Octonions

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The octonions or Cayley numbers were discovered independently by Cayley in 1845 and Graves in 1848.

An **algebra** is a vector space (today, over the reals) with an additional bilinear operation which we call multiplication and denote by juxtaposition.

There are three levels of associativity:

- An algebra is **power-associative** if every subalgebra generated by one element is associative.
- It is **alternative** if every subalgebra generated by two elements is associative.
- An algebra is associative if every subalgebra generated by three elements is associative.

An algebra is a **division algebra** if $ab = 0$ implies either $a = 0$ or $b = 0$.

A *-**algebra** is an algebra $\mathbf{A}$ with an involution ($a^{**} = a$, $\mathbf{R}$-linear, $(ab)^* = b^*a^*$).

A *-algebra $\mathbf{A}$ is **real** if $a^* = a$ for all $a \in \mathbf{A}$.

A unital *-algebra $\mathbf{A}$ is called **nicely normed** if $a + a^* \in \mathbf{R}$ and $aa^* = a^*a > 0$ for all non-zero $a \in \mathbf{A}$.

Given a *-algebra $\mathbf{A}$, its **Cayley-Dickson double** is $\mathbf{A}^2 = \mathbf{A} \times \mathbf{A}$ with the product $(a,b)(c,d) = (ac - db^*, a^*d + cb)$ and involution $(a,b)^* = (a^*, -b)$.

Example *1*.

1. $\mathbf{R}$ *is a real *-algebra and its Cayley-Dickson double is* $\mathbf{C}$.
   $1 = (1,0)$ *and* $i = (0,1)$; $(\alpha + \beta i)^* = \alpha - \beta i$

2. *The Cayley-Dickson double of* $\mathbf{C}$ *is the quaternions* $\mathbf{H}$.
   $1 = (1,0)$, $i = (0,1)$, $j = (i,0)$, $k = (0,i)$,
   $(\alpha + \beta i + \gamma j + \delta k)^* = \alpha - \beta i - \gamma j - \delta k$

3. *The Cayley-Dickson double of* $\mathbf{H}$ *is the octonions* $\mathbf{O}$.
   $1 = (1,0)$, $b_1 = (0,1)$, $b_2 = (i,0)$,
   $b_3 = (0,i)$, $b_4 = (j,0)$, $b_5 = (0,j)$,
   $b_6 = (k,0)$, $b_7 = (0,-k)$
   $b_j^* = b_j$

4. *The Cayley-Dickson double of* $\mathbf{O}$ *is the sedenions* $\mathbf{O}^2$.

A simple calculation gives (#)

$$(a,b)^*(a,b) = (a^*a + bb^*, 0)$$
$$(a,b)(a,b)^* = (aa^* + bb^*, 0)$$

Proposition *2. Let* $\mathbf{A}$ *be a unital *-algebra.*

1. $\mathbf{A}^2$ *is a unital *-algebra with unit* $(1,0)$.
2. *The map* $a \mapsto (a,0) : \mathbf{A} \to \mathbf{A}^2$ *is a homomorphic embedding.*
3. *If* $\mathbf{A}$ *is nicely normed, then*
   a. $\|a\| = \sqrt{aa^*}$ *defines a norm on* $\mathbf{A}$.
   b. *non-zero elements have multiplicative inverses.*
4. *If* $\mathbf{A}$ *is nicely normed and alternative, then*
   a. $\|ab\| = \|a\|\|b\|$
   b. $\mathbf{A}$ *is a division algebra.*

Theorem *3. Let* $\mathbf{A}$ *be a *-algebra.*

1. $\mathbf{A}^2$ is never real.
2. $\mathbf{A}$ is nicely normed $\Leftrightarrow$ $\mathbf{A}^2$ is nicely normed.
3. $\mathbf{A}$ is real (and thus commutative) $\Leftrightarrow$ $\mathbf{A}^2$ is commutative.
4. $\mathbf{A}$ is commutative and associative $\Leftrightarrow$ $\mathbf{A}^2$ is associative.
5. $\mathbf{A}$ is associative and nicely normed $\Leftrightarrow$ $\mathbf{A}^2$ is alternative & nicely normed.

Corollary *4.*

1. $\mathbf{R}, \mathbf{C}, \mathbf{H}$ *and* $\mathbf{O}$ *are all multiplicatively normed division algebras.*
2. $\mathbf{O}$ *is not real or commutative or associative, but it is alternative.*
3. $\mathbf{O}^2$ *is not real or commutative or alternative, (but it is power associative).*

Theorem *5.*

1. (Hurwitz 1898) $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathbf{O}$ are the only multiplicatively normed division algebras.
2. (Zorn 1930) $\mathbf{R}, \mathbf{C}, \mathbf{H}$ and $\mathbf{O}$ are the only alternative division algebras.
3. (Kervaire, Bott-Milnor 1958) All division algebras have dimension $1, 2, 4$ or 8.

Comparison with Clifford Algebras

The **Clifford algebra** $\mathbf{Cl(n)} = \mathbf{Cl(0,n,0)}$ is a unital associative algebra with generators $e_1, e_2, \ldots e_n$ satisfying the relations $e_j^2 = -1$ and $e_j e_k = -e_k e_j$ for $j \neq k$.

Proposition *6.*

1. $\mathbf{Cl(0)} = \mathbf{R}$
2. $\mathbf{Cl(1)} = \mathbf{C}$
3. $\mathbf{Cl(2)} = \mathbf{H}$
4. $\mathbf{Cl(3)} \neq \mathbf{O}$

Remark *7.*

1. $\mathbf{O}$ and $\mathbf{Cl(3)}$ each have generators e_1, e_2, e_3.
2. $\mathbf{O}$ and $\mathbf{Cl(3)}$ each have basis elements $1, e_1, e_2, e_1e_2, e_3, e_1e_3, e_2e_3, (e_1e_2)e_3$.
3. Renaming these elements respectively $1, b_1, b_2, b_3, b_4, b_5, b_6, b_7$ we obtain multiplication tables for the basis elements of $\mathbf{O}$ and $\mathbf{Cl(3)}$ respectively as follows:

1	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	-1	b_3	$-b_2$	b_5	$-b_4$	$-b_7$	b_6
b_2	$-b_3$	-1	b_1	b_6	b_7	$-b_4$	$-b_5$
b_3	b_2	$-b_1$	-1	b_7	$-b_6$	b_5	$-b_4$
b_4	$-b_5$	$-b_6$	$-b_7$	-1	b_1	b_2	b_3
b_5	b_4	$-b_7$	b_6	$-b_1$	-1	$-b_3$	b_2
b_6	b_7	b_4	$-b_5$	$-b_2$	b_3	-1	$-b_1$
b_7	$-b_6$	b_5	b_4	$-b_3$	$-b_2$	b_1	-1

1	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	-1	b_3	$-b_2$	b_5	$-b_4$	b_7	$-b_6$
b_2	$-b_3$	-1	b_1	b_6	$-b_7$	$-b_4$	b_5
b_3	b_2	$-b_1$	-1	b_7	b_6	$-b_5$	$-b_4$
b_4	$-b_5$	$-b_6$	b_7	-1	b_1	b_2	$-b_3$
b_5	b_4	$-b_7$	$-b_6$	$-b_1$	-1	b_3	b_2
b_6	b_7	b_4	b_5	$-b_2$	$-b_3$	-1	$-b_1$
b_7	$-b_6$	b_5	$-b_4$	$-b_3$	b_2	$-b_1$	1

4. $\mathbf{O}$ is an alternative division algebra
   $\mathbf{Cl(3)}$ is associative but has zero divisors, for example $(1 + b_7)(1 - b_7) = 0$.

5. In $\mathbf{O}$ the involution satisfies $b_j^* = -b_j$ for all j.
   In $\mathbf{Cl(3)}$ an involution satisfies $b_j^* = -b_j$ for all $j \neq 7$ and $b_7^* = b_7$.

Twisted group algebras

Consider the additive group $\mathbf{Z}_2^3$ whose elements we label

$1 = (0,0,0), b_1 = (1,0,0), b_2 = (0,1,0),$
$b_3 = (1,1,0), b_4 = (0,0,1), b_5 = (1,0,1),$
$b_6 = (0,1,1), b_7 = (1,1,1).$

This group has multiplication table

1	b_1	b_2	b_3	b_4	b_5	b_6	b_7
b_1	1	b_3	b_2	b_5	b_4	b_7	b_6
b_2	b_3	1	b_1	b_6	b_7	b_4	b_5
b_3	b_2	b_1	1	b_7	b_6	b_5	b_4
b_4	b_5	b_6	b_7	1	b_1	b_2	b_3
b_5	b_4	b_7	b_6	b_1	1	b_3	b_2
b_6	b_7	b_4	b_5	b_2	b_3	1	b_1
b_7	b_6	b_5	b_4	b_3	b_2	b_1	1

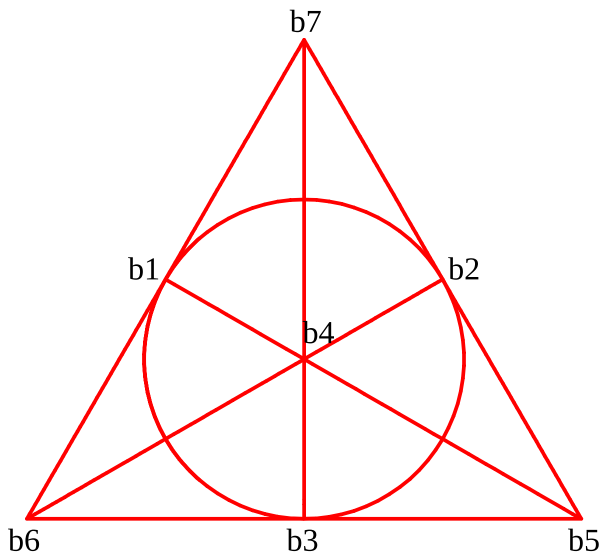
The group algebra $\mathbf{R}(\mathbf{Z}_2^3) = span\{1, b_1, b_2, b_3, b_4, b_5, b_6, b_7\}$ is the associative algebra of all formal linear combinations with product induced by the product in $G = \mathbf{Z}_2^3$. Note that $\mathbf{R}(\mathbf{Z}_2^3)$ is commutative, but not a division ring because $(1 + b_1)(1 - b_1) = 0$.

Any function $\alpha : G \times G \to \{\pm 1\}$ induces a new product on $\mathbf{R}(G)$ given by $a \times b = \alpha(a,b)ab$ for $a, b \in G$. The new algebra, not necessarily associative, is called a **twisted group algebra**. So

Proposition *8.* $\mathbf{O}$ *and* $\mathbf{Cl}(3)$ *are twisted group algebras of the group* $\mathbf{Z}_2^3$ *over* $\mathbf{R}$.

The Fano plane and O

The non-zero elements of $\mathbf{Z}_2^3$ are also the points of the seven point projective plane.



Copies of pure quaternions:

(b_7, b_2, b_5)
(b_5, b_3, b_6)
(b_6, b_1, b_7)
(b_1, b_2, b_3)
(b_1, b_4, b_5)
(b_2, b_4, b_6)
(b_3, b_4, b_7)

Left regular representations

If $\mathbf{A} = \mathbf{R}, \mathbf{C}, \mathbf{H}, \mathbf{O}, \mathbf{Cl}(\mathbf{3})$ or $\mathbf{R}(\mathbf{Z}_2^3)$ and $n = \dim(\mathbf{A})$, the **coordinate map** $\mu : \mathbf{A} \to \mathbf{R}^n$ is a linear isomorphism. For each $a \in \mathbf{A}$ there is a unique matrix $L_a \in M_n(\mathbf{R})$ such that $\mu(ab) = L_a \mu(b)$. The map $\psi : \mathbf{A} \to M_n(\mathbf{R}) : a \mapsto L_a$ is the **left regular representation**.

Remark *9.*

1. $\psi : \mathbf{A} \to M_n(\mathbf{R})$ is injective and linear.
2. $\psi(1) = I$.
3. For the associative algebras, ψ is an algebra homomorphism.
4. For the *-algebras, ψ is a *-map.
5. For the division algebras, $\psi(aa^*) = \psi(a)\psi(a^*) = \|a\|^2 I$.
6. For the division algebras, $\psi(\mathbf{A})$ is a linear subspace consisting of invertible matrices and zero.

Example *10.*

1. *If* $z = \alpha + \beta i \in \mathbf{C}$ *then* $\mu(z) = (\alpha, \beta)$,

$$\psi(z) = \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix} \text{ and}$$

$$\psi(z^*) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \psi(z)^T.$$

2. *If* $z = \alpha + \beta i + \gamma j + \delta k \in \mathbf{H}$ *then*
$\mu(z) = (\alpha, \beta, \gamma, \delta)$,

$$\psi(z) = \begin{bmatrix} \alpha & -\beta & -\gamma & -\delta \\ \beta & \alpha & -\delta & \gamma \\ \gamma & \delta & \alpha & -\beta \\ \delta & -\gamma & \beta & \alpha \end{bmatrix}$$

and again $\psi(z^*) = \psi(z)^T$.

3. *If* $z = \alpha_0 + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_7 b_7 \in \mathbf{O}$
then $\mu(z) = (\alpha_0, \alpha_1, \ldots, \alpha_7)$,

$\psi(z) =$

$$\begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & -\alpha_7 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 & -\alpha_5 & \alpha_4 & \alpha_7 & -\alpha_6 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 & -\alpha_6 & -\alpha_7 & \alpha_4 & \alpha_5 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 & -\alpha_7 & \alpha_6 & -\alpha_5 & \alpha_4 \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_5 & -\alpha_4 & \alpha_7 & -\alpha_6 & \alpha_1 & \alpha_0 & \alpha_3 & -\alpha_2 \\ \alpha_6 & -\alpha_7 & -\alpha_4 & \alpha_5 & \alpha_2 & -\alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_6 & -\alpha_5 & -\alpha_4 & \alpha_3 & \alpha_2 & -\alpha_1 & \alpha_0 \end{bmatrix}$$

and again $\psi(z^*) = \psi(z)^T$.

4. *If* $z = \alpha_0 + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_7 b_7 \in \mathbf{Cl(3)}$
then $\mu(z) = (\alpha_0, \alpha_1, \ldots, \alpha_7)$,

$\psi(z) =$

$$\begin{bmatrix} \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 & -\alpha_5 & -\alpha_6 & \alpha_7 \\ \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 & -\alpha_5 & \alpha_4 & -\alpha_7 & -\alpha_6 \\ \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 & -\alpha_6 & \alpha_7 & \alpha_4 & \alpha_5 \\ \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 & -\alpha_7 & -\alpha_6 & \alpha_5 & -\alpha_4 \\ \alpha_4 & \alpha_5 & \alpha_6 & -\alpha_7 & \alpha_0 & -\alpha_1 & -\alpha_2 & -\alpha_3 \\ \alpha_5 & -\alpha_4 & \alpha_7 & \alpha_6 & \alpha_1 & \alpha_0 & -\alpha_3 & \alpha_2 \\ \alpha_6 & -\alpha_7 & -\alpha_4 & -\alpha_5 & \alpha_2 & \alpha_3 & \alpha_0 & -\alpha_1 \\ \alpha_7 & \alpha_6 & -\alpha_5 & \alpha_4 & \alpha_3 & -\alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}$$

and again $\psi(z^*) = \psi(z)^T$.

5. *If* $z = \alpha_0 + \alpha_1 b_1 + \alpha_2 b_2 + \ldots + \alpha_7 b_7 \in \mathbf{R}(\mathbf{Z}_2^3)$

then $\mu(z) = (\alpha_0, \alpha_1, \ldots, \alpha_7)$,

$$\psi(z) = \begin{bmatrix} \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 \\ \alpha_1 & \alpha_0 & \alpha_3 & \alpha_2 & \alpha_5 & \alpha_4 & \alpha_7 & \alpha_6 \\ \alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 & \alpha_6 & \alpha_7 & \alpha_4 & \alpha_5 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 & \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 \\ \alpha_4 & \alpha_5 & \alpha_6 & \alpha_7 & \alpha_0 & \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_5 & \alpha_4 & \alpha_7 & \alpha_6 & \alpha_1 & \alpha_0 & \alpha_3 & \alpha_2 \\ \alpha_6 & \alpha_7 & \alpha_4 & \alpha_5 & \alpha_2 & \alpha_3 & \alpha_0 & \alpha_1 \\ \alpha_7 & \alpha_6 & \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \end{bmatrix}$$

References

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