Euler Characteristic

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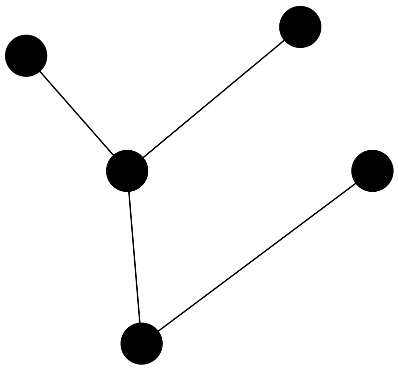
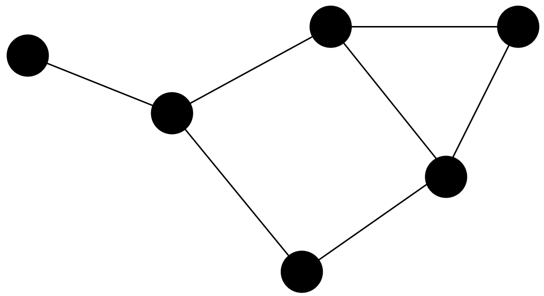
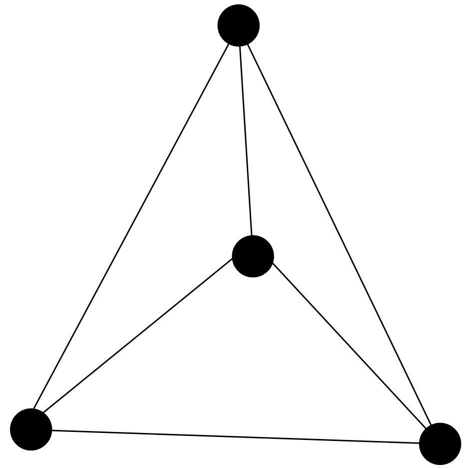
1 Planar graphs



$v = 5, e = 4, f = 1$
$v - e + f = 2$



$v = 6, e = 7, f = 3$
$v - e + f = 2$



$v = 4, e = 6, f = 4$
$v - e + f = 2$

Euler characteristic: $\chi = v - e + f$

If a finite, connected, planar graph is drawn in the plane without any edge intersections, and:

- v is the number of vertices,
- e is the number of edges, and
- f is the number of faces

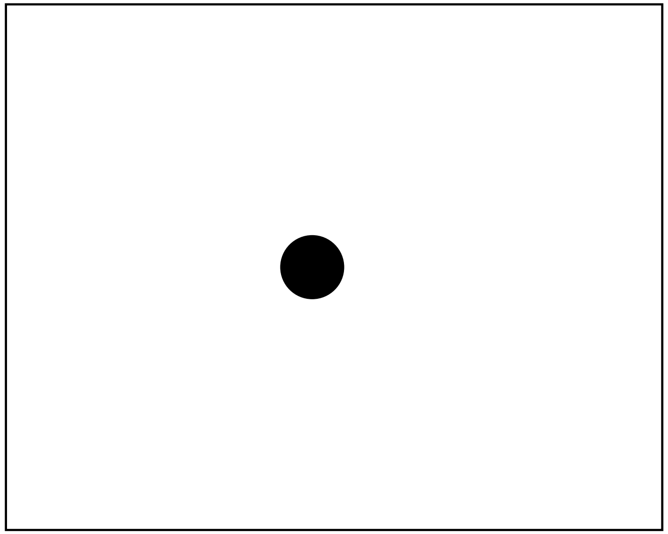
then:

$$\chi = v - e + f = 2$$

ie. the Euler characteristic is 2 for planar surfaces.

Proof.

Start with smallest possible graph:



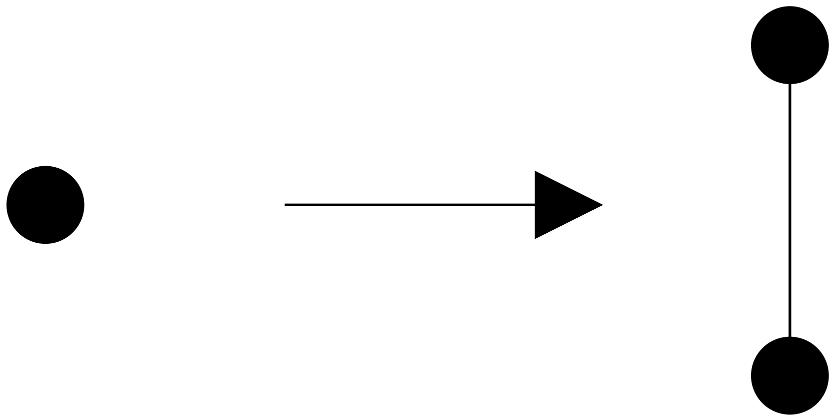
$v = 1, e = 0, f = 1$

$v - e + f = 2$

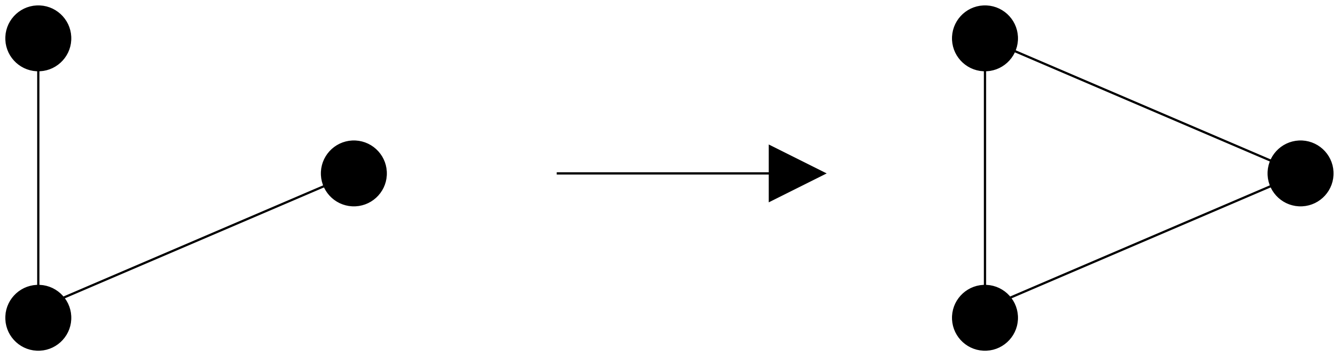
Holds for base case

Increase size of graph:

- either add a new edge and a new vertex, keeping the number of faces the same:



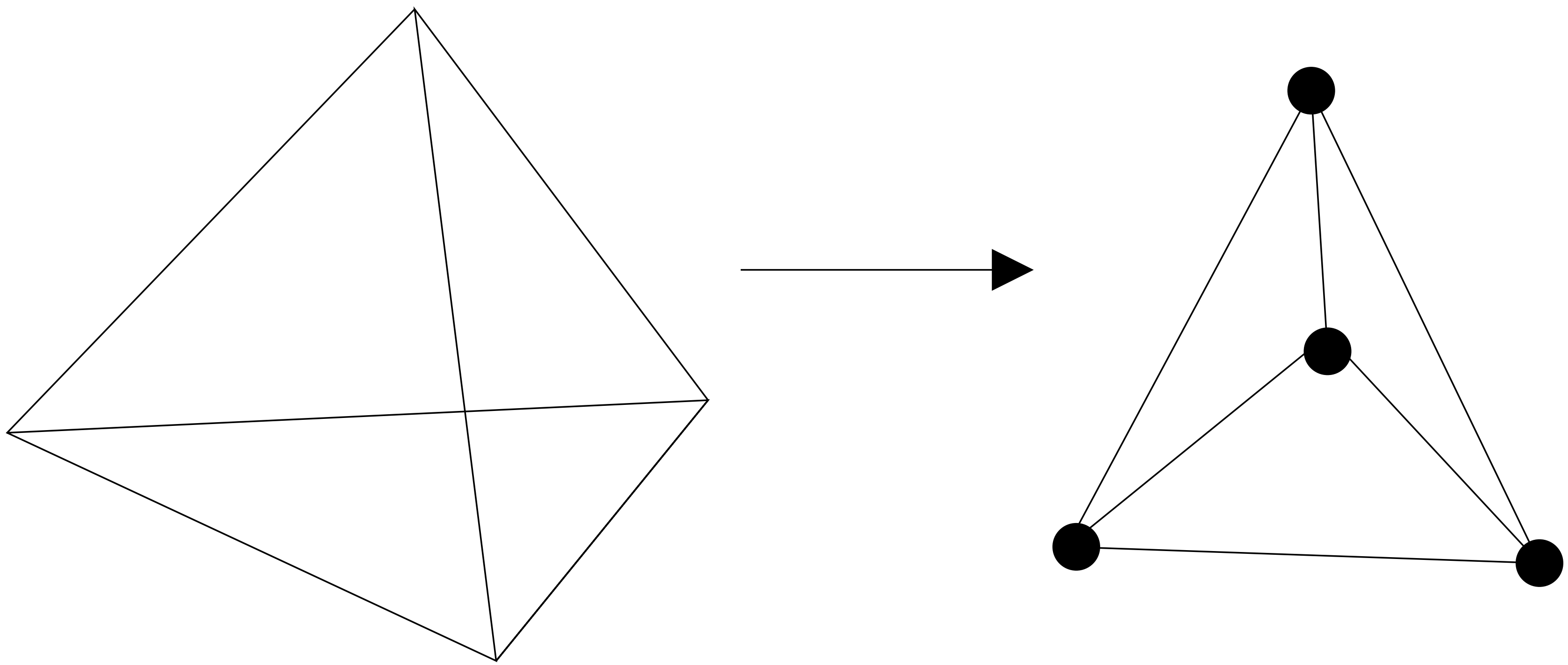
- or add a new edge but no new vertex, thus completing a new cycle and increasing the number of faces:



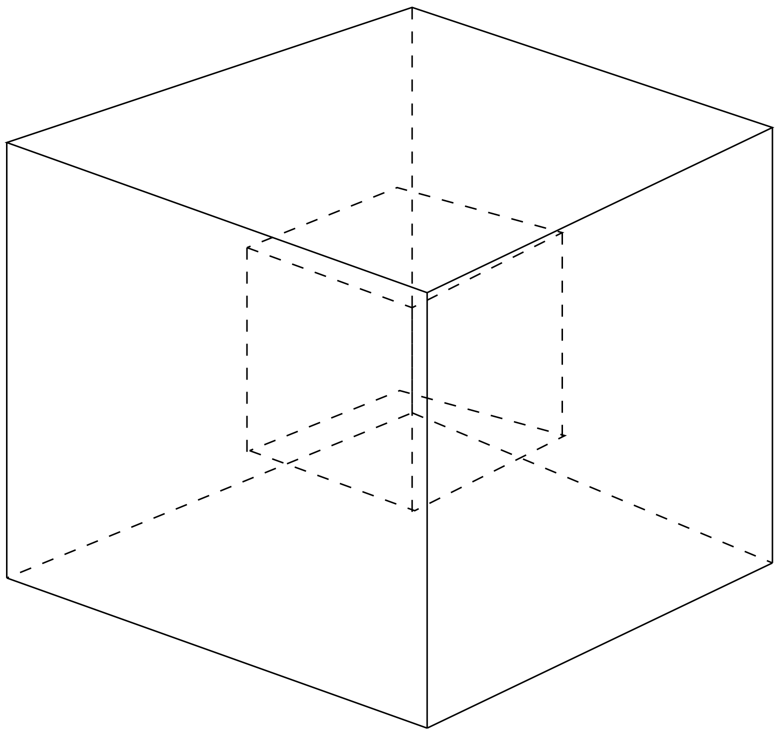
2 Polyhedra

- Euler first noticed this property applied to polyhedra
- He first mentions the formula $v - e + f = 2$ in a letter to Goldbach in 1750
- Proved the result for convex polyhedra in 1752

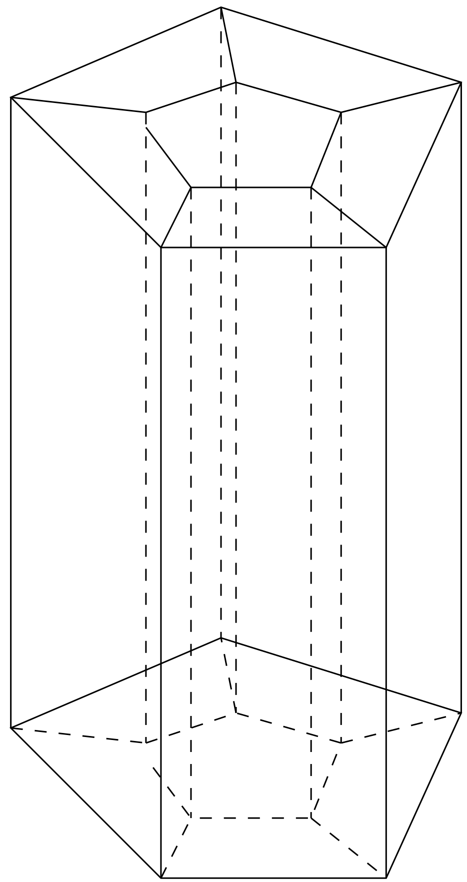
- Holds for polyhedra where the vertices, edges and faces correspond to the vertices, edges and faces of a connected, planar graph



- In 1813 Lhuilier drew attention to polyhedra which did not fit this formula



$v = 16, e = 24, f = 12$
$v - e + f = 4$



$v = 20, e = 40, f = 20$
$v - e + f = 0$

Euler's theorem. (Von Staudt, 1847) Let P be a polyhedron which satisfies:

(a) Any two vertices of P can be connected by a chain of edges.

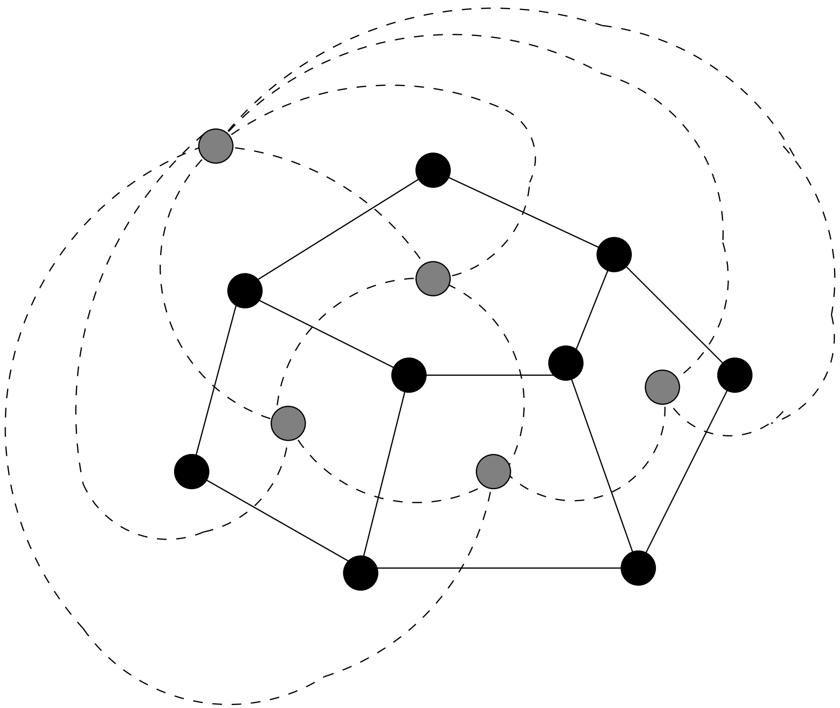
(b) Any loop on P which is made up of straight line segments (not necessarily edges) separates P into two pieces.

Then $v - e + f = 2$ for P.

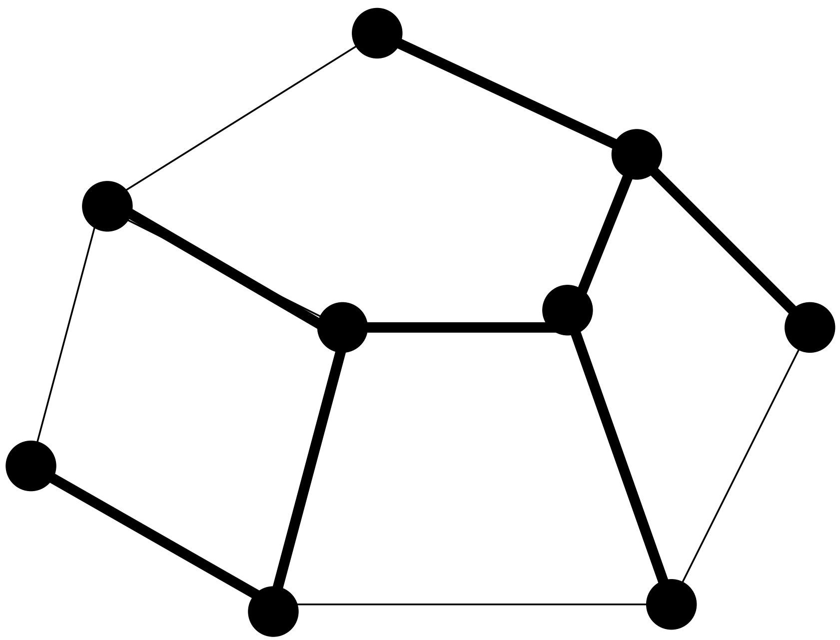
Von Staudt's proof:

For a connected, planar graph G, define the *dual graph* G' as follows:

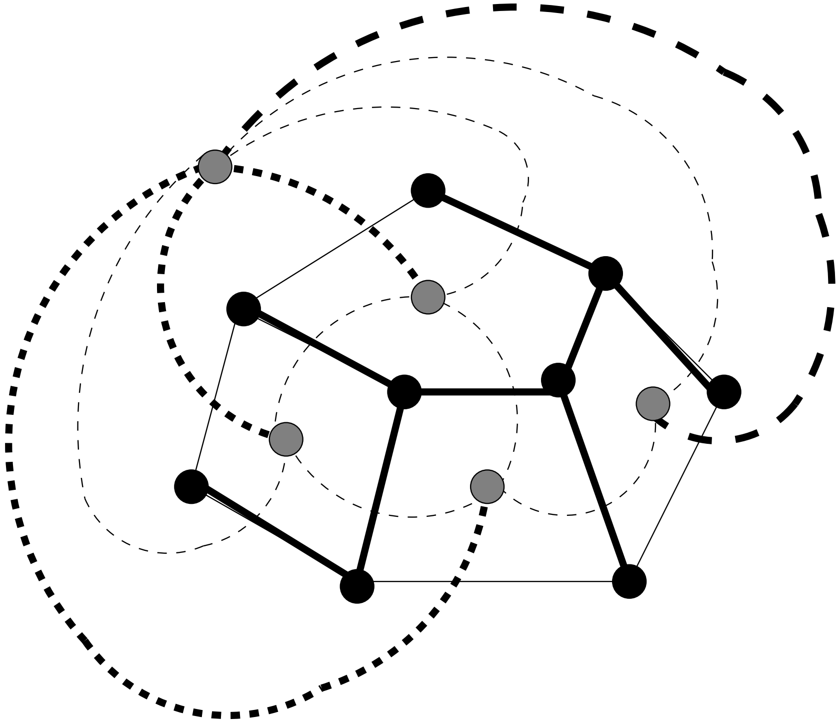
- add a vertex for each face of G; and
- add an edge for each edge in G that separates two neighbouring faces.



Choose a spanning tree T in G.



Now look at the edges in the dual graph G' of $T's$ complement ($G - T$).



The resulting graph T' is a spanning tree of G'.

- Number of vertices in any tree $=$ number of edges $+1$.

$|V(T)| - |E(T)| = 1$ and $|V(T')| - |E(T')| = 1$

$|V(T)| - [|E(T)| + |E(T')|] + |V(T')| = 2$

$|V(T)| = |V(G)|$, since T is a spanning tree of G

$|V(T')| = |F(G)|$, since T' is a spanning tree of G's dual

$|E(T)| + |E(T')| = |E(G)|$

- Therefore $V - E + F = 2$.

- *Platonic solid:* a convex, regular polyhedron, i.e. one whose faces are identical and which has the same number of faces around each vertex.
- Euler characteristic can be used to show there are exactly five Platonic solids.

Proof.

Let n be the number of edges and vertices on each face. Let d be the degree of each vertex.

$$nF = 2E = dV$$

Rearrange:

$e = dV/2$, $f = dV/n$

By Euler's formula:

$V - dV/2 + dV/n = 2$

$V(2n + 2d - nd) = 4n$

Since n and V are positive:

$2n + 2d - nd > 0$

$(n-2)(d-2) < 4$

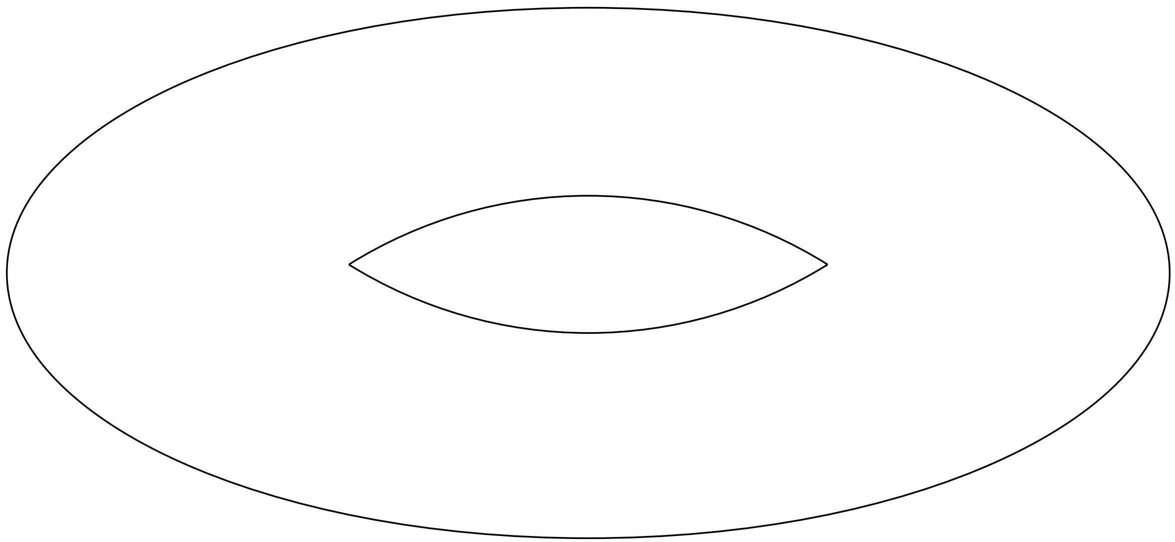
Thus there are five possibilities for (d, n):

$(3, 3)$ (tetrahedron), $(3, 4)$ (cube), $(3, 5)$ (dodecahedron), $(4, 3)$ (octahedron), $(5, 3)$ (icosahedron).

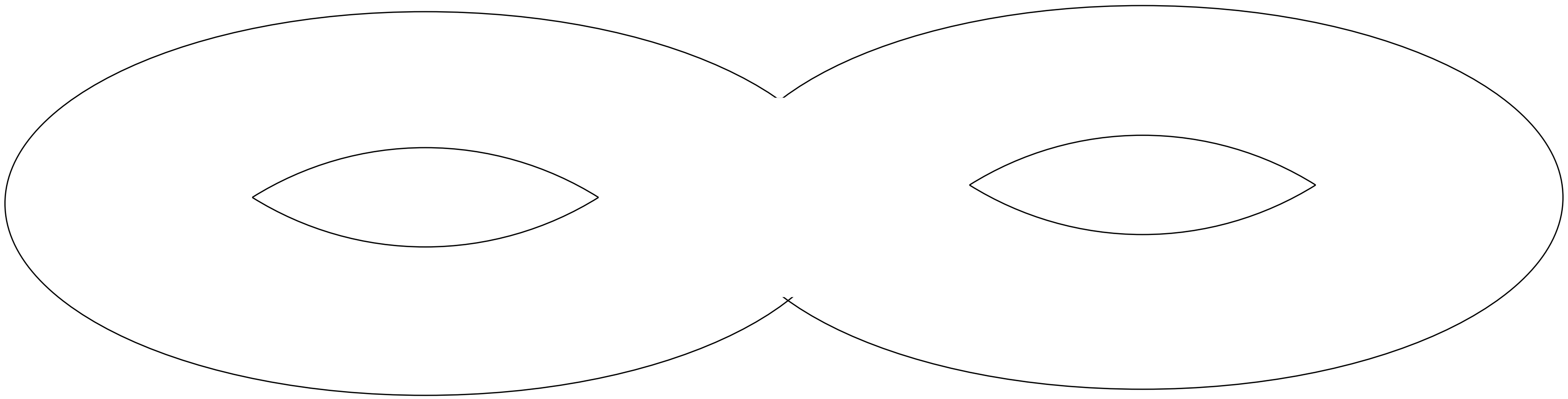
3 Non-planar surfaces

- $\chi = v - e + f = 2$ applies for graphs drawn on the plane - what about other surfaces?
- *Genus* of a graph: a number representing the maximum number of cuttings that can be made along a surface without disconnecting it - the number of *handles* of the surface.
- In general: $\chi = 2 - 2g$, where g is the genus of the surface
- Plane has genus 0, so $2 - 2g = 2$

Torus (genus 1): $v - e + f = 0$



Double torus (genus 2): $v - e + f = -2$



- *Topological equivalence:* two surfaces are topologically equivalent (or *homeomorphic*) if one can be 'deformed' into the other without cutting or gluing.
- Examples: the sphere is topologically equivalent to any convex polyhedron; a torus is topologically equivalent to a 'coffee cup' shape.
- Topologically equivalent surfaces have the same Euler number: the Euler characteristic is called a *topological invariant*