Quaternions

Alan Pryde 24/11/11

1. Introduction

The set **H** of quaternions was first described by William Hamilton in 1843. It is defined to be the associative algebra over the reals generated by the four elements $1, i, j, k$ with the relations $i^2 = j^2 = k^2 = ijk = -1$ and 1 is an identity element.

Claim *1*.

(i) $ij = k = -ji$

(ii) $jk = i = -kj$

(iii) $ki = j = -ik$

So quaternions are objects of the form $X = x_0 + x_1 i + x_2 j + x_3 k = x_0 + x$ where the coefficients $x_j \in \mathbf{R}$. The coefficient $x_0 = Re(X)$ is called the scalar or real part of X and $x = x_1 i + x_2 j + x_3 k = Pu(X)$ is the vector or purely quaternionic part. As vector spaces, $\mathbf{H} = \mathbf{R}^4 = \mathbf{R} + \mathbf{R}^3$, and we can interpret i, j, k as the standard basis vectors of $\mathbf{R}^3$.

H inherits the standard Euclidean **norm** and **inner product** from $\mathbf{R}^4$. So $\|X\| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ and $X \bullet Y = x_0 y_0 + x_1 y_1 + x_2 y_2 + x_3 y_3$.

There is also an **involution** given by $X^* = x_0 - x_1 i - x_2 j - x_3 k = x_0 - x$.

On pure quaternions there is also the standard vector cross product

$$x \times y = \det\begin{bmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}$$

Claim *2*.

(i) $Re(X) = \frac{1}{2}(X + X^*)$.

(ii) $Pu(X) = \frac{1}{2}(X - X^*)$.

(iii) *X is real if and only if $X = X^*$.*

(iv) *X is purely quaternionic if and only if $X = -X^*$.*

Claim *3*. *Take $x, y, z \in \mathbf{R}^3$.*

(i) $xy = -x \bullet y + x \times y$.

(ii) *Every $x \in S^2 \subset \mathbf{R}^3$ satisfies $x^2 = -1$.*

(iii) $x \times y = \frac{1}{2}(xy - yx)$.

(iv) $xyx = \|x\|^2 y - 2(x \cdot y)x$

(v) $\mathrm{Re}(xyz) = -x \cdot (y \times z) = -\det(x, y, z)$.

Claim *4.*

(i) $(XY)^* = Y^* X^*$.

(ii) $X^* X = \|X\|^2$.

(iii) $\|XY\| = \|X\| \|Y\|$.

(iv) *Each non-zero X is invertible with* $X^{-1} = X^*/\|X\|^2$.

(v) *Each quaternion X is the product of 2 pure quaternions.*

Claim *5.*

(i) $\mathbf{H}$ *is a four-dimensional associative division algebra over the reals.*

(ii) $\mathbf{H}$ *is a* C^**-algebra (a Banach algebra with involution satisfying* $\|X^* X\| = \|X\|^2$*).*

Theorem *6. (**Frobenius**, **1878**) .The only finite dimensional associative division algebras over the reals are* $\mathbf{R}, \mathbf{C}$ *and* $\mathbf{H}$.

Theorem *7. (**Hurwitz**, **1898**) .The only finite dimensional multiplicatively normed division algebras over the reals are* $\mathbf{R}, \mathbf{C}, \mathbf{H}$ *and* $\mathbf{O}$.

Claim *8.*

(i) *The set* $\mathbf{H}^{\#}$ *of non-zero quaternions is a group.*

(ii) *The set of unit quaternions coincides with the unit sphere* S^3 *in* $\mathbf{R}^4$ *and is a subgroup of* $\mathbf{H}^{\#}$.

2. Matrix representations

Define the maps $\varphi : \mathbf{H} \to \mathbf{M}_2(\mathbf{C})$ by $\varphi(X) = \begin{bmatrix} x_0 + x_1 i & x_2 + x_3 i \\ -x_2 + x_3 i & x_0 - x_1 i \end{bmatrix}$

and $\psi : \mathbf{H} \to \mathbf{M}_4(\mathbf{R})$ by $\psi(X) = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 \\ -x_1 & x_0 & -x_3 & x_2 \\ -x_2 & x_3 & x_0 & -x_1 \\ -x_3 & -x_2 & x_1 & x_0 \end{bmatrix}$.

Claim *9.*

(i) $\varphi : \mathbf{H} \to \mathbf{M}_2(\mathbf{C})$ *is an injective *-homomorphism of algebras.*

(ii) $\det(\varphi(X)) = \|X\|^2$.

(iii) *Restricted to the unit quaternions we get a *-isomorphism* $\varphi : S^3 \to SU(2)$.

Claim *10.*

(i) $\psi : \mathbf{H} \to \mathbf{M}_4(\mathbf{R})$. *is an injective *-homomorphism of algebras.*

(ii) $\det(\psi(X)) = \|X\|^4$.

(iii) *Restricted to the unit quaternions we get an injective *-homomorphism* .$\psi : S^3 \to SO(4)$.

3. Quaternions and rotations in 3-space.

Theorem ***11***. *(**Cartan–Dieudonné**) An element of $O(3)$ is a rotation if and only if it is the composite of two planar reflections.*

Take $X \in S^3 \subset \mathbf{H}$. .So,. X is a unit quaternion and $XX^* = X^*X = \|X\|^2 = 1$. .Now define a map $\rho_X : \mathbf{H} \to \mathbf{H}$ by $\rho_X(Y) = XYX^*$.

Claim ***12***. *Let X be a unit quaternion.*

(i) $\rho_X : \mathbf{H} \to \mathbf{H}$ *is an injective *-homomorphism of algebras.*

(ii) $\rho_X : \mathbf{R}^3 \to \mathbf{R}^3$.

(iii) *If x is a unit pure quaternion, then* $-\rho_x : \mathbf{R}^3 \to \mathbf{R}^3$ *is reflection in the plane* $x^\perp$.

(iv) *In general,* $\rho_X : \mathbf{R}^3 \to \mathbf{R}^3$ *is a rotation, that is* $\rho_X \in SO(3)$.

Now define the map .$\rho : S^3 \to SO(3)$ .by. $\rho(X) = \rho_X$.

Claim ***13***. $\rho : S^3 \to SO(3)$ *.is a surjective group homomorphism with kernel.* $S^0 = \{1, -1\}$.

So the group of unit quaternions .S^3 is a double cover of the special orthogonal group. $SO(3)$. This is the definition of the spin group $Spin(3)$. .So $Spin(3) = S^3 = SU(2)$. Note that $Spin(3)$ is a simply connected Lie group. On the other hand $SO(3)$ is connected but not simply connected. Its fundamental group is $\mathbf{Z}_2 = S^0$.

Finally, .for a unit quaternion X, we can write
$X = x_0 + x = x_0 + \|x\|\,\hat{x} = \cos\alpha + \sin\alpha\,\hat{x}$ where $0 \le \alpha < \pi$.

Claim ***14***. *For* $X \in S^3$, *the map* $\rho_X : \mathbf{R}^3 \to \mathbf{R}^3$ *is rotation through the angle* $2\alpha = 2\arccos(\mathrm{Re}\,X)$ *about the axis given by* $x = Pu(X)$.

Reference

I.R. Porteous "Clifford algebras and the classical groups" (Cambridge, 1995)