Introduction to Priestley duality

Outline

What is a distributive lattice?

Priestley duality for finite distributive lattices

Using the duality: an example

Priestley duality for infinite distributive lattices

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Three classes of algebras

1. Groups $(*, {}^{-1}, e)$

Defining equations

$(x * y) * z = x * (y * z)$

$x * e = x$

$x * x^{-1} = e$

Representation

A collection of permutations of a set, closed under

- composition $(*)$,
- inverse $({}^{-1})$,
- identity (e).

Three classes of algebras

2. Semigroups $(*)$

Defining equations

$(x * y) * z = x * (y * z)$

Representation

A collection of self-maps of a set, closed under

- composition $(*)$.

Three classes of algebras

3. Distributive lattices $(\vee, \wedge)$

Defining equations

$$(x \vee y) \vee z = x \vee (y \vee z)$$
$$(x \wedge y) \wedge z = x \wedge (y \wedge z)$$

$$x \vee y = y \vee x$$
$$x \wedge y = y \wedge x$$

$$x \vee x = x$$
$$x \wedge x = x$$

$$x \vee (x \wedge y) = x$$
$$x \wedge (x \vee y) = x$$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$
$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

Representation

A collection of subsets of a set, closed under

- union $(\vee)$,
- intersection $(\wedge)$.

Concrete examples of distributive lattices

1. All subsets of a set S:

   $\langle \wp(S); \cup, \cap \rangle$.

2. Finite and cofinite subsets of $\mathbb{N}$:

   $\langle \wp_{FC}(\mathbb{N}); \cup, \cap \rangle$.

3. Open subsets of a topological space $\mathbf{X}$:

   $\langle \mathcal{O}(\mathbf{X}); \cup, \cap \rangle$.

More examples of distributive lattices

4. $\langle \{T, F\}; \mathit{or}, \mathit{and} \rangle$.

5. $\langle \mathbb{N} \cup \{0\}; \mathit{lcm}, \mathit{gcd} \rangle$.

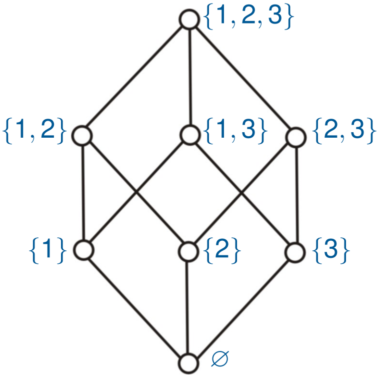
   (Represent a number as its set of prime-power divisors.)

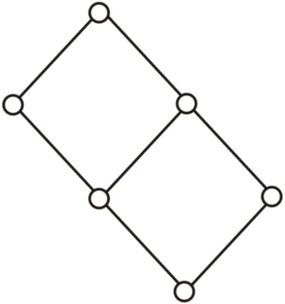
6. Subgroups of a cyclic group **G**,

   $\langle \mathrm{Sub}(\mathbf{G}); \vee, \cap \rangle$, where $H \vee K := \langle H \cup K \rangle$.

Drawing distributive lattices

Any distributive lattice $\langle L; \vee, \wedge \rangle$ has a natural order corresponding to set inclusion: $a \leqslant b \iff a \vee b = b$.





$\vee$ union
$\wedge$ intersection
$\leqslant$ inclusion

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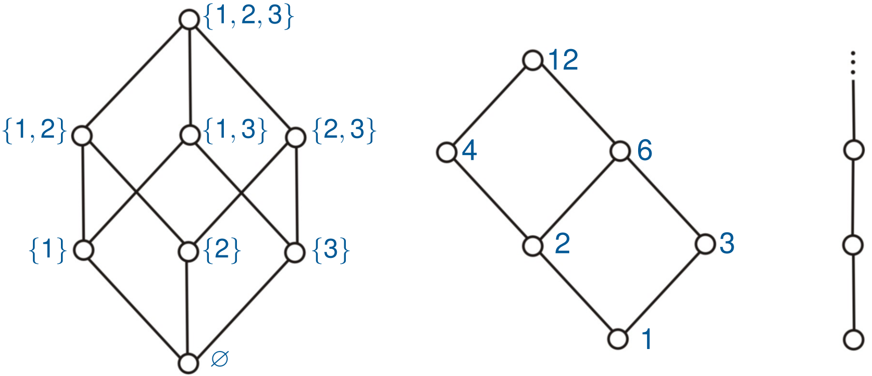

$\vee$ union
$\wedge$ intersection
$\leqslant$ inclusion

$\vee$ lcm
$\wedge$ gcd
$\leqslant$ division

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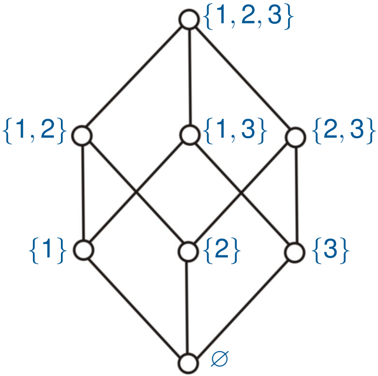
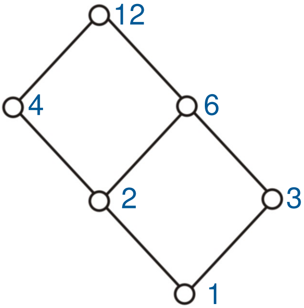


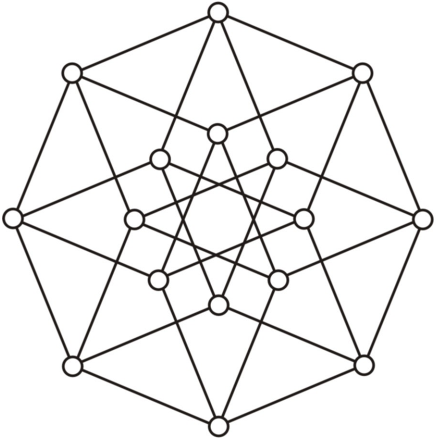


$\vee$ union
$\wedge$ intersection
$\leqslant$ inclusion

$\vee$ lcm
$\wedge$ gcd
$\leqslant$ division

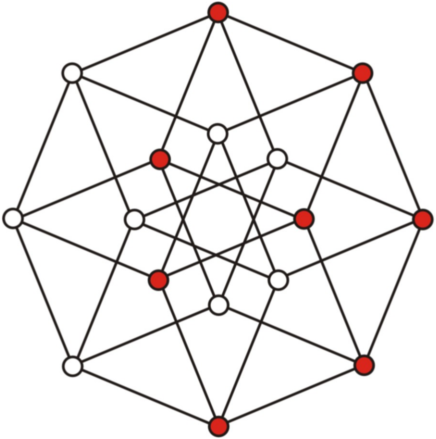
$\vee$ max
$\wedge$ min
$\leqslant$ usual

More pictures of distributive lattices



$\underline{\mathbf{2}}^4$

More pictures of distributive lattices



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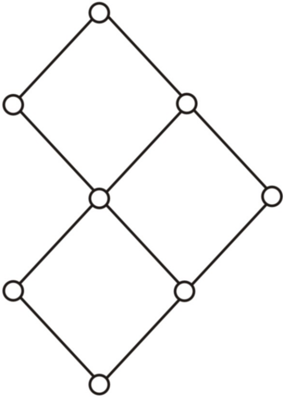

Note: Every distributive lattice embeds into $\underline{\mathbf{2}}^S$, for some set S.

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Representing finite distributive lattices

Original representation

A collection of subsets of a set,
closed under union and intersection.

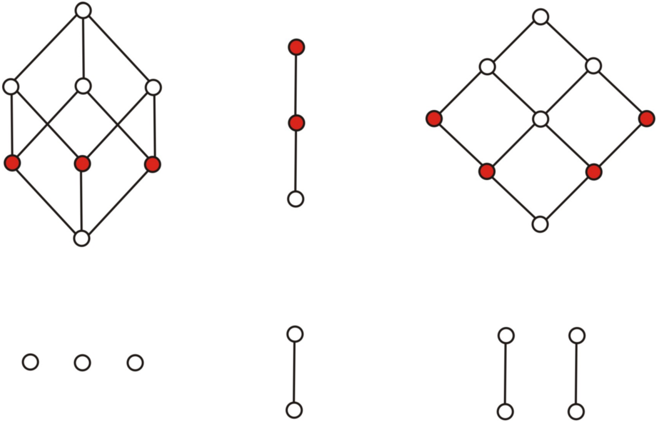
New representation

The collection of all down-sets of an ordered set,
under union and intersection.

More examples

Distributive lattice

Ordered set

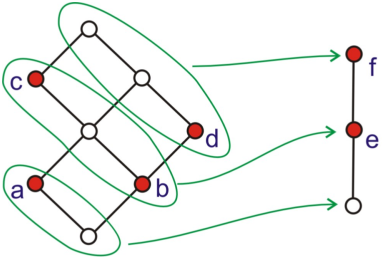


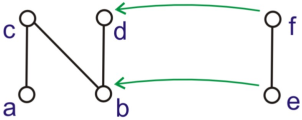
Duality for finite distributive lattices

The classes of

finite distributive lattices and finite ordered sets

are dually equivalent.





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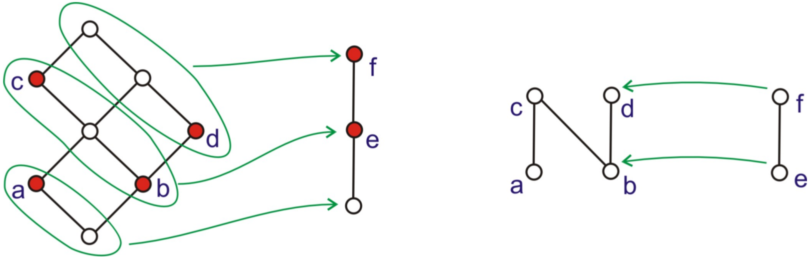

surjections	$\longleftrightarrow$	embeddings
embeddings	$\longleftrightarrow$	surjections
products	$\longleftrightarrow$	disjoint unions

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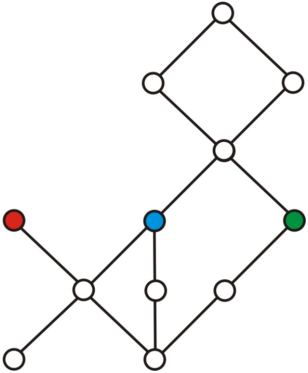
Dilworth's Theorem for Ordered Sets

Let **P** *be a finite ordered set.*

The minimum number of chains needed to cover **P** *is equal to the width of* **P** *(i.e. the maximum size of an anti-chain in* **P***).*



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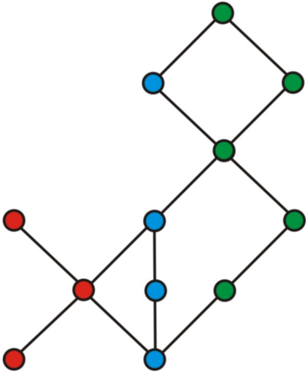
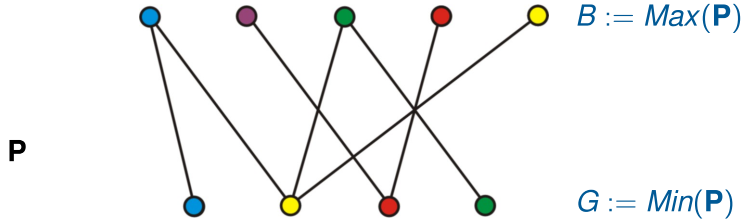


Aside: Hall's Marriage Theorem

Let **P** *be an ordered set of height* 1*.*



Assume that $|S| \leqslant |{\uparrow}S \setminus S|$*, for each* $S \subseteq G$*.*

Then **P** *can be covered by* $|B|$ *chains*

(i.e., each girl can be paired with a boy she likes).

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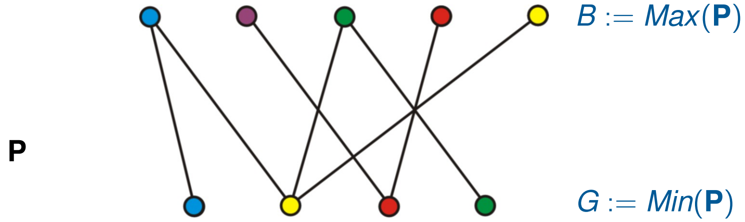

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Then **P** *can be covered by* $|B|$ *chains*

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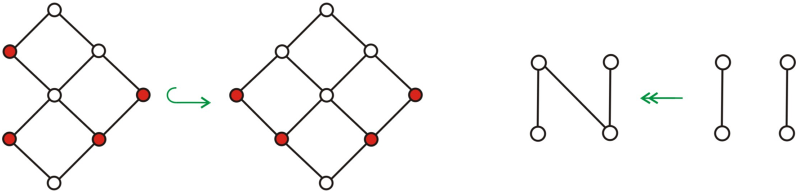
Proof.

Using Dilworth's Theorem, we just need to show that **P** has width $|B|$. □

Dual version of Dilworth's Theorem

Let **L** *be a finite distributive lattice.*

The smallest n such that **L** *embeds into a product of n chains is exactly the width of the join-irreducibles of* **L**.



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Infinite distributive lattices

Example

The finite-cofinite lattice $\langle \wp_{FC}(\mathbb{N}); \cup, \cap \rangle$ cannot be obtained as the down-sets of an ordered set.

Infinite distributive lattices

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The finite-cofinite lattice $\langle \wp_{FC}(\mathbb{N}); \cup, \cap \rangle$ cannot be obtained as the down-sets of an ordered set.

Proof.

- The ordered set would have to be an anti-chain.

Infinite distributive lattices

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The finite-cofinite lattice $\langle \wp_{FC}(\mathbb{N}); \cup, \cap \rangle$ cannot be obtained as the down-sets of an ordered set.

Proof.

- The ordered set would have to be an anti-chain.
- The ordered set would have to be infinite.

Infinite distributive lattices

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The finite-cofinite lattice $\langle \wp_{FC}(\mathbb{N}); \cup, \cap \rangle$ cannot be obtained as the down-sets of an ordered set.

Proof.

- The ordered set would have to be an anti-chain.
- The ordered set would have to be infinite.
- So there would be at least $2^{\mathbb{N}}$ down-sets.

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- But $\wp_{FC}(\mathbb{N})$ is countable. □

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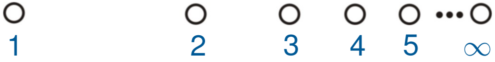
- The ordered set would have to be an anti-chain.
- The ordered set would have to be infinite.
- So there would be at least $2^{\mathbb{N}}$ down-sets.
- But $\wp_{FC}(\mathbb{N})$ is countable. $\square$

But it can be obtained as the clopen down-sets of a topological ordered set.



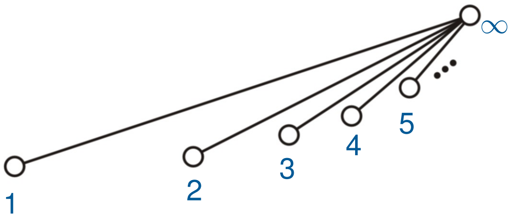
More examples

Distributive lattice:

All finite subsets of $\mathbb{N}$, as well as $\mathbb{N}$ itself,

$\langle \wp_{fin}(\mathbb{N}) \cup \{\mathbb{N}\}; \cup, \cap \rangle$.

Topological ordered set:

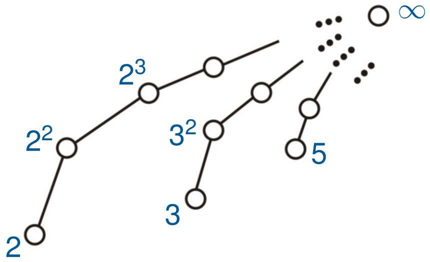


More examples

Distributive lattice:

$$\langle \mathbb{N} \cup \{0\};\ lcm, gcd \rangle.$$

Topological ordered set:



Setting up Priestley duality

1. From distributive lattices to topological ordered sets

Let $\mathbf{L} = \langle L; \vee, \wedge \rangle$ be a distributive lattice.

Define the dual of $\mathbf{L}$ by

$$D(\mathbf{L}) := \text{hom}(\mathbf{L}, \underline{\mathbf{2}}) \leqslant \underset{\sim}{\mathbf{2}}^{L},$$

where

- $\underline{\mathbf{2}}$ is the two-element lattice with $0 \leqslant 1$,
- $\underset{\sim}{\mathbf{2}}$ is the two-element discrete ordered set with $0 \leqslant 1$.

Setting up Priestley duality

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- $\underline{\mathbf{2}}$ is the two-element lattice with $0 \leqslant 1$,
- $\underset{\sim}{\mathbf{2}}$ is the two-element discrete ordered set with $0 \leqslant 1$.

Note
The topological ordered sets obtained in this way are called Priestley spaces.

Setting up Priestley duality, continued

2. From Priestley spaces to distributive lattices

Let $\mathbf{X} = \langle X; \leqslant, \mathcal{T} \rangle$ be a Priestley space.

Define the dual of $\mathbf{X}$ by

$$E(\mathbf{X}) := \hom(\mathbf{X}, \utilde{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^X.$$

Setting up Priestley duality, continued

2. From Priestley spaces to distributive lattices

Let $\mathbf{X} = \langle X; \leqslant, \mathcal{T} \rangle$ be a Priestley space.

Define the dual of **X** by

$$E(\mathbf{X}) := \hom(\mathbf{X}, \underset{\sim}{\mathbf{2}}) \leqslant \underline{\mathbf{2}}^X.$$

3. The duality

Every distributive lattice is encoded by a Priestley space:

$$ED(\mathbf{L}) \cong \mathbf{L} \quad \text{and} \quad DE(\mathbf{X}) \cong \mathbf{X},$$

for each distributive lattice **L** and Priestley space **X**.

Indeed, the classes of distributive lattices and Priestley spaces are dually equivalent.

Natural dualities in general

1. Distributive lattices $\leftrightarrow$ Priestley spaces
   $\underline{\mathbf{2}} = \langle \{0,1\}; \vee, \wedge \rangle$ $\qquad$ $\underset{\sim}{\mathbf{2}} = \langle \{0,1\}; \leqslant, \mathcal{T} \rangle$
2. Abelian groups $\leftrightarrow$ Compact abelian groups
   $\underline{\mathbf{A}} = \langle S^1; \cdot, ^{-1}, 1 \rangle$ $\qquad$ $\underset{\sim}{\mathbf{A}} = \langle S^1; \cdot, ^{-1}, 1, \mathcal{T} \rangle$
3. Boolean algebras $\leftrightarrow$ Boolean spaces
   $\underline{\mathbf{B}} = \langle \{0,1\}; \vee, \wedge, \neg \rangle$ $\qquad$ $\underset{\sim}{\mathbf{B}} = \langle \{0,1\}; \mathcal{T} \rangle$