Diagrammatically maximal and geometrically maximal knots

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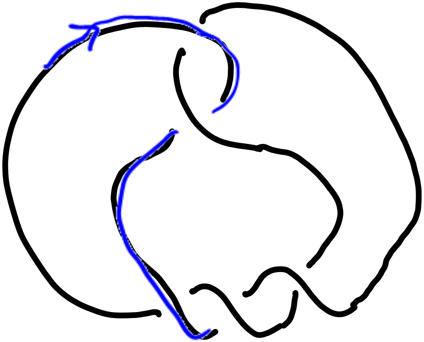
Part 1: Knots from a combinatorial point of view

Knot diagram



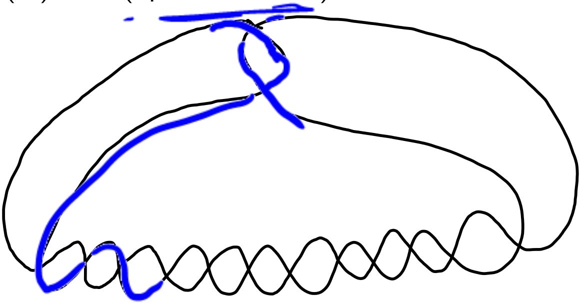
Knot diagram:
4–valent planar graph
with over–under
crossing info at
each vertex.

Alternating knot: Crossings alternate between over and under.

Diagram graph: Graph obtained by dropping crossing decoration on K, denoted $G(K)$

Diagram graph of alternating knot

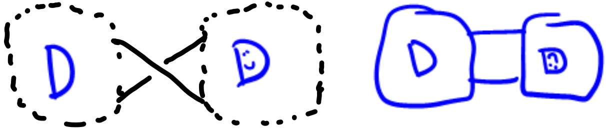
Given any 4–valent graph G, $\exists!$ alternating knot K with $G(K) = G$ (up to reflection).

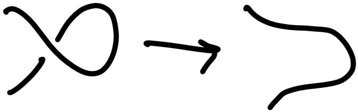


Reduced alternating diagram

Throughout, all diagrams are connected (i.e. $G(K)$ connected)

Diagram is *reduced* if it has no *nugatory crosings*:

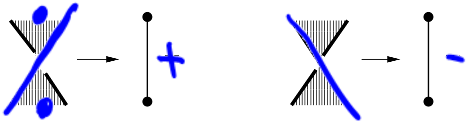




Undo nugatory crossings

Associated to any diagram K is a *Tait graph* Γ_K:

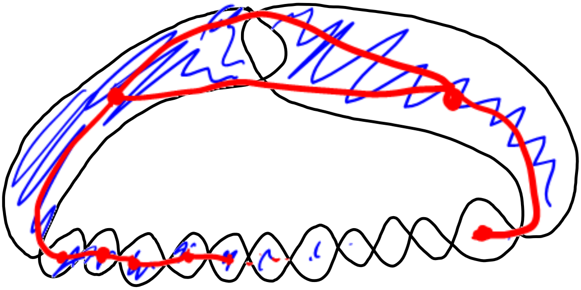
- Checkerboard color complementary regions of K.
- Assign a vertex to every shaded region,
- edge to every crossing,
- $\pm$ sign to every edge:



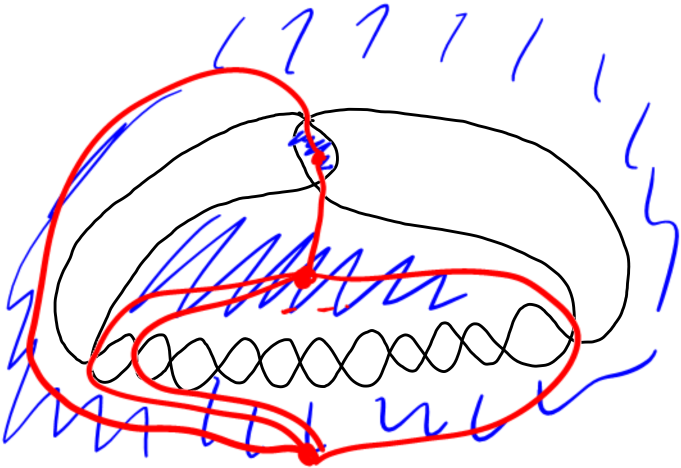
Note:

- $e(\Gamma_K) = c(K)$ crossing number of diagram of $K = v(G(K))$.
- Signs agree on all edges of $\Gamma_K \Leftrightarrow K$ is alternating.

Example: Twist knot

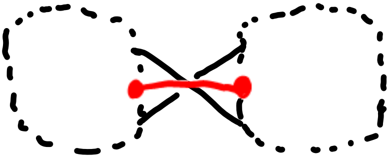


Example: Twist knot



Reduced diagrams and Tait graphs

K reduced $\Leftrightarrow \Gamma_K$ has no loops, no bridges.



Determinant of a knot

Let $\tau(K)$ = number spanning trees of Tait graph Γ_K.

Definition

- *If K is alternating,* $\det(K) = \tau(K)$.
- *More generally, let $s_n(K)$ be number of spanning trees of Γ_K with n positive edges.*

$$\det(K) = \left|\sum_n (-1)^n s_n(K)\right|$$

(Not usual definition of determinant, but equivalent)

$|\Delta_K(-1)|$

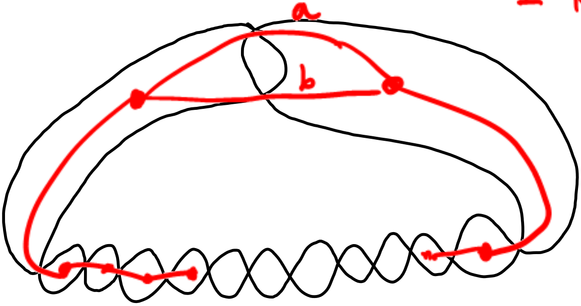
Example: Twist knot

Spanning tree:

+1 Not a & not b

a or b & remove one other edge

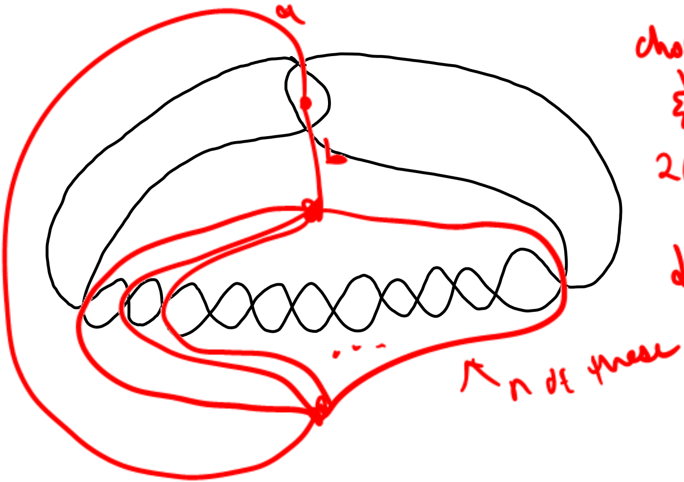
2n



← n crossings

$\det(K) = 2n+1$

Example: Twist knot



+1 keep a & b

choose a or b & one of n edges

2(n) edges

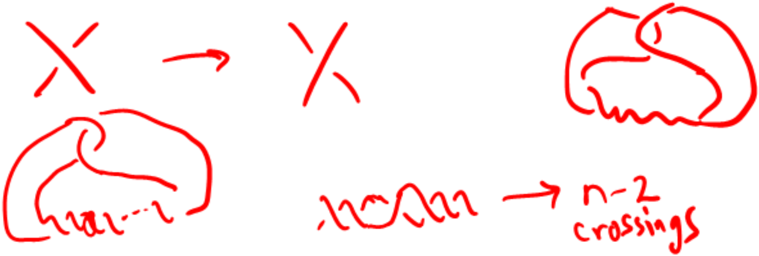
det = 2n+1

Determinant under crossing change

Proposition

Let K be a reduced alternating link diagram, K' obtained by changing any proper subset of crossings of K. Then

$$\det(K') < \det(K).$$



Proposition

Let K be a reduced alternating link diagram, K' obtained by changing any proper subset of crossings of K. Then

$$\det(K') < \det(K).$$

Proof. If only one crossing is switched, let e be corresponding edge. Note e is only negative edge in $\Gamma_{K'}$.

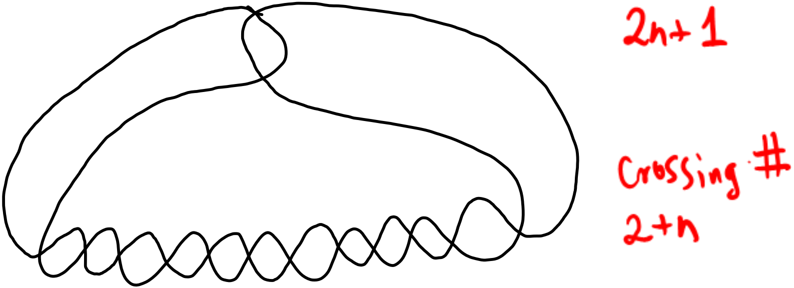
- K has no nugatory crossings $\Rightarrow$ e is not a bridge or loop $\Rightarrow \exists$ spanning trees T_1, T_2 such that $e \in T_1$ and $e \notin T_2$.
- Add 2 to $\det(K)$ for T_1, T_2. Add 1 subtract 1 to $\det(K')$.

+1 +1 +1 T_2 −1 T_1

Similarly if more than one crossing is switched. □

How big can det(K) be?

det(K) can be arbitrarily large.



However, note it grows by crossing number.

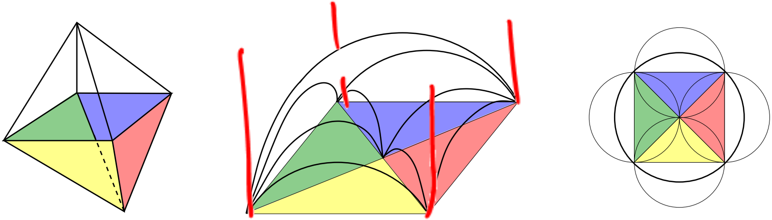
Determinant density conjecture

Conjecture

If K is any knot or link,

$$\frac{2\pi \log \det(K)}{c(K)} \leq v_{\mathrm{oct}}. \simeq 3.66$$

Here v_{oct} is the volume of a regular hyperbolic ideal octahedron.



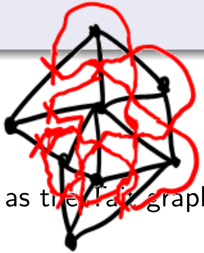
Equivalent to Conjecture of Kenyon

Conjecture (Kenyon, 1996)

If G is any finite planar graph,

$$\frac{\log \tau(G)}{e(G)} \leq 2C/\pi,$$

where $C \approx 0.916$ is Catalan's constant.



Equivalence:

- $4C = v_{\mathrm{oct}}$
- Any finite planar graph G can be realized as the Tait graph Γ_K of an alternating link K
- $e(\Gamma_K) = c(K)$

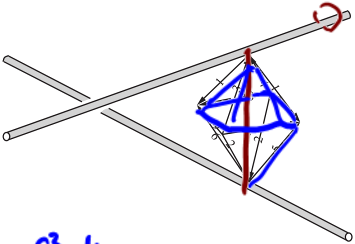
Part 1.5: Geometric Interlude

Some geometry of knots

$S^3 - K$

Can build the complement of K out of octahedra:

$S^3 - K$
3-manifold
$\cong S^3 - \overline{N(K)}$
admits geometry
typically hyperbolic.



Choose $\pm\infty$ in $S^3 - K$

$$S^3 - K = \bigcup_{\text{crossings}} \text{octahedra.}$$

Some geometry of knots

$$v_{\text{oct}} = \text{vol of regular hyperbolic ideal octahedron}$$
$$= \text{max vol of any hyperbolic octahedron}$$

$$\Longrightarrow \qquad \frac{\mathrm{vol}(K)}{c(K)} < v_{\text{oct}}.$$

Relations between vol(K) and det(K)

Known	**Conjectured**
$\dfrac{\mathrm{vol}(K)}{c(K)} \leq v_{\mathrm{oct}}$	$\dfrac{2\pi \log \det(K)}{c(K)} \leq v_{\mathrm{oct}}$
$\det(K') < \det(K)$	$\mathrm{vol}(K') < \mathrm{vol}(K)$

Kenyon

2nd line: K' obtained from alternating K by changing crossings.

Conjectures verified experimentally for 10.7 million knots.

Relations between $\mathrm{vol}(K)$ and $\det(K)$

Known	**Conjectured**
$\dfrac{\mathrm{vol}(K)}{c(K)} \leq v_{\mathrm{oct}}$	$\dfrac{2\pi \log \det(K)}{c(K)} \leq v_{\mathrm{oct}}$
$\det(K') < \det(K)$	$\mathrm{vol}(K') < \mathrm{vol}(K)$

2nd line: K' obtained from alternating K by changing crossings.

Conjectures verified experimentally for 10.7 million knots.

Conjecture

For any alternating hyperbolic knot,

$$\mathrm{vol}(K) < 2\pi \log \det(K)$$

How sharp is the upper bound?

Is v_{oct} the best possible? I.e. does $\exists$ sequence of knots K_n with

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} \to v_{\mathrm{oct}}?$$

How sharp is the upper bound?

Is v_{oct} the best possible? I.e. does $\exists$ sequence of knots K_n with

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} \to v_{\mathrm{oct}}?$$

Answer: Yes.

Part 2: Sequences of knots.

Sequences of knots

When does a sequence of knots "converge"?

- Geometrically: Metric space $S^3 - K$ converges in Gromov-Hausdorff sense.
- Combinatorially: Diagrams converge.

Følner sequences of graphs

Let G be any possibly infinite graph, H a finite subgraph.

$\partial H = \{$vertices of H that share an edge with a vertex not in $H\}$

$|\cdot|$ = number of vertices in a graph.

An exhaustive nested sequence of countably many graphs

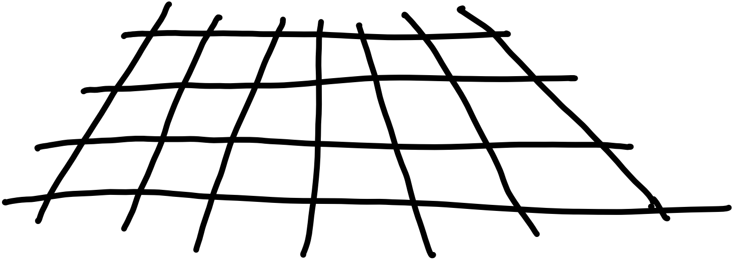
$$\{H_n \subset G \mid H_n \subset H_{n+1} \text{ and } \cup_n H_n = G\}$$

is a *Følner sequence* for G if

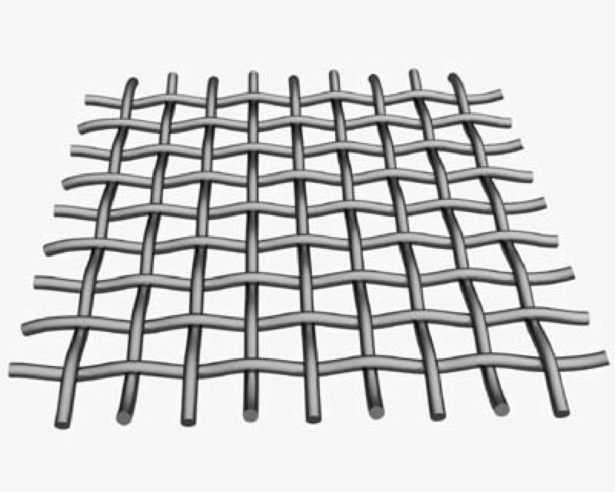
$$\lim_{n\to\infty} \frac{|\partial H_n|}{|H_n|} = 0.$$

G is *amenable* if a Følner sequence for G exists.

Example: infinite square lattice



Make alternating: infinite weave



Call this the *infinite weave*, denoted $\mathcal{W}$.

Combinatorial theorem

Theorem (Champanerkar–Kofman–P)

Let K_n be a sequence of alternating link diagrams such that

1. $\exists$ *subgraphs* $G_n \subset G(K_n)$ *that form a Følner sequence for* $G(\mathcal{W})$*, the infinite square lattice*
2. $\lim_{n\to\infty} \frac{|G_n|}{c(K_n)} = 1$

Then

$$\lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = v_{\text{oct}}.$$

I.e. maximum in conjecture is a small as possible.

We say sequence K_n is *diagrammatically maximal*.

Geometric

Theorem (Champanerkar–Kofman–P)

Let K_n be a sequence of alternating link diagrams with no cycle of tangles such that

1. $\exists$ *subgraphs* $G_n \subset G(K_n)$ *that form a Følner sequence for* $G(\mathcal{W})$, *the infinite square lattice*
2. $\lim_{n\to\infty} \frac{|G_n|}{c(K_n)} = 1$

Then

$$\lim_{n\to\infty} \frac{\mathrm{vol}(K_n)}{c(K_n)} = v_{\mathrm{oct}}.$$

We say K_n is *geometrically maximal*.

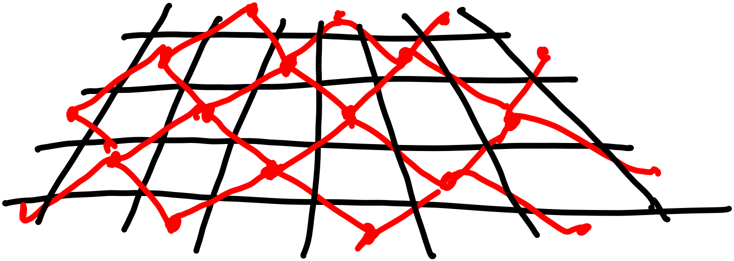
Question: Is K_n geometrically maximal $\Leftrightarrow$ diagrammatically maximal?

Proof of combinatorial theorem

Let K_n satisfy

1. $\exists$ subgraphs $G_n \subset G(K_n)$ that form a Følner sequence for $G(\mathcal{W})$, the infinite square lattice,
2. $\lim_{n\to\infty} \frac{|G_n|}{c(K_n)} = 1$

Tait graphs of K_n also form a Følner sequence for the infinite square lattice.



Proof of combinatorial theorem

Lyons (2005): Graphs H_n that approach $G(\mathcal{W})$ satisfy

$$\lim_{n\to\infty} \frac{\log \tau(H_n)}{|H_n|} = \frac{4C}{\pi}$$

Two to one correspondence, vertices to crossings

$$\Longrightarrow \lim_{n\to\infty} \frac{2\pi \log \det(K_n)}{c(K_n)} = 4C = v_{\mathrm{oct}}.$$

A word on Gromov–Hausdorff convergence

Question: Does $K_n \to G$, i.e. Følner convergence of diagrams, imply $S^3 - K_n \to S^3 - G$ Gromov–Hausdorff convergence of spaces?

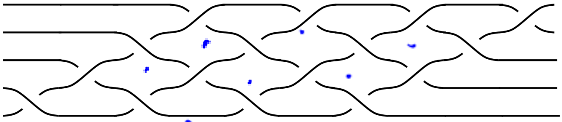
A word on Gromov–Hausdorff convergence

Question: Does $K_n \to G$, i.e. Følner convergence of diagrams, imply $S^3 - K_n \to S^3 - G$ Gromov–Hausdorff convergence of spaces?

Unknown – difficult to show in general.

Weaving knots

Start w/ (p,q) torus knot ⇝ make alternating.



W(5,4)

Diagrams converge to infinite weave

$$\Longrightarrow \frac{2\pi \log \det(W(p,q))}{c(W(p,q))} \to v_{\rm oct}, \quad \text{and} \quad \frac{\mathrm{vol}(W(p,q))}{c(W(p,q))} \to v_{\rm oct}$$

Theorem (Champanerkar–Kofman–P)

Complements of weaving knots converge in the Gromov–Hausdorff sense to $S^3 - \mathcal{W}$.

Other knots

Question:

Let K_n be a sequence of geoemtrically maximal, diagrammatically maximal knots.

Does $S^3 - K_n$ always converge to $\mathbb{R}^3 - \mathcal{W}$ in the Gromov–Hausdorff sense?