Diagrammatically maximal and geometrically maximal knots

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Part 1: Knots from a combinatorial point of view
Knot diagram: 4–valent planar graph with over–under crossing info at each vertex.

Alternating knot: Crossings alternate between over and under.

Diagram graph: Graph obtained by dropping crossing decoration on $K$, denoted $G(K)$. 
Given any 4-valent graph $G$, $\exists !$ alternating knot $K$ with $G(K) = G$ (up to reflection).
Reduced alternating diagram

Throughout, all diagrams are connected (i.e. $G(K)$ connected)

Diagram is *reduced* if it has no *nugatory crossings*:

Undo nugatory crossings
Associated to any diagram $K$ is a *Tait graph* $\Gamma_K$:

- Checkerboard color complementary regions of $K$.
- Assign a vertex to every shaded region, edge to every crossing,
- $\pm$ sign to every edge:

$$\begin{align*}
\text{+} & \quad \rightarrow \quad \text{+} \\
\text{−} & \quad \rightarrow \quad \text{−}
\end{align*}$$

Note:

- $e(\Gamma_K) = c(K)$ crossing number of diagram of $K = v(G(K))$.
- Signs agree on all edges of $\Gamma_K \iff K$ is alternating.
Example: Twist knot
Example: Twist knot
Reduced diagrams and Tait graphs

\[ K \text{ reduced } \iff \Gamma_K \text{ has no loops, no bridges.} \]
Let $\tau(K) = \text{number spanning trees of Tait graph } \Gamma_K$.

**Definition**

- If $K$ is alternating, $\det(K) = \tau(K)$.
- More generally, let $s_n(K)$ be number of spanning trees of $\Gamma_K$ with $n$ positive edges.

$$\det(K) = \left| \sum_{n} (-1)^n s_n(K) \right|$$

(Not usual definition of determinant, but equivalent)

$$\left| \Delta_K(-1) \right|$$
Example: Twist knot

$\det(K) = 2n + 1$

Spanning tree:

$+1$ Not $a$ & not $b$

$a$ or $b$ & remove one other edge

$2n$

$n$ crossings
Example: Twist knot

+1 keep a \& b
choose a or b
3/7 one of n
2(n)
edges

\[ \det = 2n+1 \]
Proposition

Let $K$ be a reduced alternating link diagram, $K'$ obtained by changing any proper subset of crossings of $K$. Then

$$\det(K') < \det(K).$$
Proposition

Let \( K \) be a reduced alternating link diagram, \( K' \) obtained by changing any proper subset of crossings of \( K \). Then

\[
\text{det}(K') < \text{det}(K).
\]

Proof. If only one crossing is switched, let \( e \) be corresponding edge. Note \( e \) is only negative edge in \( \Gamma_{K'} \).

- \( K \) has no nugatory crossings \( \Rightarrow \) \( e \) is not a bridge or loop
  \( \Rightarrow \) \( \exists \) spanning trees \( T_1, T_2 \) such that \( e \in T_1 \) and \( e \notin T_2 \).
- Add 2 to \( \text{det}(K) \) for \( T_1, T_2 \). Add 1 subtract 1 to \( \text{det}(K') \).

Similarly if more than one crossing is switched.

\[\square\]
det($K$) can be arbitrarily large.

However, note it grows by crossing number.
Conjecture

If $K$ is any knot or link,

\[
\frac{2\pi \log \det(K)}{c(K)} \leq v_{\text{oct}}. \quad \approx 3.66
\]

Here $v_{\text{oct}}$ is the volume of a regular hyperbolic ideal octahedron.
Conjecture (Kenyon, 1996)

If $G$ is any finite planar graph,

$$\frac{\log \tau(G)}{e(G)} \leq 2C/\pi,$$

where $C \approx 0.916$ is Catalan’s constant.

Equivalence:

- $4C = \nu_{\text{oct}}$
- Any finite planar graph $G$ can be realized as the Tait graph $\Gamma_K$ of an alternating link $K$
- $e(\Gamma_K) = c(K)$
Part 1.5: Geometric Interlude
Some geometry of knots

Can build the complement of $K$ out of octahedra:

\[ S^3 - K = \bigcup \text{crossings} \]

Choose \( \pm \infty \) in \( S^3 - K \)
Some geometry of knots

\[ v_{\text{oct}} = \text{vol of regular hyperbolic ideal octahedron} \]
\[ = \text{max vol of any hyperbolic octahedron} \]

\[ \implies \frac{\text{vol}(K)}{c(K)} < v_{\text{oct}}. \]
## Relations between $\text{vol}(K)$ and $\text{det}(K)$

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2nd line: $K'$ obtained from alternating $K$ by changing crossings.

Conjectures verified experimentally for 10.7 million knots.
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2nd line: $K'$ obtained from alternating $K$ by changing crossings.

Conjectures verified experimentally for 10.7 million knots.

**Conjecture**

For any alternating hyperbolic knot,

$$\text{vol}(K) < 2\pi \log \det(K)$$
Is $v_{\text{oct}}$ the best possible? I.e. does $\exists$ sequence of knots $K_n$ with

$$
\lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} \to v_{\text{oct}}?
$$

Answer: Yes.
Is $v_{\text{oct}}$ the best possible? I.e. does $\exists$ sequence of knots $K_n$ with

$$\lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} \to v_{\text{oct}}?$$

Answer: Yes.
Part 2: Sequences of knots.
Sequences of knots

When does a sequence of knots “converge”?

- Geometrically: Metric space \((S^3 - K_n)\) converges in Gromov-Hausdorff sense.
- Combinatorially: Diagrams converge.
Følner sequences of graphs

Let $G$ be any possibly infinite graph, $H$ a finite subgraph.

$$\partial H = \{\text{vertices of } H \text{ that share an edge with a vertex not in } H\}$$

$|\cdot|$ = number of vertices in a graph.

An exhaustive nested sequence of countably many graphs

$$\{H_n \subset G \mid H_n \subset H_{n+1} \text{ and } \bigcup_n H_n = G\}$$

is a Følner sequence for $G$ if

$$\lim_{n \to \infty} \frac{|\partial H_n|}{|H_n|} = 0.$$  

$G$ is amenable if a Følner sequence for $G$ exists.
Example: infinite square lattice
Call this the *infinite weave*, denoted \( \mathcal{W} \).
Combinatorial theorem

Theorem (Champanerkar–Kofman–P)

Let $K_n$ be a sequence of alternating link diagrams such that

1. ∃ subgraphs $G_n \subset G(K_n)$ that form a Følner sequence for $G(W)$, the infinite square lattice

2. $\lim_{n \to \infty} \frac{|G_n|}{c(K_n)} = 1$

Then

$$\lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = \nu_{oct}.$$  

I.e. maximum in conjecture is as small as possible.

We say sequence $K_n$ is diagrammatically maximal.
Theorem (Champanerkar–Kofman–P)

Let \( K_n \) be a sequence of alternating link diagrams with no cycle of tangles such that

1. \( \exists \) subgraphs \( G_n \subset G(K_n) \) that form a Følner sequence for \( G(\mathcal{W}) \), the infinite square lattice

\[ \lim_{n \to \infty} \frac{|G_n|}{c(K_n)} = 1 \]

Then

\[ \lim_{n \to \infty} \frac{\text{vol}(K_n)}{c(K_n)} = v_{\text{oct}}. \]

We say \( K_n \) is \textit{geometrically maximal}.

**Question:** Is \( K_n \) geometrically maximal \( \iff \) diagrammatically maximal?
Proof of combinatorial theorem

Let $K_n$ satisfy

1. $\exists$ subgraphs $G_n \subseteq G(K_n)$ that form a Følner sequence for $G(\mathcal{W})$, the infinite square lattice,

2. $\lim_{n \to \infty} \frac{|G_n|}{c(K_n)} = 1$

Tait graphs of $K_n$ also form a Følner sequence for the infinite square lattice.
Proof of combinatorial theorem

Lyons (2005): Graphs $H_n$ that approach $G(\mathcal{W})$ satisfy

$$\lim_{n \to \infty} \frac{\log \tau(H_n)}{|H_n|} = \frac{4C}{\pi}$$

Two to one correspondence, vertices to crossings

$$\implies \lim_{n \to \infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = 4C = \nu_{\text{oct}}.$$
Question: Does $K_n \to G$, i.e. Følner convergence of diagrams, imply $S^3 - K_n \to S^3 - G$ Gromov–Hausdorff convergence of spaces?
A word on Gromov–Hausdorff convergence

Question: Does $K_n \to G$, i.e. Følner convergence of diagrams, imply $S^3 - K_n \to S^3 - G$ Gromov–Hausdorff convergence of spaces?

Unknown – difficult to show in general.
Weaving knots

Start w/ $(p,q)$ torus knot \(\rightarrow\) make alternating.

\[
\text{Diagrams converge to infinite weave}
\]

\[
\Rightarrow \frac{2\pi \log \det(W(p,q))}{c(W(p,q))} \to \nu_{\text{oct}}, \quad \text{and} \quad \frac{\text{vol}(W(p,q))}{c(W(p,q))} \to \nu_{\text{oct}}
\]

**Theorem (Champanerkar–Kofman–P)**

Complements of weaving knots converge in the Gromov–Hausdorff sense to $S^3 - \mathcal{W}$.
Question:
Let $K_n$ be a sequence of geometrically maximal, diagrammatically maximal knots.
Does $S^3 - K_n$ always converge to $\mathbb{R}^3 - W$ in the Gromov–Hausdorff sense?