On the distances between Latin Squares

Reshma Ramadurai

Joint work with Nick Cavenagh

University of Waikato

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Example:

0		2	
3			
		0	
1			0

P_1

0		2	
3			
		0	
1			3

P_2

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Example:

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

$P_1 \to L_1$

0		2	
3			
		0	
1			3

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3			
		0	
1		?	3

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		0	
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		0	
1			3

P_2

Latin Square: of *order* n is an array of n symbols with n rows and n columns such that each symbol appears exactly once in each row and each column.

In fact...

0		2	
3			
		0	
1			0

P_1

0		2	
3			
		0	
1			0

P_1

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0	1	2	3
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L_1

0		2	
3			
		0	
1			0

P_1

In fact...

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3	0	1	2
2	3	0	1
1	2	3	0

L_1

0	3	2	1
3	0	1	2
2	1	0	3
1	2	3	0

L_2

In fact...

0	1	2	3
3	0	1	2
2	3	0	1
1	2	3	0

L_1

0	3	2	1
3	0	1	2
2	1	0	3
1	2	3	0

L_2

Furthermore...

0	1	3	2
3	0	2	1
1	2	0	3
2	3	1	0

L_3

n	$L(n, n)$	reference
1	1	
2	1	
3	1	
4	4	
5	56	Euler (1782), Cayley (1890), MacMahon (1915; incorrect value)
6	9408	Frolov (1890) and Tarry (1900)
7	16942080	Frolov (1890; incorrect), Norton (1939; incomplete), Sade (1948), Saxena (1951)
8	535281401856	Wells (1967)
9	377597570964258816	Bammel and Rothstein (1975)
10	7580721483160132811489280	McKay and Rogoyski (1995)
11	5363937773277371298119673540771840	McKay and Wanless (2005)
12	1.62×10^{44}	McKay and Rogoyski (1995)
13	2.51×10^{56}	McKay and Rogoyski (1995)
14	2.33×10^{70}	McKay and Rogoyski (1995)
15	1.5×10^{86}	McKay and Rogoyski (1995)

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Example:

	3		1
	1		3

$L_2 \setminus L_1$

		3	2
		2	1
1	2		3
2	3	1	

$L_3 \setminus L_1$

Distance $d(L, L')$: between two (partial) Latin squares L and L' of the same order is defined to be $|L \setminus L'|$, i.e., the number of cells in which they differ.

Example:

	3		1
	1		3

$L_2 \setminus L_1$

		3	2
		2	1
1	2		3
2	3	1	

$L_3 \setminus L_1$

Question: Given L, what is the minimum distance, $dist(L)$, to a distinct Latin square of the same order?

Latin trade:

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	3		1
	1		3

$L_2 \setminus L_1$

		3	2
		2	1
1	2		3
2	3	1	

$L_3 \setminus L_1$

	1		3
	3		1

$L_1 \setminus L_2$

		2	3
		1	2
2	3		1
1	2	3	

$L_1 \setminus L_3$

Alternatively, we can define $dist(L)$ as the size of the smallest Latin trade in L.

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2	1_3	0	3_1
1	2	3	0

$L_1 \to L_2$

0	1	3_2	2_3
3	0	2_1	1_2
1_2	2_3	0	3_1
2_1	3_2	1_3	0

$L_1 \to L_3$

Alternatively, we can define $dist(L)$ as the size of the smallest Latin trade in L.

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0	3_1	2	1_3
3	0	1	2
2	1_3	0	3_1
1	2	3	0

$L_1 \rightarrow L_2$

0	1	3_2	2_3
3	0	2_1	1_2
1_2	2_3	0	3_1
2_1	3_2	1_3	0

$L_1 \rightarrow L_3$

Can be shown that $dist(L) \geq 4$.

Intercalates: are Latin trades of size 4.

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Example:

0	1	2	3	4	5	6	7
1	0	3	2	5	4	7	6
2	3	0	1	6	7	4	5
3	2	1	0	7	6	5	4
4	5	6	7	0	1	2	3
5	4	7	6	1	0	3	2
6	7	4	5	2	3	0	1
7	6	5	4	3	2	1	0

Addition table for $(\mathbb{Z}_2)^3$.

Known results:

Suppose $I(n)$ is the maximum number of intercalates in a Latin square of order n, then

- $I(n) = n^2(n-1)/4$ if $n = 2^k$.
- $I(n) = n(n-1)(n-3)/4$ if $n = 2^k - 1$.
- Browning, Cameron and Wanless, 2014:
  $I(n) \geq (n-1)(n-3)(n-15)/8$.

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- Browning, Cameron and Wanless, 2014: $I(n) \geq (n-1)(n-3)(n-15)/8$.

In fact,

Theorem (McKay, Wanless 1999): For most Latin squares,

$$I(n) \geq n^{3/2-\varepsilon}$$

for any $\varepsilon > 0$.

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- Drápal and Kepka, 1989: $dist(B_p) \geq \mathrm{e} \log p + 3$, where p is prime. (Alternate proof independently by Cavenagh, and also by Cavenagh and Wanless.)
- Szabados, 2014: $dist(B_n) \leq 5 \log_2 n + 3$.

Definition: Define $dist(n) = \max\{dist(L) : L \text{ is a Latin square of order } n\}$.

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0	1		
	0		1
1		0	
		1	0

0	1	3	2
1	2	0	3

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Example:

0	1		
	0	1	
		0	1
1			0

0	1	2	3
1	2	3	0

Definition: Define $dist(n) = \max\{dist(L) : L \text{ is a Latin square of order } n\}$.

Example: cycle trades

0_1	1_0		
	0_1	1_0	
		0_1	1_0
1_0			0_1

0_1	1_2	2_3	3_0
1_0	2_1	3_2	0_3

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	0_1	1_0	
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0_1	1_2	2_3	3_0
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Thus, $dist(n) \leq 2n$.

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0_1	1_0		
	0_1	1_0	
		0_1	1_0
1_0			0_1

0_1	1_2	2_3	3_0
1_0	2_1	3_2	0_3

Thus, $dist(n) \leq 2n$.

Conjecture (Cavenagh, 2003): $dist(n) = O(\log n)$.

Theorem (Cavenagh and R.): $dist(n) = O(\sqrt{n})$.

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Conjectured: $\left\lfloor \dfrac{n^2}{4} \right\rfloor$.

Best known previously: $n \left\lfloor \dfrac{(\log n)^{1/3}}{2} \right\rfloor$.