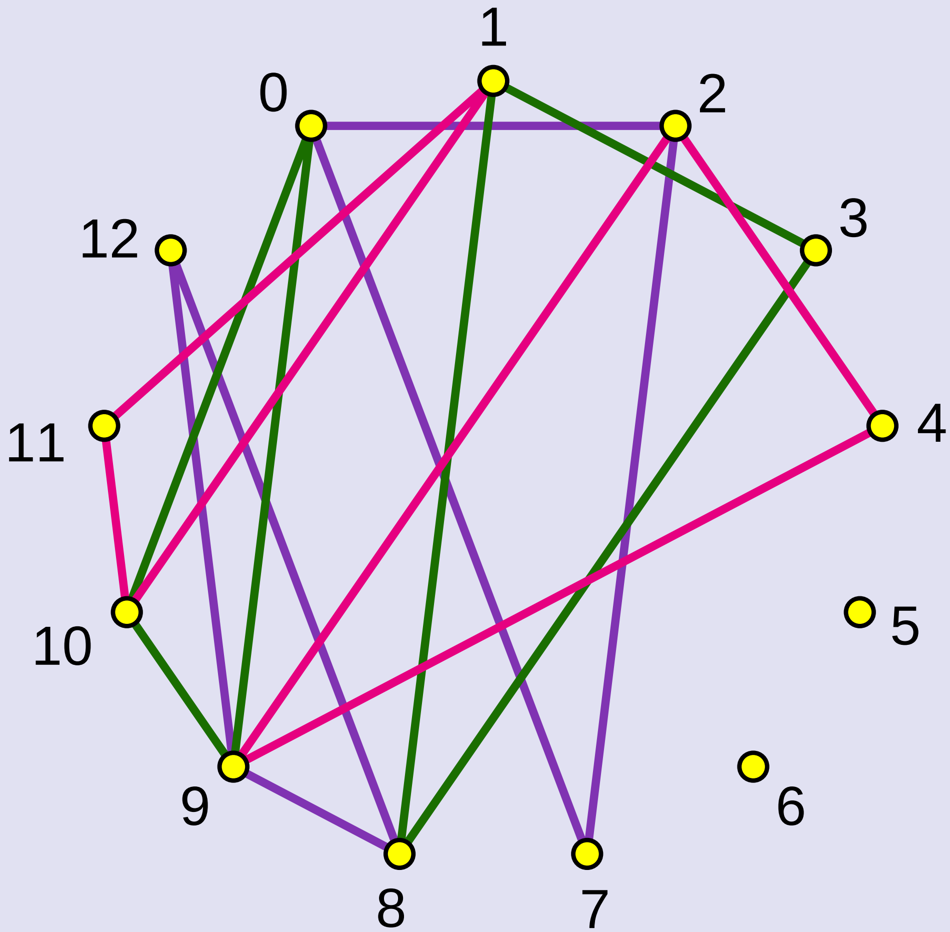

{0,2,7} {8,9,12}

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{2,4,9} {10,11,1}

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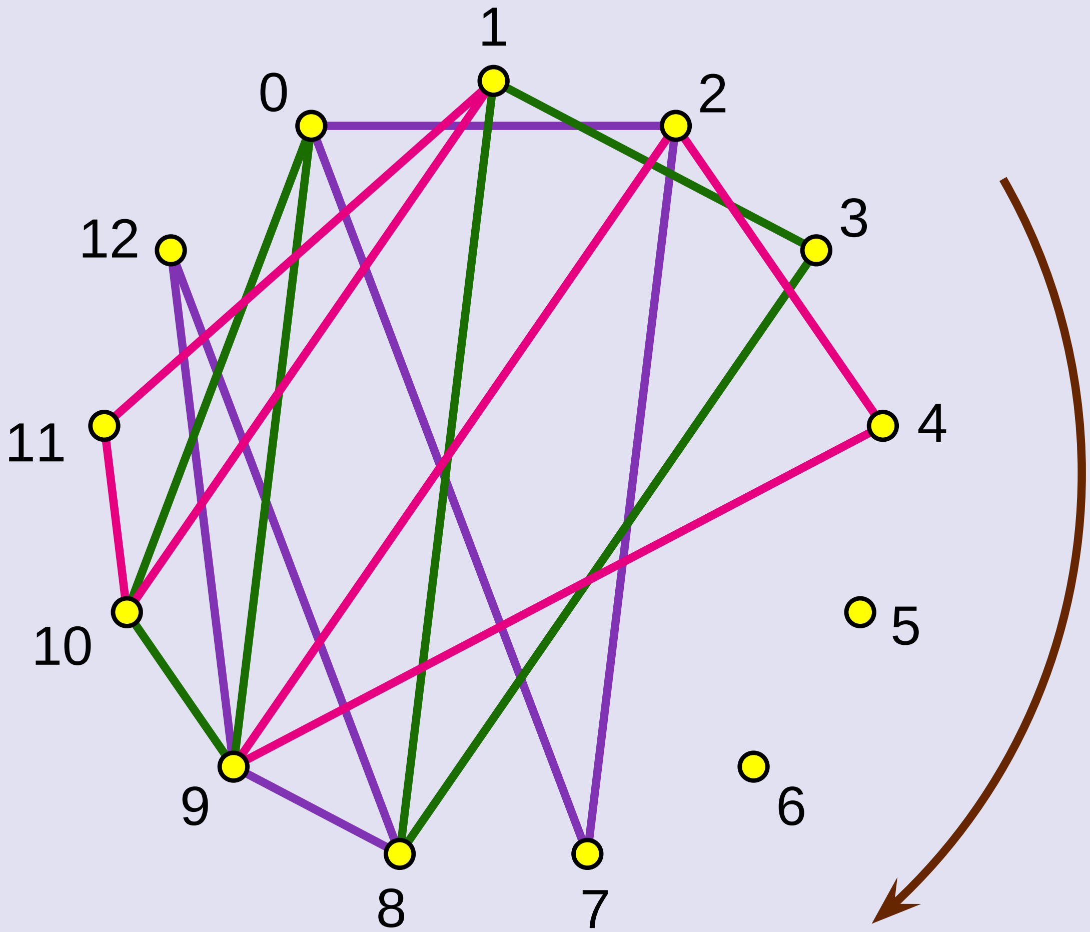

{0,2,7}	{8,9,12}
{1,3,8}	{9,10,0}
{2,4,9}	{10,11,1}
{3,5,10}	{11,12,2}
{4,6,11}	{12,0,3}
{5,7,12}	{0,1,4}
{6,8,0}	{1,2,5}
{7,9,1}	{2,3,6}
{8,10,2}	{3,4,7}
{9,11,3}	{4,5,8}
{10,12,4}	{5,6,9}
{11,0,5}	{6,7,10}
{12,1,6}	{7,8,11}

This is equivalent to a C_3-decomposition of K_{13}

Example: a Kirkman triple system of order 9:

$V = \{1,2,3,4,5,6,7,8,9\}$

Blocks:

{1,2,3}	{1,4,7}	{1,5,9}	{1,6,8}
{4,5,6}	{2,5,8}	{2,6,7}	{2,4,9}
{7,8,9}	{3,6,9}	{3,4,8}	{3,5,7}

This is equivalent to a 2-factorisation of K_9 in which each 2-factor consists of 3-cycles.

Theorem (Kirkman, 1847)

A STS(v) exists if and only if $v \equiv$ 1 or 3 (mod 6).

Theorem (Ray-Chaudhuri and Wilson, 1971)

A KTS(v) exists if and only if $v \equiv$ 3 (mod 6).

Definition:

Given a combinatorial design $\mathcal{D}$ with block set $\mathcal{B}$, the block-intersection graph of $\mathcal{D}$ is the graph having $\mathcal{B}$ as its vertex set, and in which two vertices B_1 and B_2 are adjacent if and only if $B_1 \cap B_2 \neq \emptyset$.

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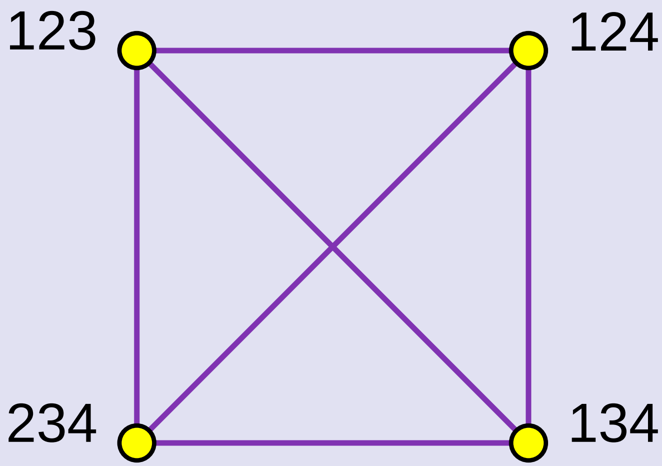
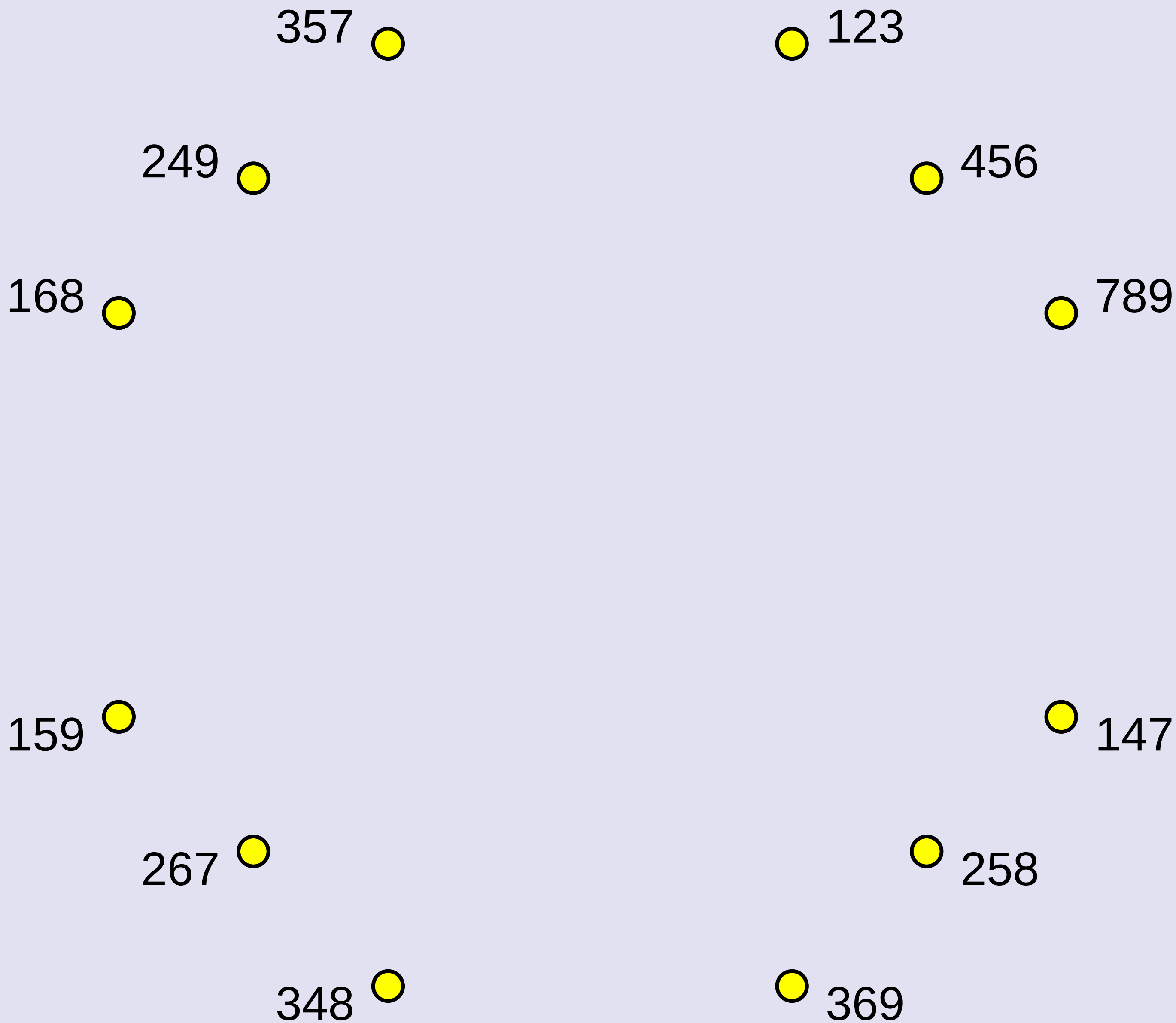
Example: A BIBD(4,3,2):

$$V = \{1, 2, 3, 4\}$$
$$\mathcal{B} = \big\{\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\big\}$$



Example: KTS(9):



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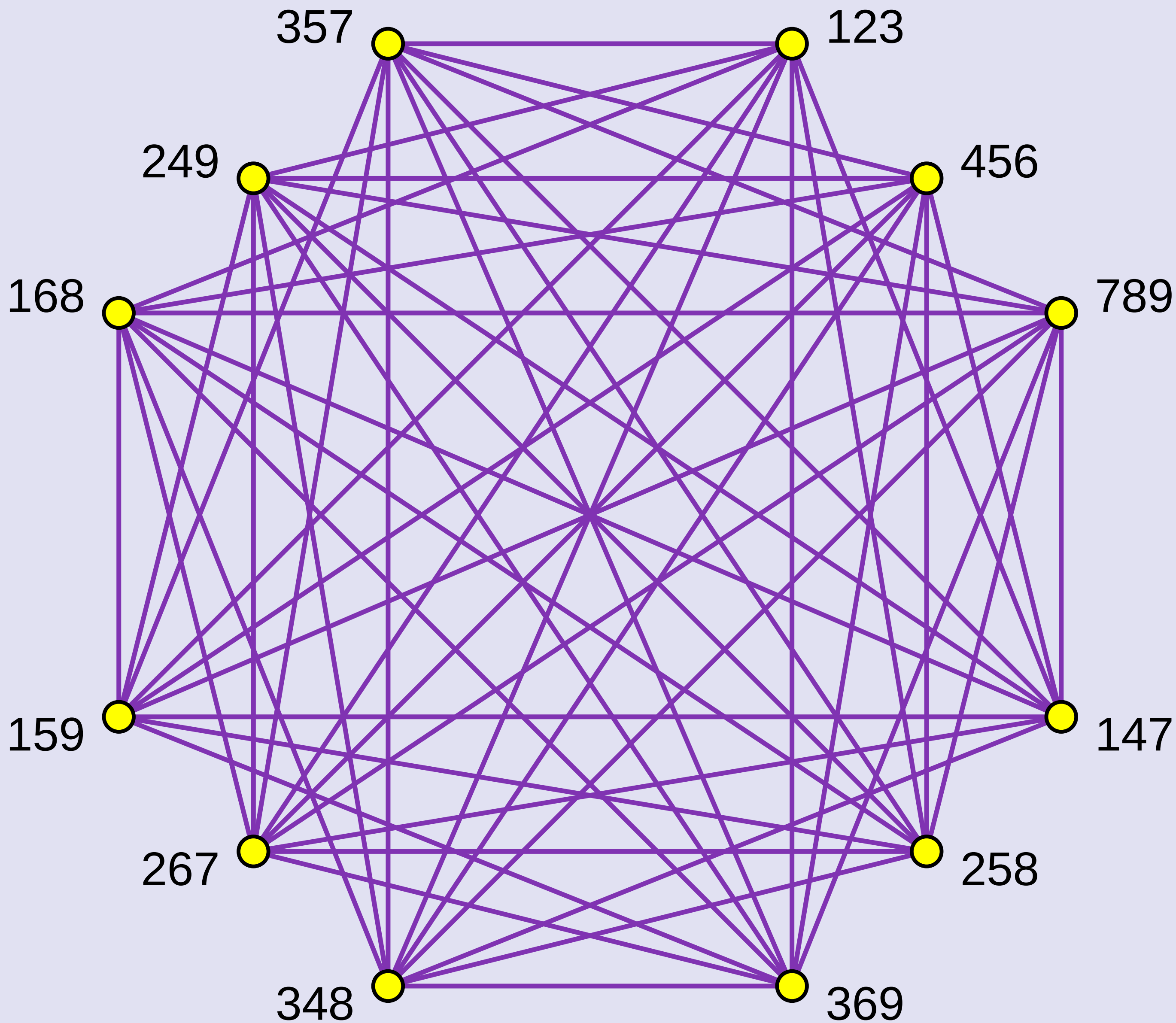

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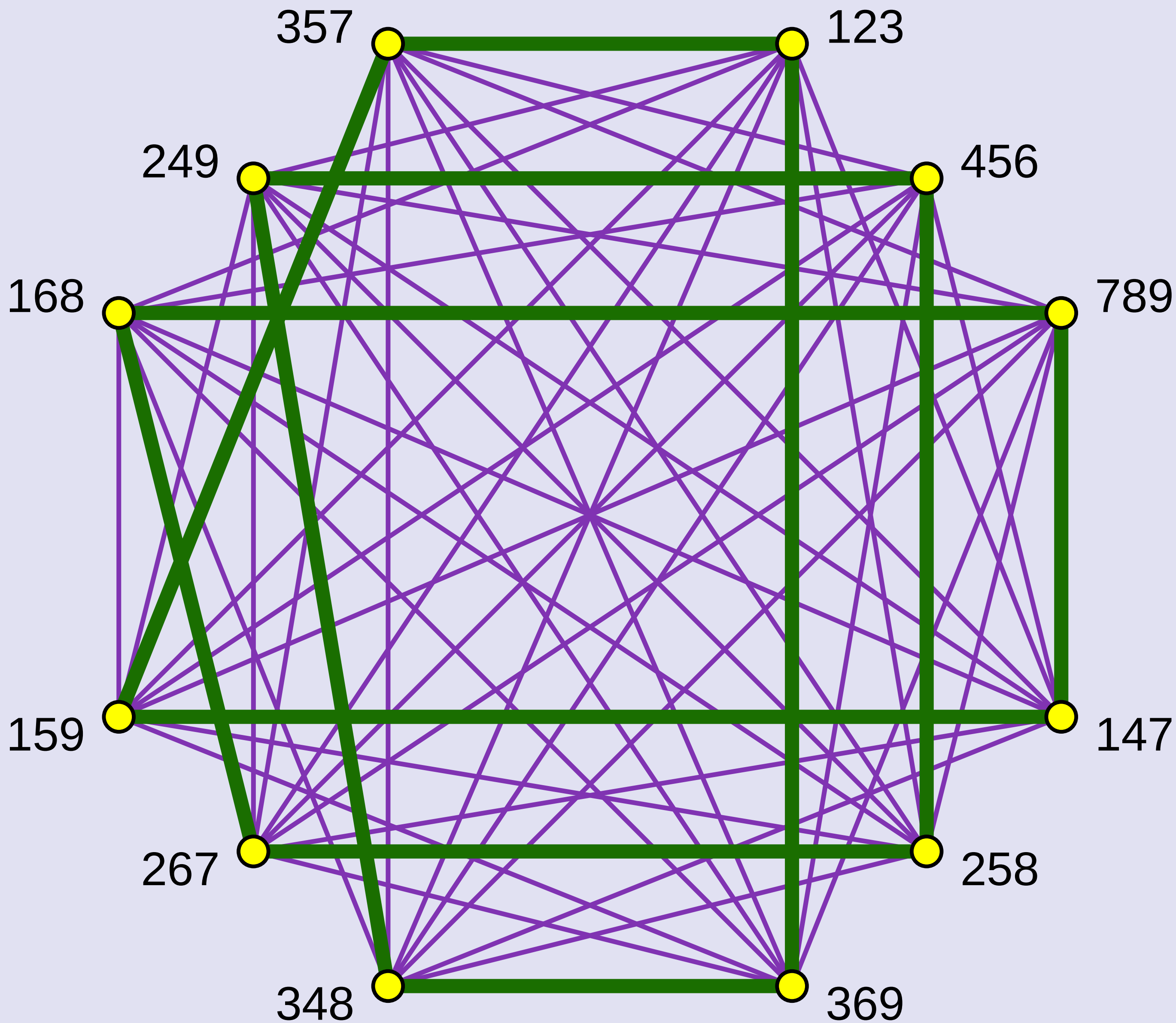

Question (Graham, 1987)

Is the block-intersection graph of a $\mathrm{STS}(v)$ Hamiltonian?

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Some Subsequent Discoveries:

- BIBD$(v, k, \lambda) \Rightarrow$ Hamiltonian (Horák and Rosa, 1988)
- BIBD$(v, k, 1)$ with $k \geqslant 3 \Rightarrow$ edge-pancyclic (Alspach and Hare, 1991)
- BIBD$(v, k, \lambda) \Rightarrow$ pancyclic (Mamut, Pike and Raines, 2004)
- BIBD$(v, k, \lambda) \Rightarrow$ cycle extendable (Abueida and Pike, 2013)

Similar results for Pairwise Balanced Designs also exist.

Definition:

A Hamilton decomposition of a Δ-regular graph G consists of a set of Hamilton cycles (plus a 1-factor if Δ is odd) that partition the edges of G.

Theorem (Pike, 1999)

Every STS(v) with $v \leqslant 15$ has a Hamilton decomposable block-intersection graph.

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A Hamilton decomposition of a Δ-regular graph G consists of a set of Hamilton cycles (plus a 1-factor if Δ is odd) that partition the edges of G.

Theorem (Pike, 1999)

Every STS(v) with $v \leqslant 15$ has a Hamilton decomposable block-intersection graph.

Question: What about $v \geqslant 19$?

Observation:

If $|V(G)|$ is even then a Hamilton decomposition yields a 1-factorisation.

An easier question than Hamilton decompositions:

Is it true that a STS has a 1-factorable block-intersection graph whenever the graph has even order?

More generally:

Determine the chromatic index of the block-intersection graph.

The chromatic index χ' of a graph G is the least number of colours that enable each edge of G to be assigned a single colour so that adjacent edges never have the same colour.

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More generally:

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The chromatic index χ' of a graph G is the least number of colours that enable each edge of G to be assigned a single colour so that adjacent edges never have the same colour.

Theorem (Vizing, 1964)

If G is a simple graph, then $\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1$.

Class 1 Class 2

Theorem (Darijani, Pike and Poulin, 20xx)

The block-intersection graph of a STS(v) is Class 2 whenever $v \equiv$ 3 or 7 (mod 12).

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The block-intersection graph of a STS(v) is Class 2 whenever $v \equiv$ 3 or 7 (mod 12).

Proof:

If $v \equiv$ 3 or 7 (mod 12),
then the block-intersection graph G has odd order.

So $\frac{|V(G)|}{2} > \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

Hence $|E(G)| = \frac{\Delta(G) \cdot |V(G)|}{2} > \Delta(G) \left\lfloor \frac{|V(G)|}{2} \right\rfloor$.

Therefore $\chi'(G)$ must exceed $\Delta(G)$.

QED

Theorem (Darijani, Pike and Poulin, 20xx)

The block-intersection graph of a KTS(v) is Class 1 whenever $v \equiv 9 \pmod{12}$.

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The block-intersection graph of a KTS(v) is Class 1 whenever $v \equiv 9 \pmod{12}$.

Proof:

$v \equiv 9 \pmod{12}$ implies that $v = 6n + 3$ where n is odd.

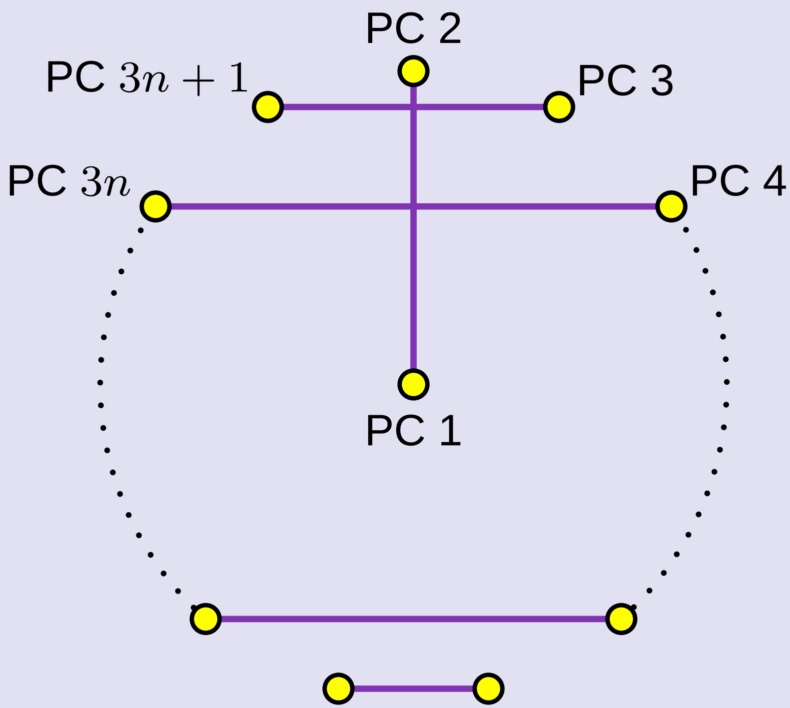
The number of parallel classes is $\frac{3b}{v} = 3n + 1$, which is even.

Partition the pairs of parallel classes into $3n$ sets $\mathcal{S}_1, \ldots, \mathcal{S}_{3n}$.

We can do this via a 1-factorisation of K_{3n+1}.

Proof (cont'd):

$\mathcal{S}_1$:



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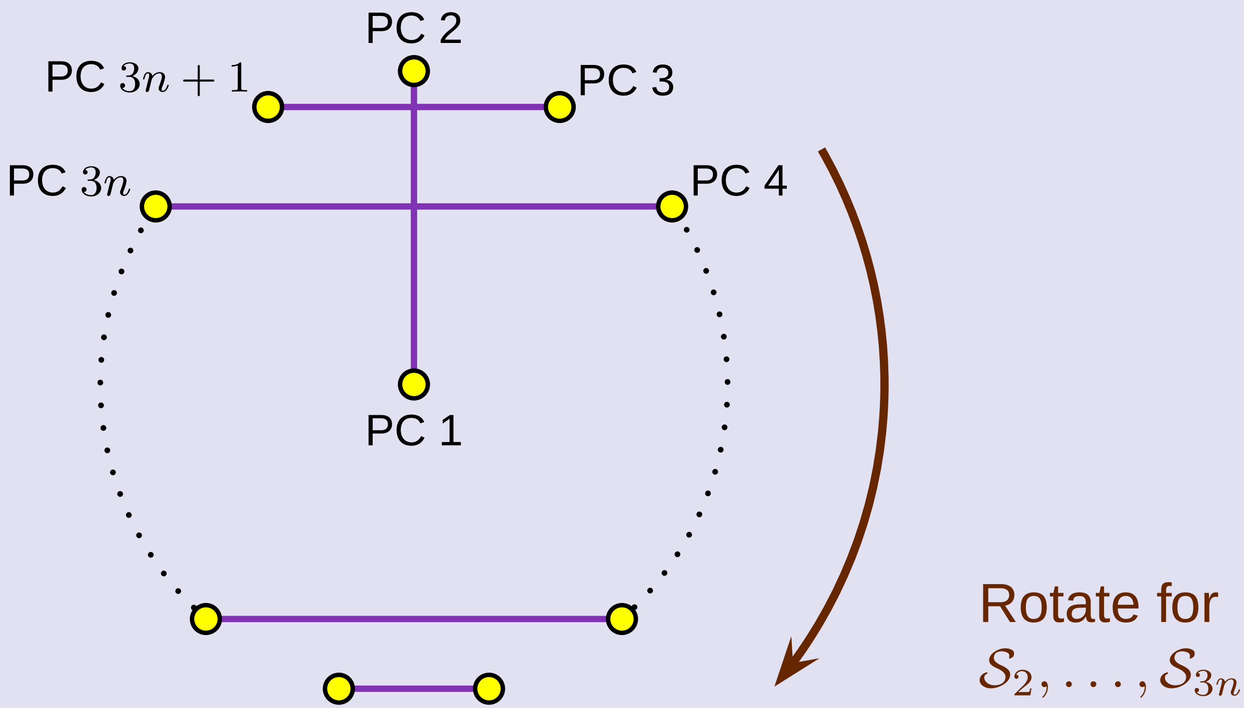
$\mathcal{S}_1$:



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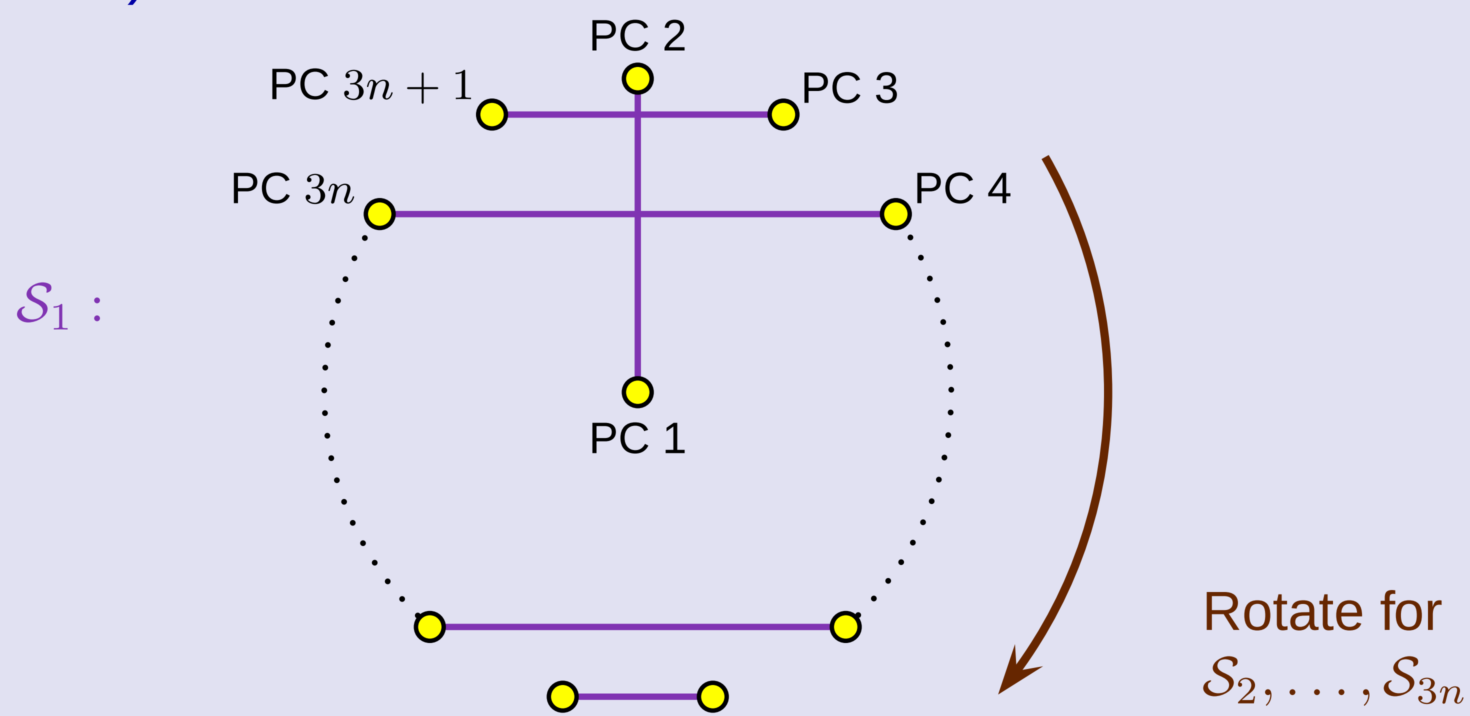

Each pair of parallel classes induces a cubic bipartite graph.

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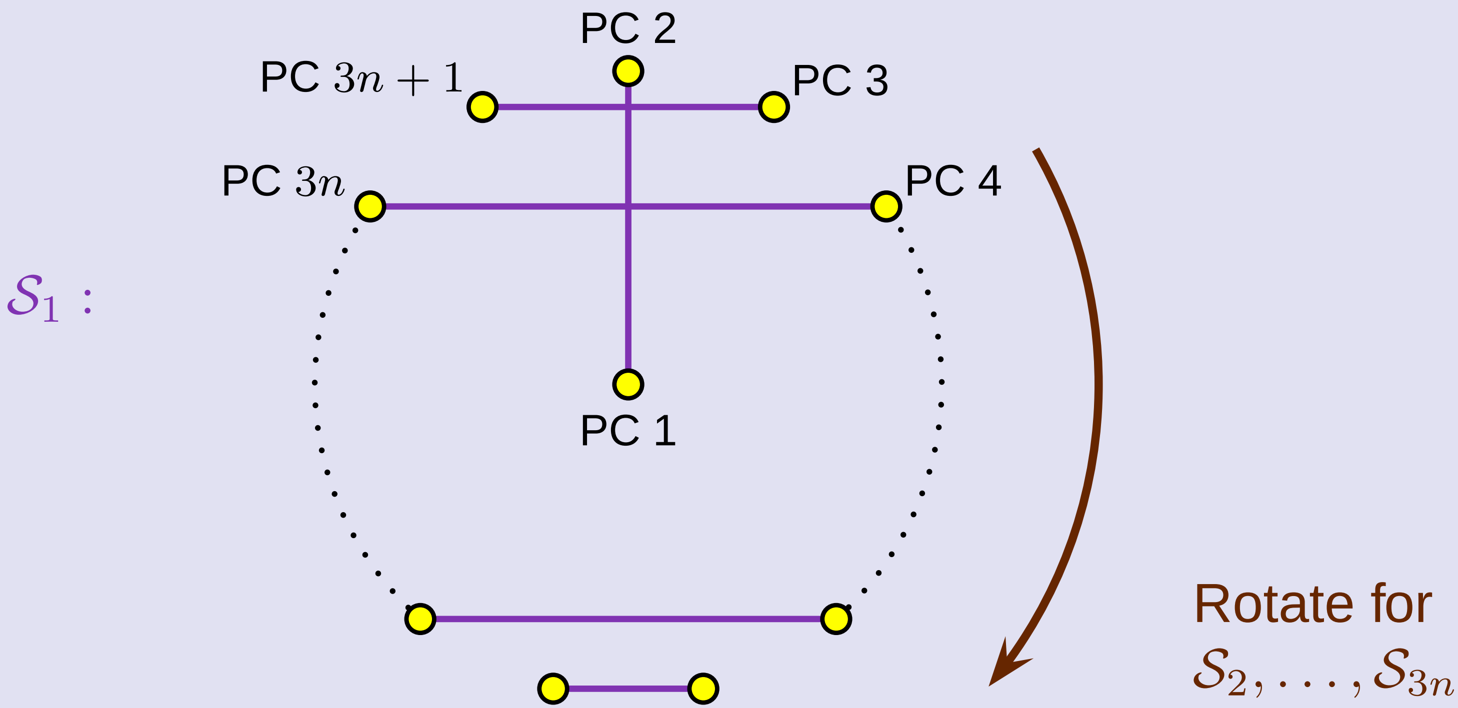

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Theorem (König, 1916): Bipartite graphs are Class 1.

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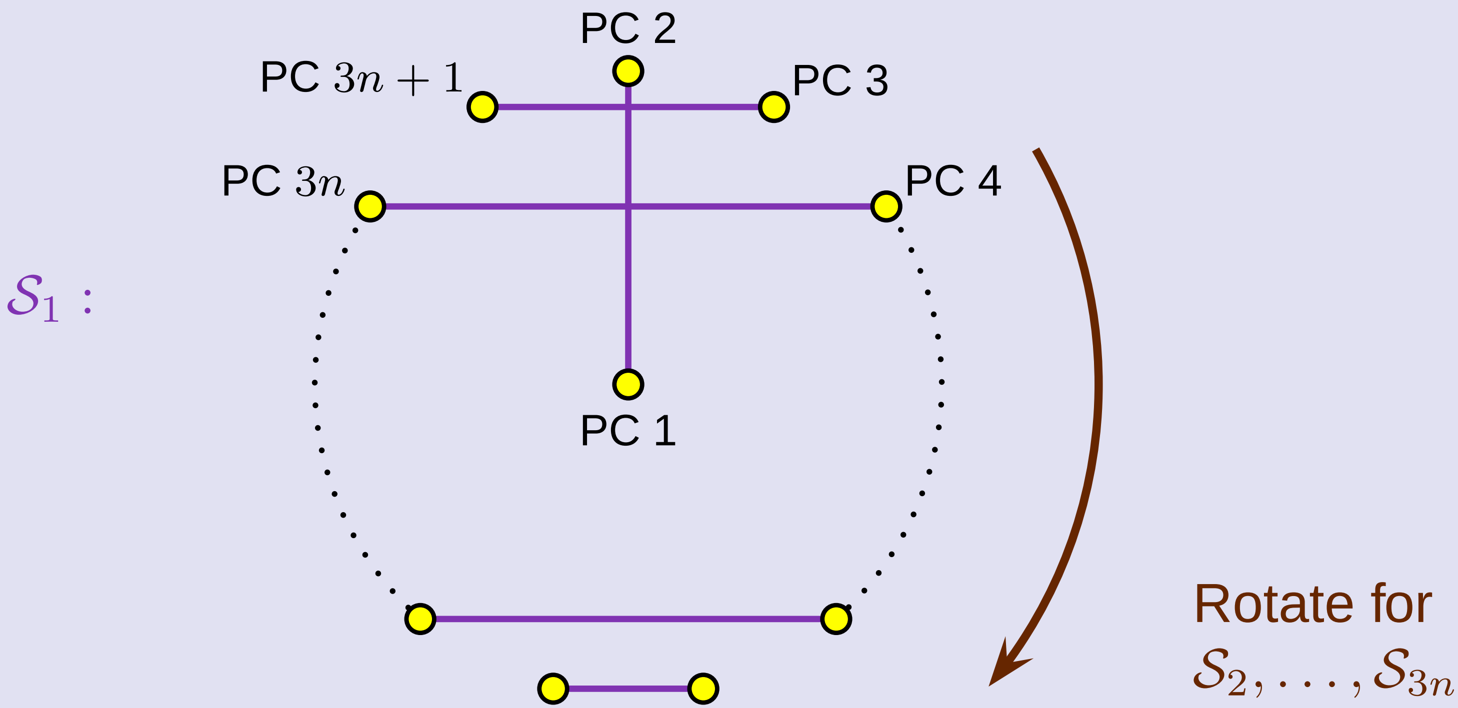

Each pair of parallel classes induces a cubic bipartite graph.

Theorem (König, 1916): Bipartite graphs are Class 1.

For each set $\mathcal{S}_i$, use colours $3i-2$, $3i-1$ and $3i$ to properly 3-edge-colour the bipartite graphs.

We obtain a proper edge-colouring with $9n = \Delta(G)$ colours. QED

Questions:

What else can we do?

What about when $v \equiv 1 \pmod{12}$?

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Definition:

A STS(v) is cyclic if its automorphism group contains a cyclic subgroup of order v.

Example:

The STS(13) presented earlier with point set $V = \mathbb{Z}_{13}$ is a cyclic STS with base blocks $\{0,2,7\}$ and $\{8,9,12\}$.

Equivalently: base blocks $\{0,2,7\}$ and $\{0,1,4\}$

Definition:

In an orbit arising from a base block $\{0, a, b\}$
the block-intersection graph will have edges between blocks
that are (with respect to the orbit) $\pm a$, $\pm b$ and $\pm(b - a)$ apart.
The three smallest of these six values (modulo v) will be called
the orbital differences for the orbit.

Observe that $\{0, a, b\}$ is adjacent to $\{\pm a, a \pm a, b \pm a\}$,
$\{\pm b, a \pm b, b \pm b\}$ and $\{\pm(b - a), a \pm (b - a), b \pm (b - a)\}$.

Example:

For the cyclic STS(13) with base blocks $\{$0,2,7$\}$ and $\{$0,1,4$\}$:

$\{$0,2,7$\}$ yields an orbit with orbital differences 2, 5 and 6

$\{$0,1,4$\}$ yields an orbit with orbital differences 1, 3 and 4

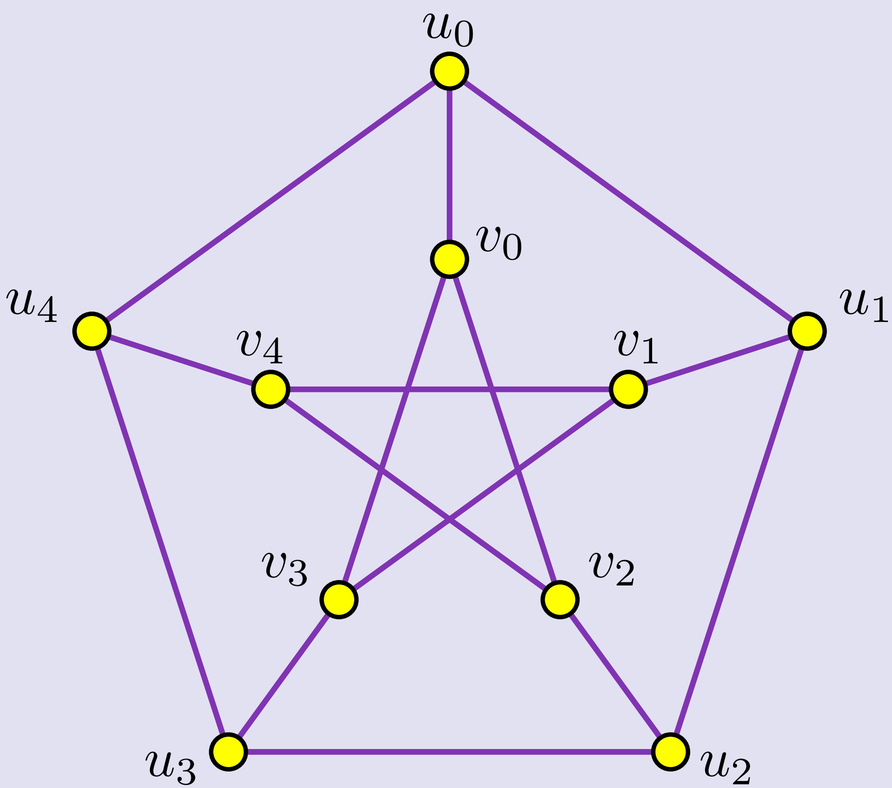
Definition:

The generalised Petersen graph $P(n,k)$ is the graph on $2n$ vertices and $3n$ edges as follows:

$$V = \{u_0, u_1, \ldots, u_{n-1}\} \cup \{v_0, v_1, \ldots, v_{n-1}\}$$

For each $i \in \mathbb{Z}_n$, $P(n,k)$ has edges: $\{u_i, u_{i+1}\}$, $\{u_i, v_i\}$, $\{v_i, v_{i+k}\}$

Example: $P(5,2)$



Theorem (Castagna and Prins, 1972)

Every generalised Petersen graph except for $P(5, 2)$ has a proper 3-edge-colouring and hence is Class 1.

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Lemma (Darijani, Pike and Poulin, 20xx)

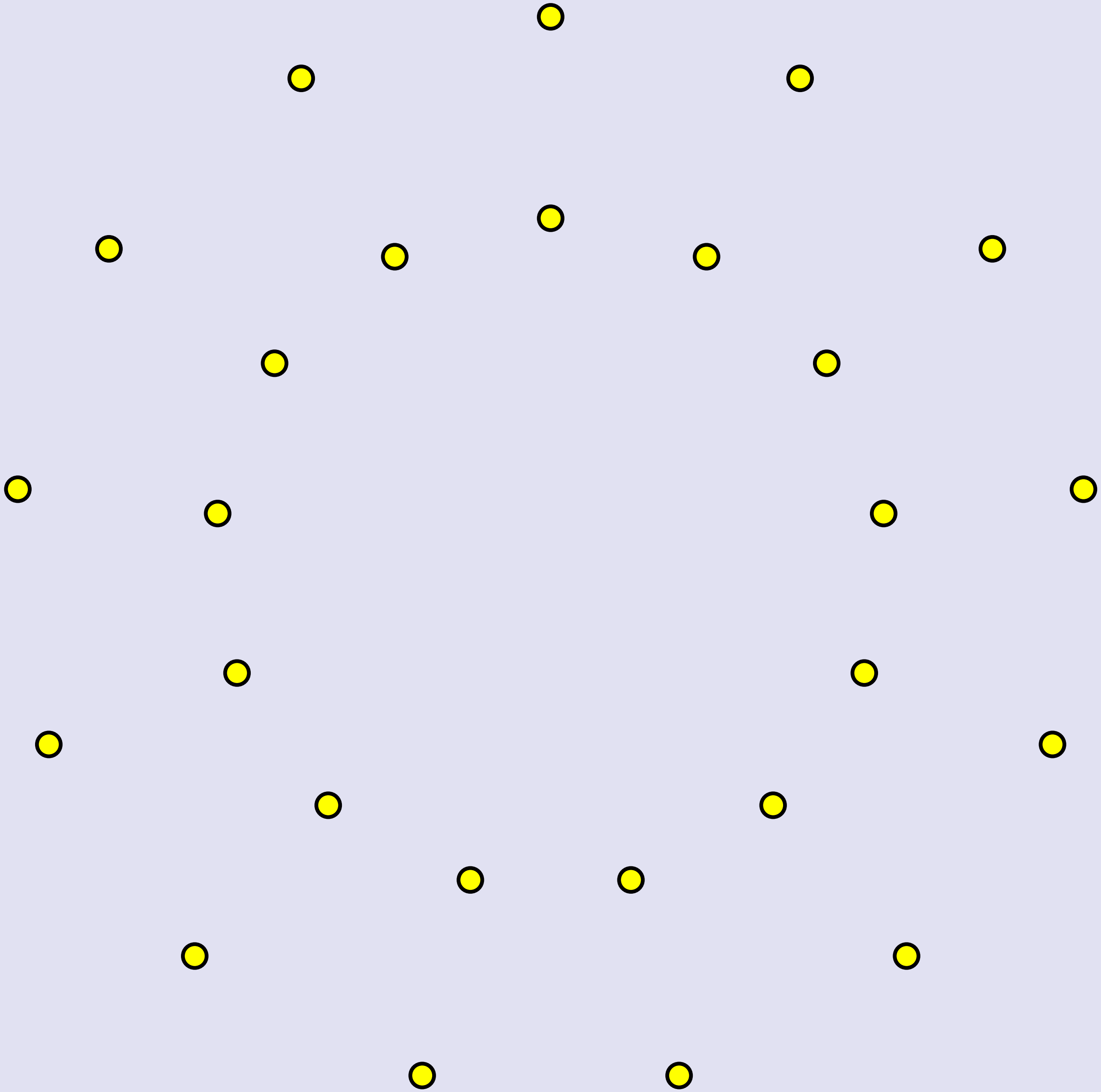
Given two orbits of size v in a cyclic STS(v),
if one of them has an orbital difference k that is co-prime to v
and ℓ is an orbital difference of the other orbit,
then the edges of difference k and ℓ in these two orbits,
together with a cyclic 1-factor between them, form a $P(v, k^{-1}\ell)$.

Example: Our cyclic STS(13):

$\{$0,1,4$\}$ yields orbital differences 1, 3 and 4 $\qquad k = 3$

$\{$0,2,7$\}$ yields orbital differences 2, 5 and 6 $\qquad \ell = 2$

Example of a $P(13, 5)$ in the BIG of our cyclic STS(13):



Example of a $P(13,5)$ in the BIG of our cyclic STS(13):

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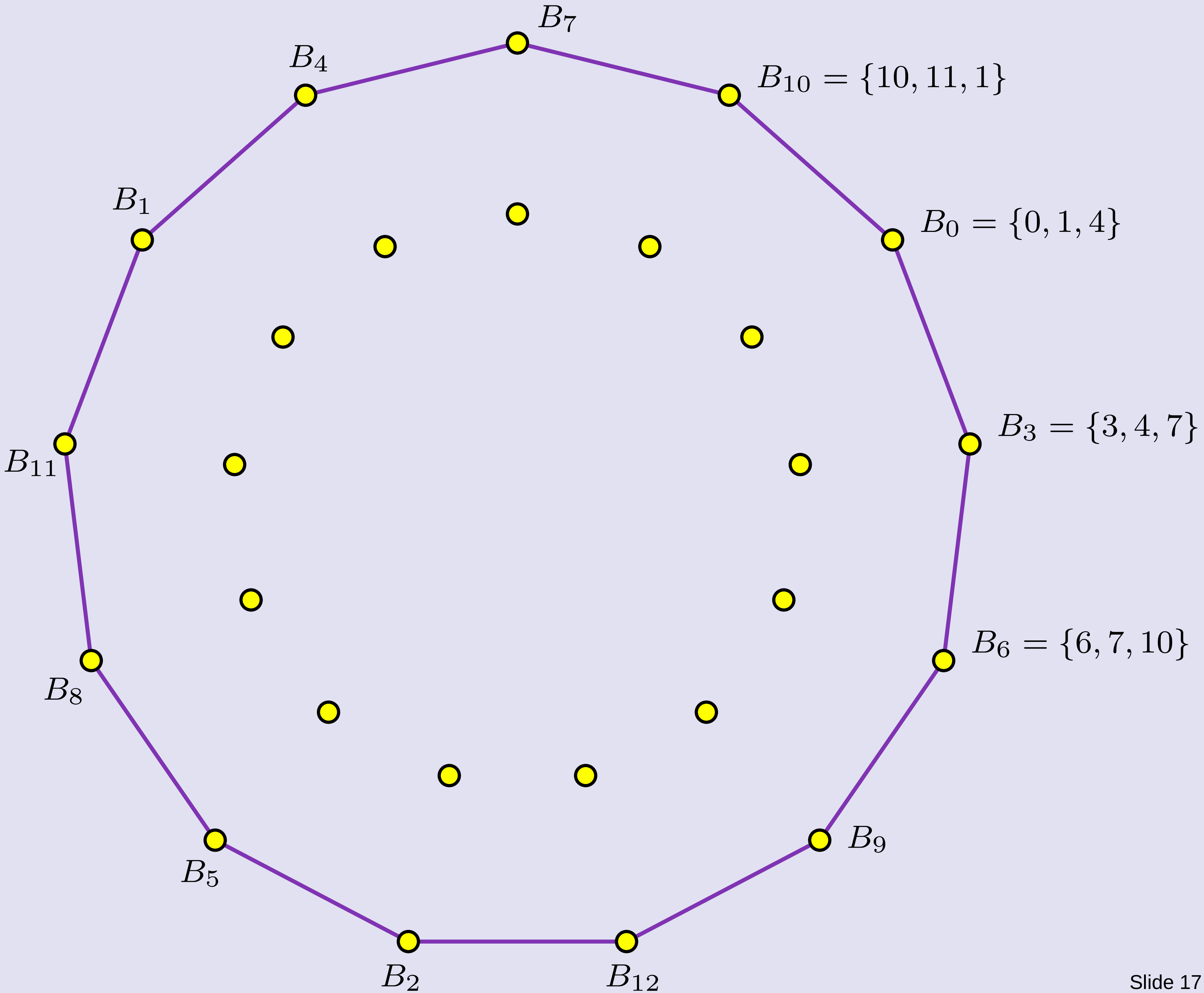

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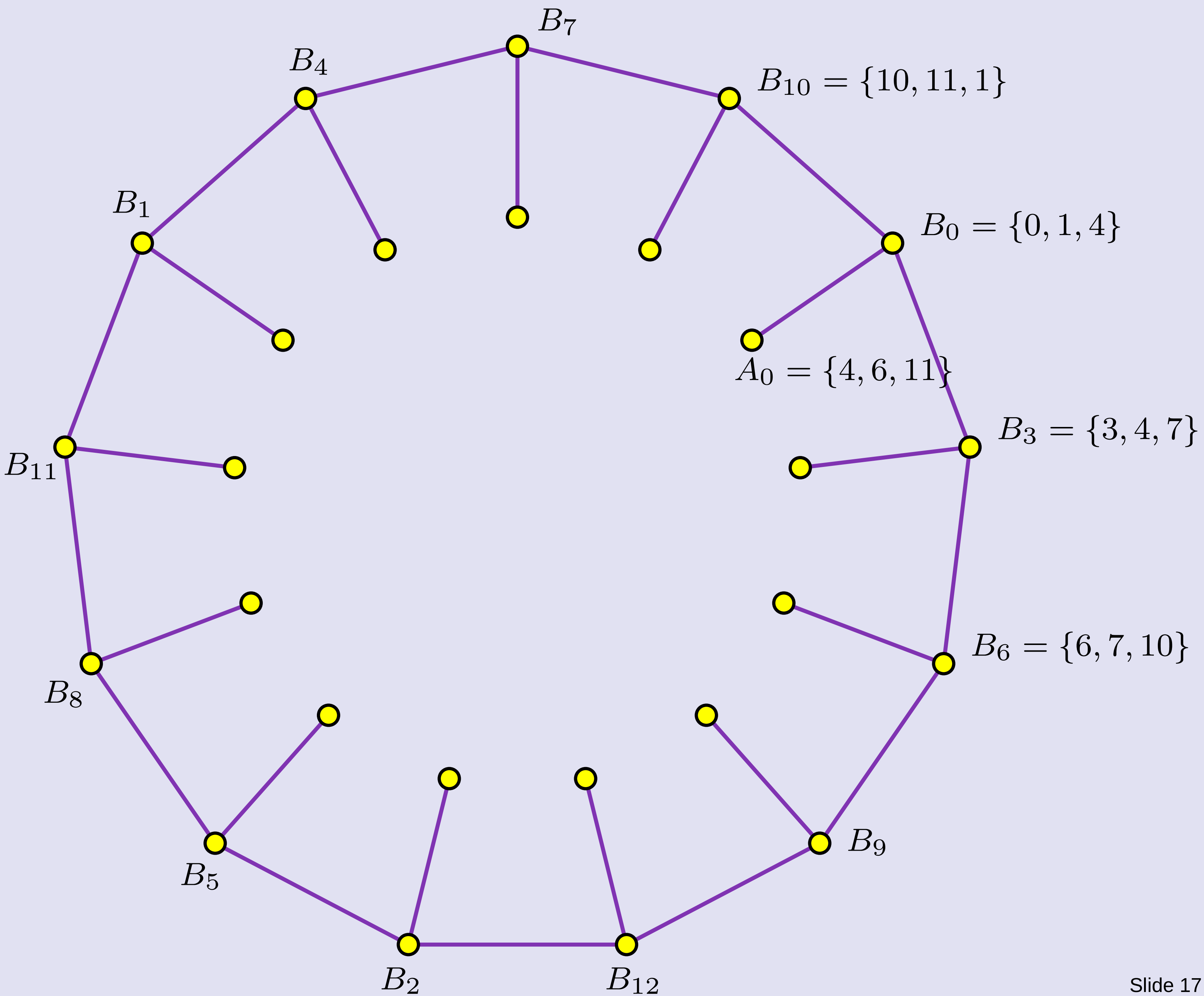

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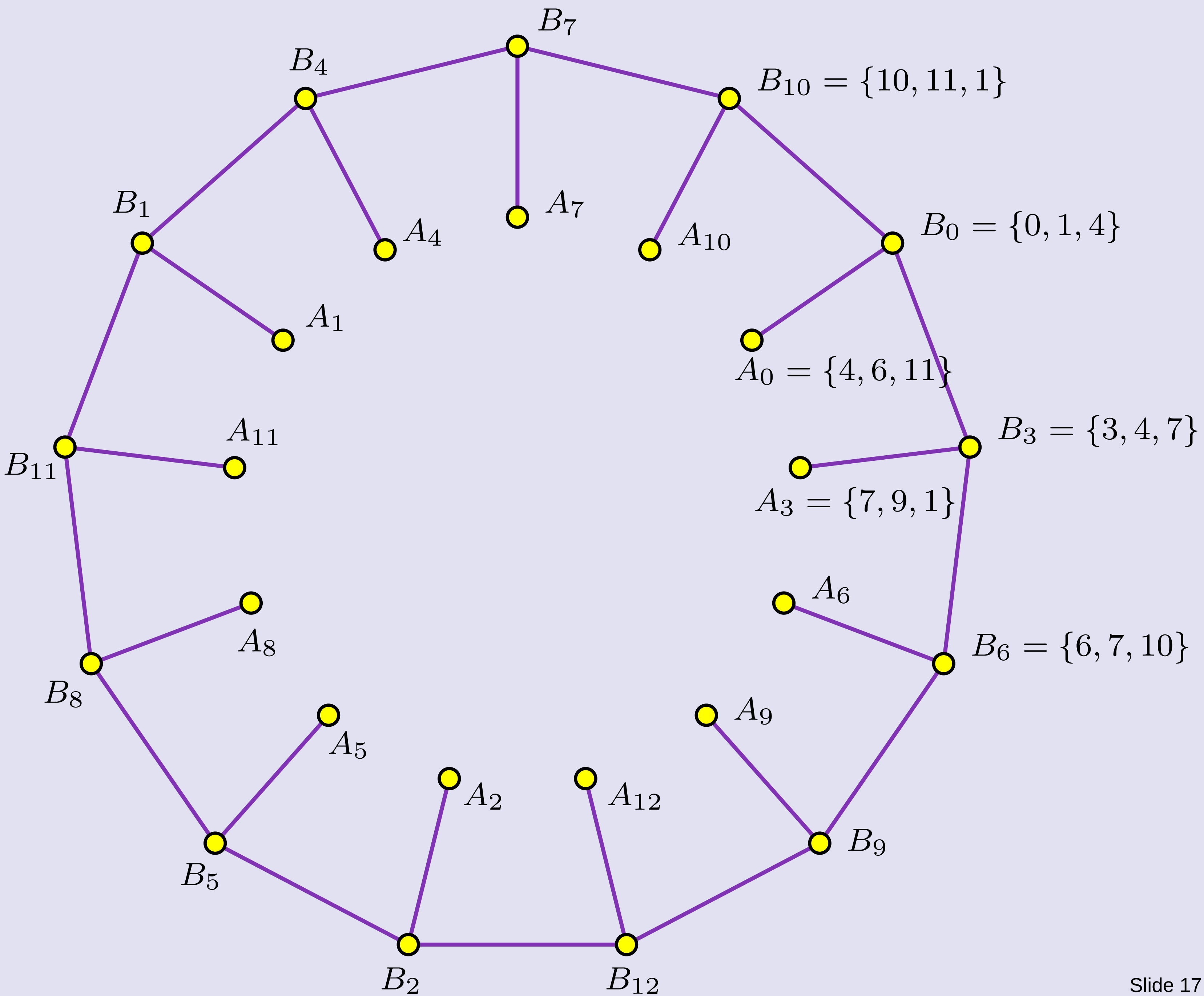
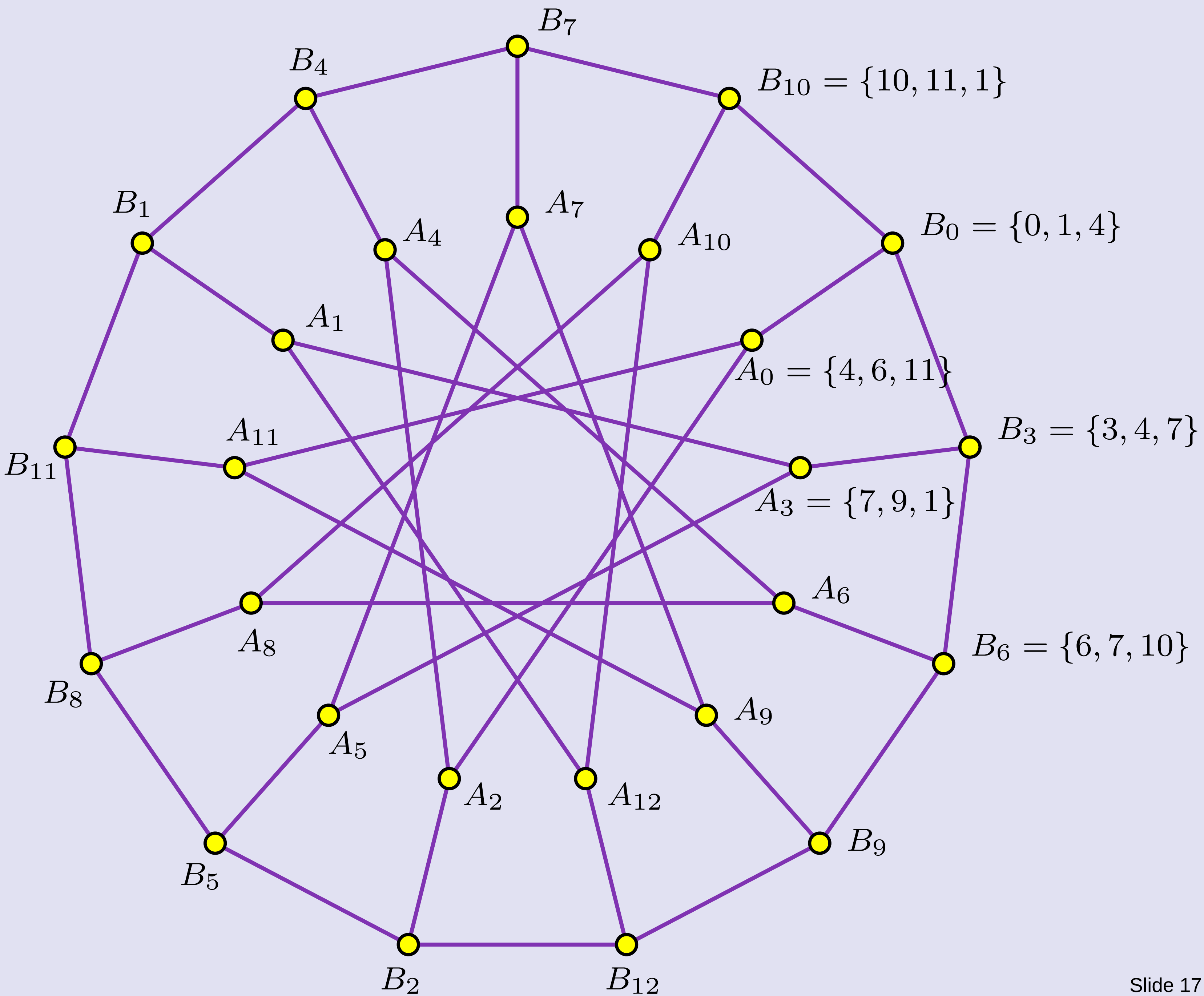


Example of a $P(13,5)$ in the BIG of our cyclic STS(13):

$k = 3$

$\ell = 2$



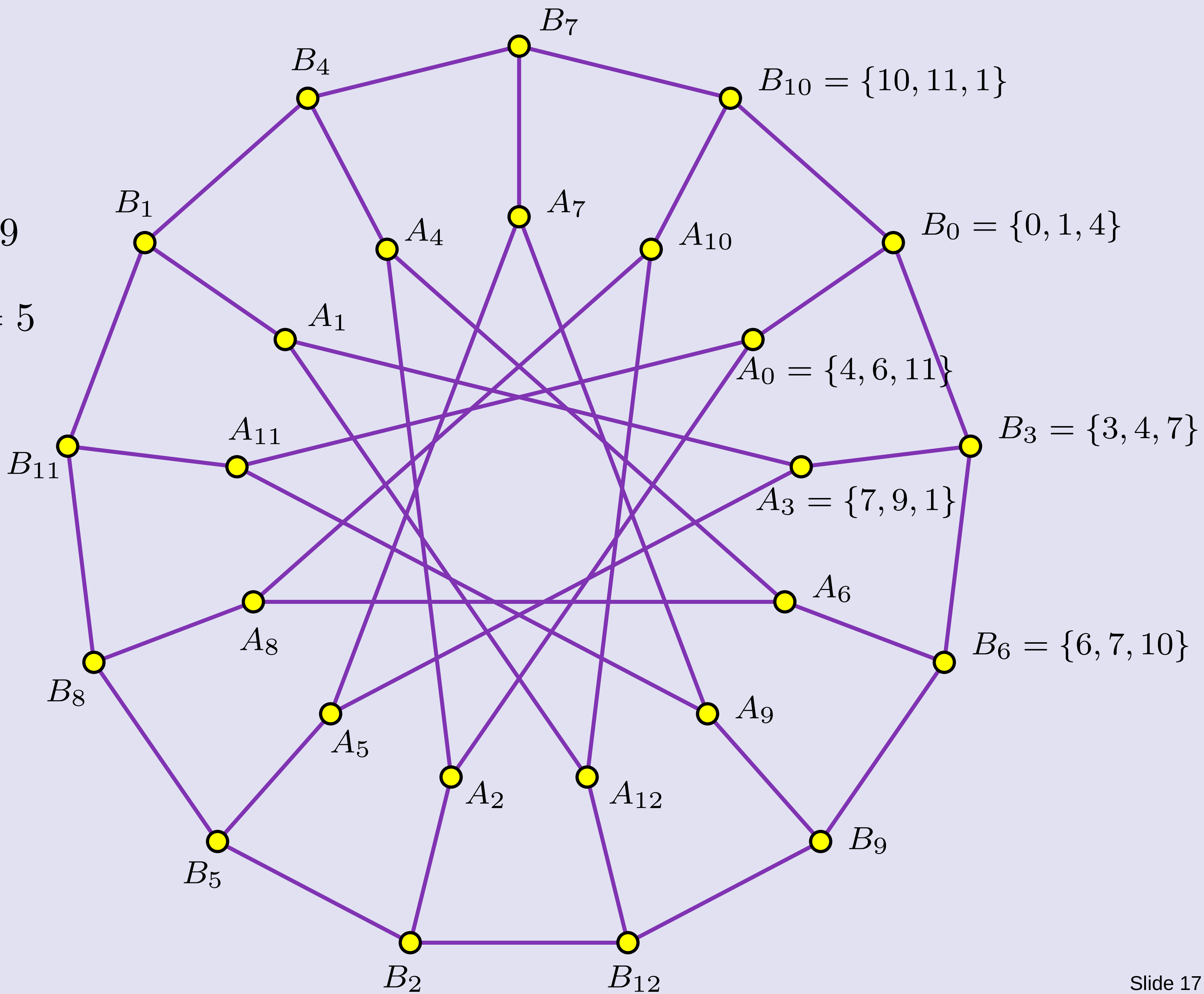
Example of a $P(13,5)$ in the BIG of our cyclic STS(13):

$k = 3$

$\ell = 2$

$k^{-1} = 9$

$k^{-1}\ell = 5$



To obtain a Class 1 colouring for the BIG of our cyclic STS(13):

- Use orbital differences $k = 3$ and $\ell = 2$ to get a $P(13, 5)$. Colour it with colours 1, 2, 3.

- Use orbital differences $k = 1$, $\ell = 6$ and different spokes to get a $P(13, 6)$. Colour it with colours 4, 5, 6.

- Use orbital differences $k = 4$, $\ell = 5$ and remaining spokes to get a $P(13, 6)$. Colour it with colours 7, 8, 9.

- All edges joining blocks of the same orbit have now been used as well as a 3-regular subgraph consisting of edges joining blocks in different orbits.

- What remains is a 6-regular bipartite graph with edges joining blocks in different orbits. Colour it with colours 10 through 15.

Definition:

Let $\rho(v)$ denote the proportion of the orbital differences in the set $\{1, 2, \ldots, \frac{v-1}{2}\}$ that are co-prime to v.

So $\rho(v) = \dfrac{\varphi(v)}{v-1}$, where φ is Euler's totient function.

Theorem (Darijani, Pike and Poulin, 20xx)

Any cyclic STS(v) with $v \equiv 1 \pmod{12}$ and $\rho(v) \geqslant \frac{2}{3}$ has a Class 1 block-intersection graph.

Overview of the Proof:

- The number of orbits is $N = \frac{v-1}{6}$, which is even.
- We will make use of a 1-factorisation of K_N in which each vertex represents an orbit.
- The initial 1-factor is special.
  For it, we want to pair each orbit with another one so that at least three of their six orbital differences are co-prime to v.
  Such a 1-factor can always be found when $\rho(v) \geqslant \frac{2}{3}$.
  For each pair of orbits from this 1-factor we obtain three generalised Petersen graphs and a 6-regular bipartite graph (similar to our STS(13) example)
- For each remaining 1-factor, each pair of orbits yields a 9-regular bipartite graph.

QED

Question:

What about cyclic STS(v) with $v \equiv 9 \pmod{12}$?

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Theorem (Darijani, Pike and Poulin, 20xx)

Any cyclic STS(v) with $v \equiv 9 \pmod{12}$
has a Class 1 block-intersection graph.

Note that these are cyclic STS with short orbits.

Example of a cyclic STS(21)

$\{0,1,5\}$ $\{0,2,10\}$ $\{0,3,9\}$ $\{0,7,14\}$

Example of a cyclic STS(21)

$\{0,1,5\}$	$\{0,2,10\}$	$\{0,3,9\}$	$\{0,7,14\}$
$\{1,2,6\}$	$\{1,3,11\}$	$\{1,4,10\}$	$\{1,8,15\}$

Example of a cyclic STS(21)

$\{0,1,5\}$	$\{0,2,10\}$	$\{0,3,9\}$	$\{0,7,14\}$
$\{1,2,6\}$	$\{1,3,11\}$	$\{1,4,10\}$	$\{1,8,15\}$
$\{2,3,7\}$	$\{2,4,12\}$	$\{2,5,11\}$	$\{2,9,16\}$

Example of a cyclic STS(21)

$\{0,1,5\}$	$\{0,2,10\}$	$\{0,3,9\}$	$\{0,7,14\}$
$\{1,2,6\}$	$\{1,3,11\}$	$\{1,4,10\}$	$\{1,8,15\}$
$\{2,3,7\}$	$\{2,4,12\}$	$\{2,5,11\}$	$\{2,9,16\}$
$\{3,4,8\}$	$\{3,5,13\}$	$\{3,6,12\}$	$\{3,10,17\}$

Example of a cyclic STS(21)

{0,1,5}	{0,2,10}	{0,3,9}	{0,7,14}
{1,2,6}	{1,3,11}	{1,4,10}	{1,8,15}
{2,3,7}	{2,4,12}	{2,5,11}	{2,9,16}
{3,4,8}	{3,5,13}	{3,6,12}	{3,10,17}
{4,5,9}	{4,6,14}	{4,7,13}	{4,11,18}

Example of a cyclic STS(21)

{0,1,5}	{0,2,10}	{0,3,9}	{0,7,14}
{1,2,6}	{1,3,11}	{1,4,10}	{1,8,15}
{2,3,7}	{2,4,12}	{2,5,11}	{2,9,16}
{3,4,8}	{3,5,13}	{3,6,12}	{3,10,17}
{4,5,9}	{4,6,14}	{4,7,13}	{4,11,18}
{5,6,10}	{5,7,15}	{5,8,14}	{5,12,19}

Example of a cyclic STS(21)

{0,1,5}	{0,2,10}	{0,3,9}	{0,7,14}
{1,2,6}	{1,3,11}	{1,4,10}	{1,8,15}
{2,3,7}	{2,4,12}	{2,5,11}	{2,9,16}
{3,4,8}	{3,5,13}	{3,6,12}	{3,10,17}
{4,5,9}	{4,6,14}	{4,7,13}	{4,11,18}
{5,6,10}	{5,7,15}	{5,8,14}	{5,12,19}
{6,7,11}	{6,8,16}	{6,9,15}	{6,13,20}

Example of a cyclic STS(21)

{0,1,5}	{0,2,10}	{0,3,9}	{0,7,14}
{1,2,6}	{1,3,11}	{1,4,10}	{1,8,15}
{2,3,7}	{2,4,12}	{2,5,11}	{2,9,16}
{3,4,8}	{3,5,13}	{3,6,12}	{3,10,17}
{4,5,9}	{4,6,14}	{4,7,13}	{4,11,18}
{5,6,10}	{5,7,15}	{5,8,14}	{5,12,19}
{6,7,11}	{6,8,16}	{6,9,15}	{6,13,20}
{7,8,12}	{7,9,17}	{7,10,16}	{7,14,0}

Example of a cyclic STS(21)

$\{0,1,5\}$	$\{0,2,10\}$	$\{0,3,9\}$	$\{0,7,14\}$
$\{1,2,6\}$	$\{1,3,11\}$	$\{1,4,10\}$	$\{1,8,15\}$
$\{2,3,7\}$	$\{2,4,12\}$	$\{2,5,11\}$	$\{2,9,16\}$
$\{3,4,8\}$	$\{3,5,13\}$	$\{3,6,12\}$	$\{3,10,17\}$
$\{4,5,9\}$	$\{4,6,14\}$	$\{4,7,13\}$	$\{4,11,18\}$
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{8,9,13}	{8,10,18}	{8,11,17}	
{9,10,14}	{9,11,19}	{9,12,18}	
⋮	⋮	⋮	
{19,20,3}	{19,0,8}	{19,1,7}	
{20,0,4}	{20,1,9}	{20,2,8}	

More Generally:

Suppose $v = 6n + 3$ where n is odd.

Then there are n full orbits $\mathcal{O}_1, \mathcal{O}_2, \ldots, \mathcal{O}_n$
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With an even number $N = n + 1$ of orbits, we can represent each orbit as a vertex of K_N and make use of a 1-factorisation.

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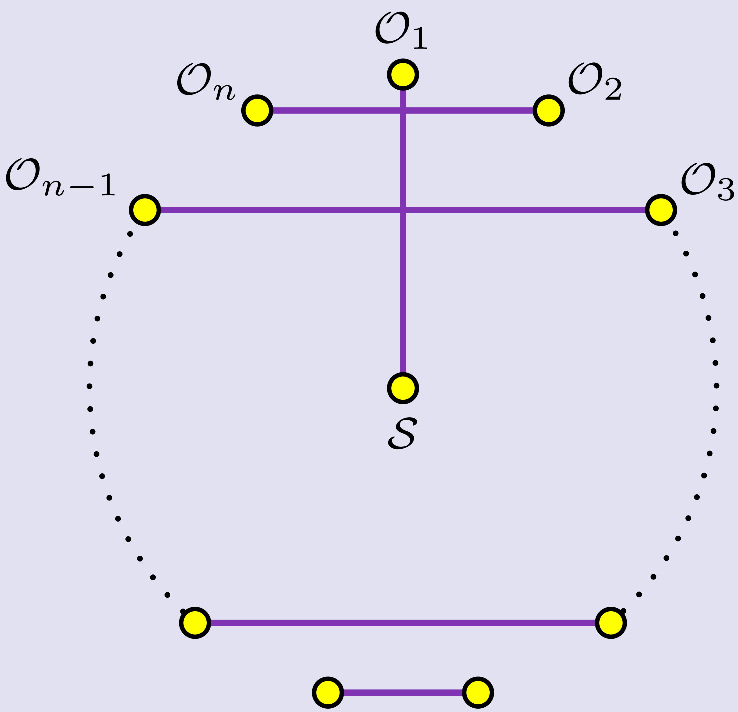

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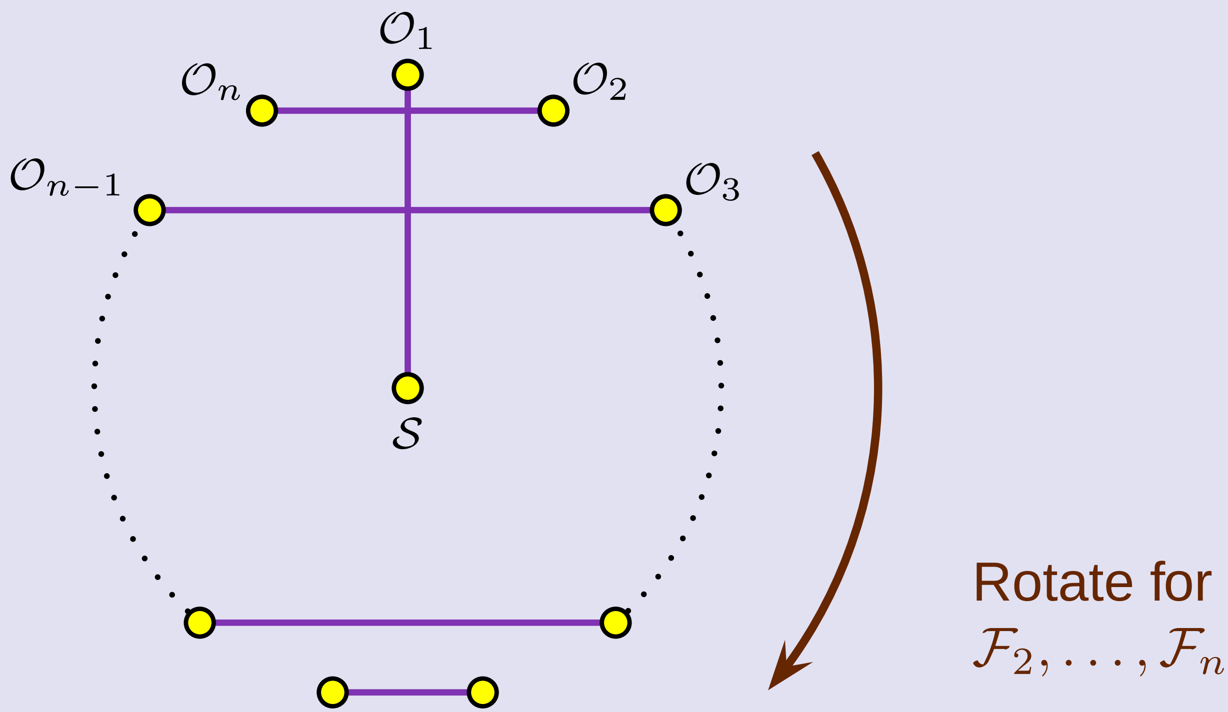

For each 1-factor $\mathcal{F}_i$

- For each instance of paired full orbits $\{\mathcal{O}_x, \mathcal{O}_y\}$ of $\mathcal{F}_i$, consider the subgraph of the BIG on the vertex set $\mathcal{O}_x \cup \mathcal{O}_y$ having edges between $\mathcal{O}_x$ and $\mathcal{O}_y$.

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- For the pair $\{\mathcal{O}_i, \mathcal{S}\}$, consider the subgraph of the BIG on the vertex set $\mathcal{O}_i \cup \mathcal{S}$, with edges between $\mathcal{O}_i$ and $\mathcal{S}$, as well as the edges within $\mathcal{O}_i$.

  This is also a 9-regular graph, but it is not bipartite, so it gets special attention...

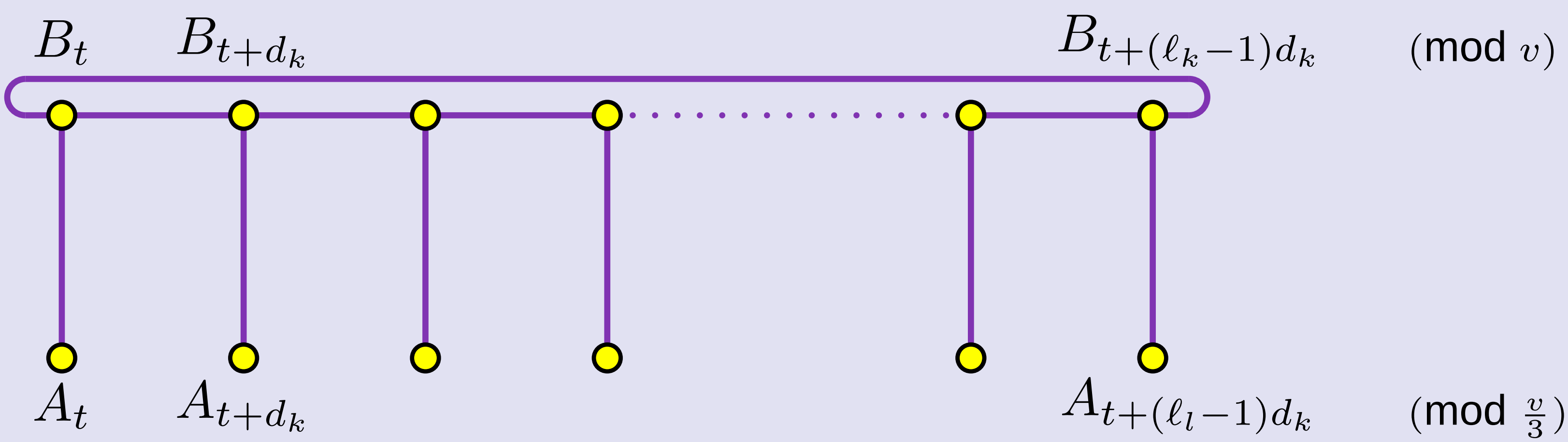
The orbits $\mathcal{O}_i$ and $\mathcal{S}$

Suppose $\mathcal{O}_i$ has orbital differences d_1, d_2 and d_3.

Consider the edges between blocks of $\mathcal{O}_i$ corresponding to a single difference d_k.

Lemma: these edges produce $\gcd(v, d_k)$ cycles of the form $(B_t, B_{t+d_k}, B_{t+2d_k}, \dots, B_{t+(\ell_k-1)d_k})$, where $\ell_k = \frac{v}{\gcd(v,d_k)}$.

Let $A_t \in \mathcal{S}$ be a neighbour of B_t.
Then we obtain a configuration:



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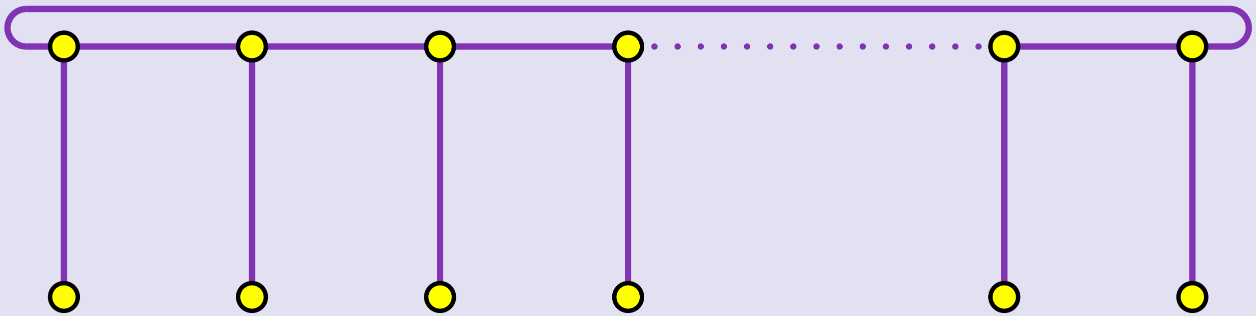


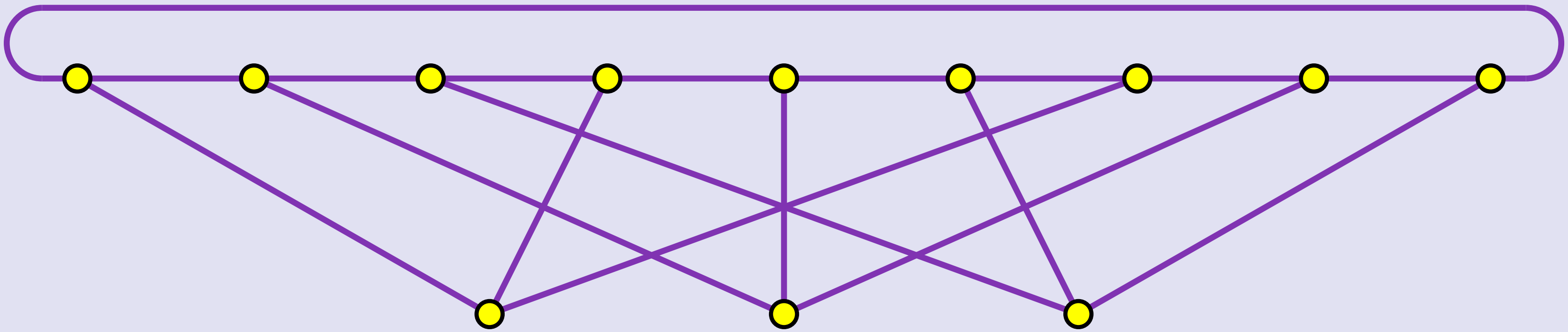
When $A_t, A_{t+d_k}, \ldots, A_{t+(\ell-1)d_k}$ are distinct, then it is easy to 3-colour the configurations arising from d_k.

Using colours 1, 2 and 3, first colour the cycle and then colour the pendant edges.

Apply the permutation $\sigma = (1, 2, 3)$ to the next configuration with the same "A" blocks and σ^2 to the remaining such configuration.

The orbits $\mathcal{O}_i$ and $\mathcal{S}$

When $A_t, A_{t+d_k}, \ldots, A_{t+(\ell-1)d_k}$ are not distinct, then the configuration looks something like this:



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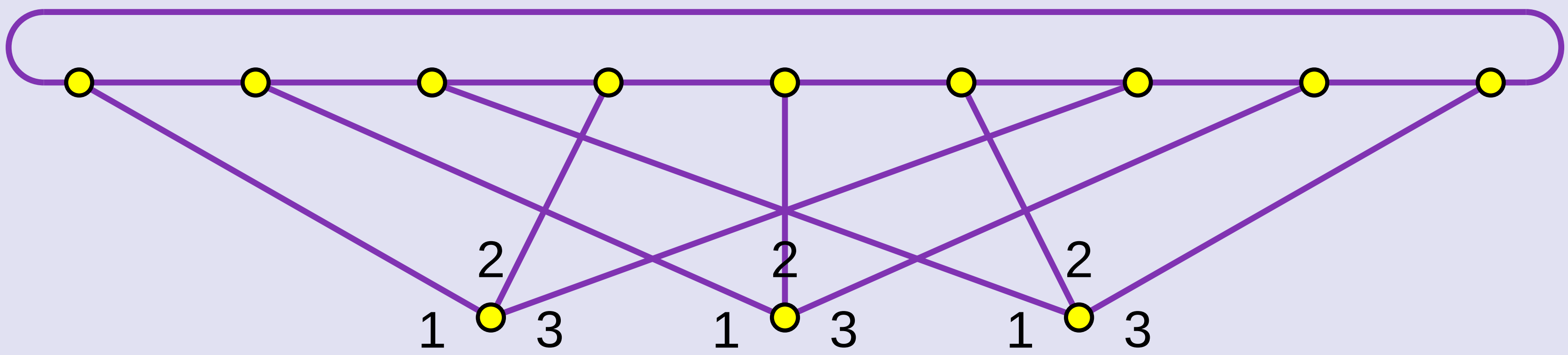

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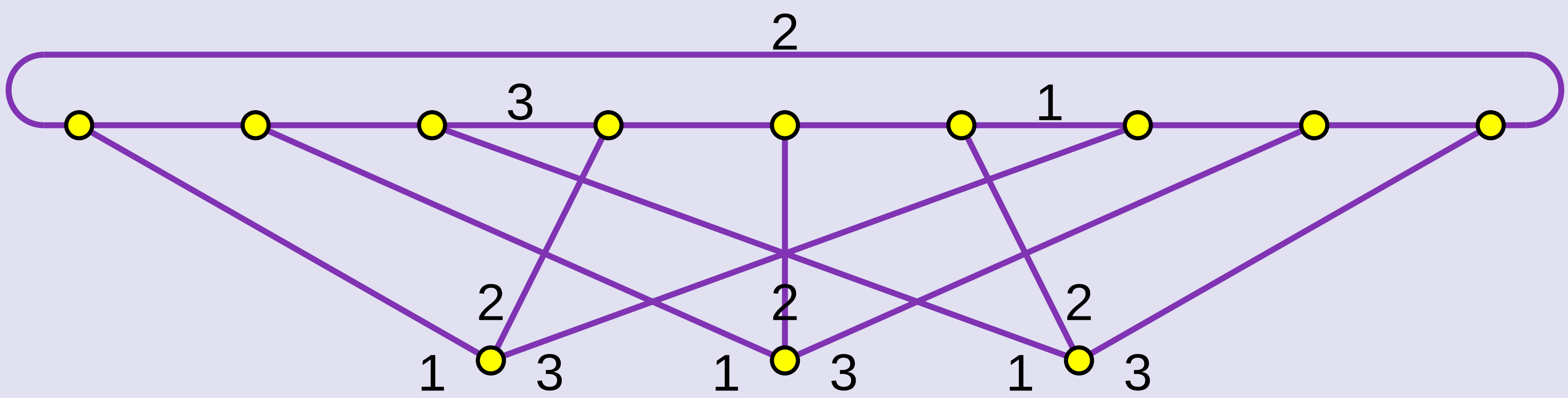

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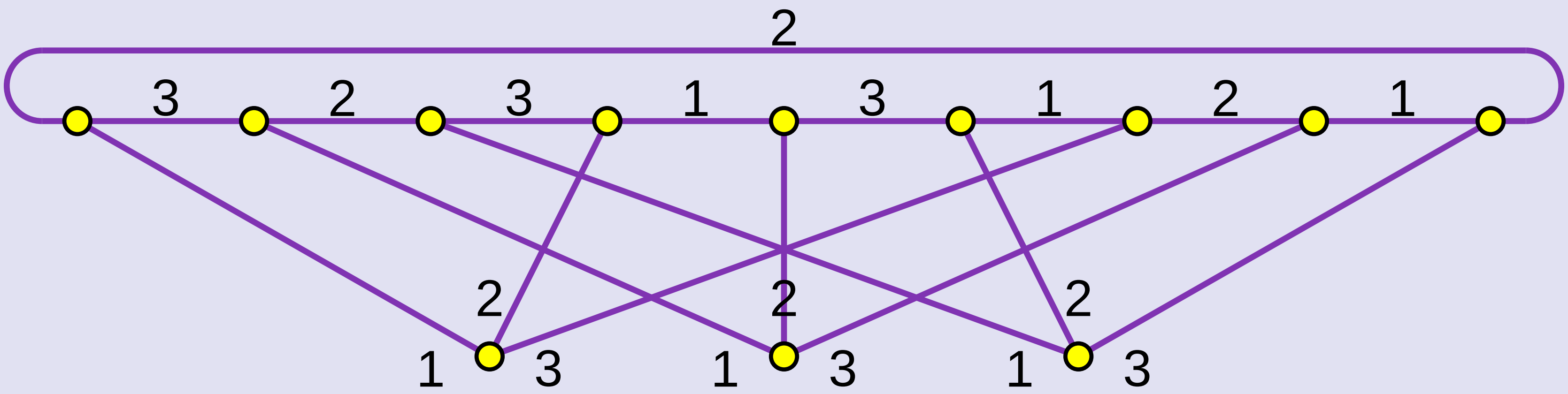

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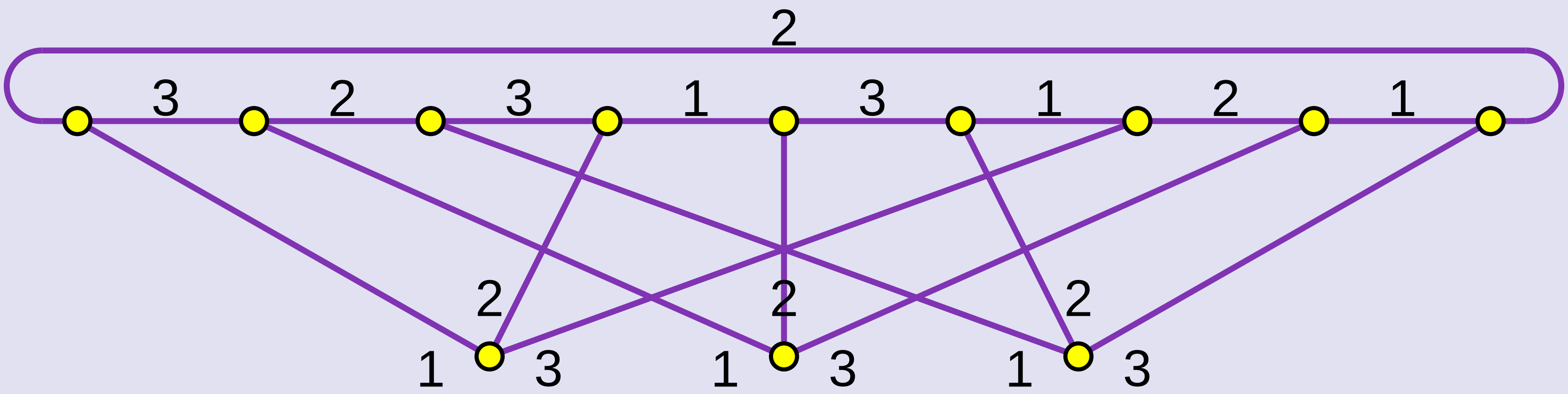

So we can colour the configurations for d_1 with colours 1, 2, 3.
We can colour the configurations for d_2 with colours 4, 5, 6.
And we can colour the configurations for d_3 with colours 7, 8, 9.

QED

Further Research:

- What about the remaining cyclic STS(v) with $v \equiv 1 \pmod{12}$, namely those for which $\frac{1}{2} \leqslant \rho(v) < \frac{2}{3}$?
- What about non-cyclic STS(v)?
- What can be said about BIBDs and strongly regular graphs?

Thank You

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Memorial University of Newfoundland

PROVEHITO IN ALTUM

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