Topological containment of the 5-clique minus an edge in 4-connected graphs

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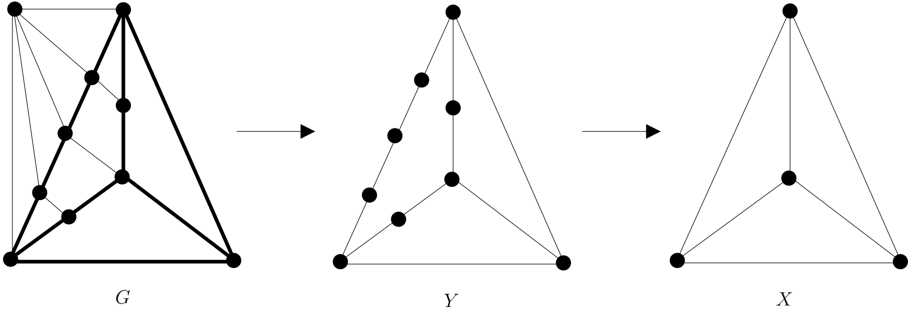
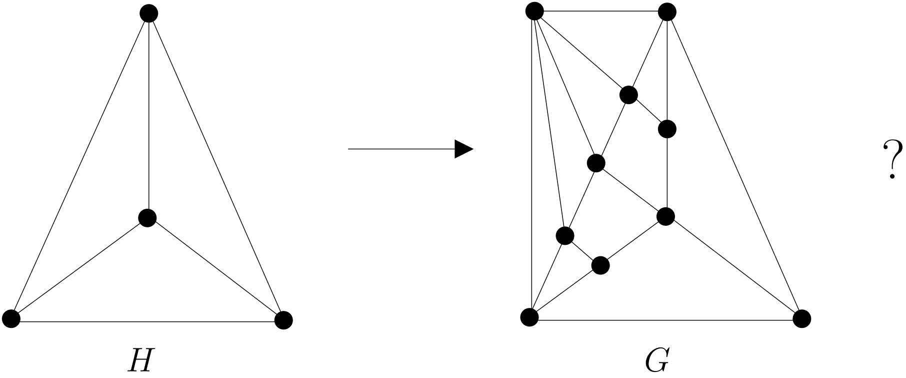
Topological containment



- Formally: G topologically contains X *iff G contains some subgraph Y such that X can be obtained from Y by performing a series of contractions limited to edges that have at least one endvertex of degree 2.*
- Also: Y is an *X-subdivision*; G *contains* an X-subdivision

Problem of topological containment:

- *TC*(*H*): For some fixed *pattern graph H* — given a graph *G*, does *G* contain an *H*-subdivision?



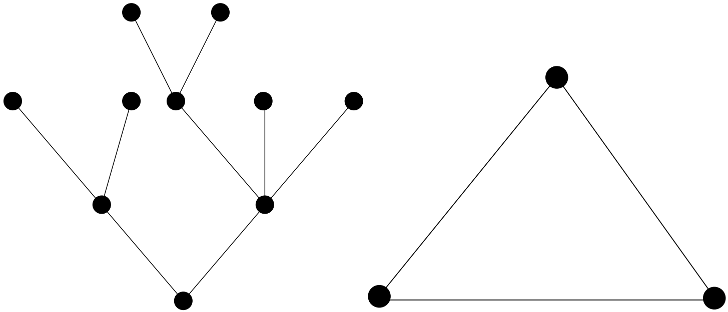
DISJOINT PATHS and Topological Containment

DISJOINT PATHS (DP)
Input: Graph G; pairs $(s_1, t_1), ..., (s_k, t_k)$ of vertices of G.
Question: Do there exist paths $P_1, ..., P_k$ of G, mutually vertex-disjoint, such that P_i joins s_i and t_i $(1 \leq i \leq k)$?

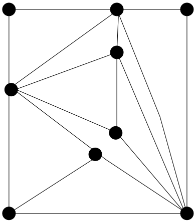
- DISJOINT PATHS in P for any fixed k (Robertson & Seymour, 1995).
- $\Longrightarrow$ TC(H) is also in P — use DP repeatedly.
- Doesn't give practical algorithms.
- We still want characterisations for particular pattern graphs.

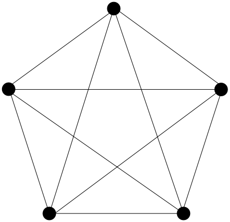
Examples of good characterizations

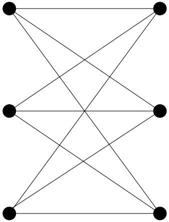
- Trees — K_3



- Kuratowski (1930) — K_5 or $K_{3,3}$ in non-planar graphs

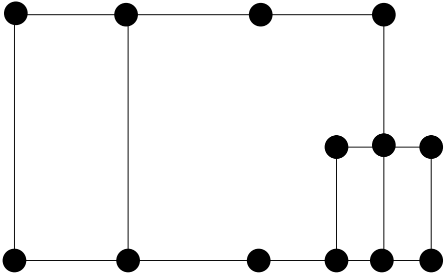


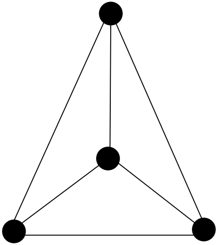




- Wagner (1937) and Hall (1943) strengthened this result to characterize graphs with no $K_{3,3}$-subdivisions

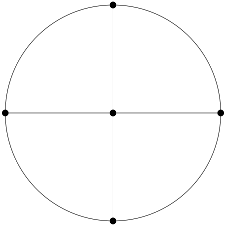
- Duffin (1965) — K_4 in non-series-parallel graphs



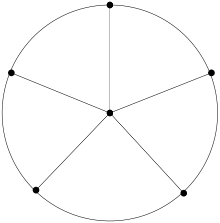


Results for wheel graphs

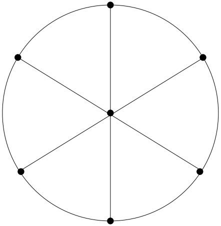
- W_4 (Farr, 1988) — If G is 3-connected, G contains a W_4-subdivision iff G has a vertex of degree ≥ 4
- W_5 (Farr, 1988) — If G is 3-connected with no internal 3-edge-cutsets, G contains a W_5-subdivision iff G has a vertex v of degree ≥ 5 and a circuit of length ≥ 5 that does not contain v.
- More recently, characterisations obtained for:
  - graphs with no W_6-subdivision (Robinson & Farr, 2009); and
  - graphs with no W_7-subdivision (Robinson & Farr, 2014).



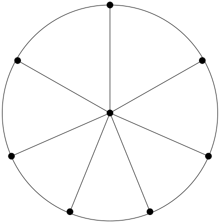
W_4



W_5



W_6



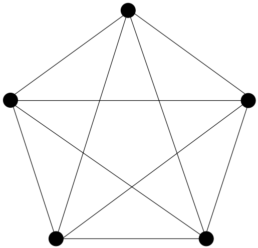
W_7

Connections with Hajós' Conjecture

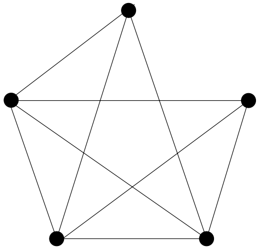
- Of particular interest: solving TC(K_5).
- Conjectured by Hajós, 1940s: no K_k-subdivision $\Rightarrow$ $(k-1)$-colourable.
- Proved for $k \leq 4$ (Hadwiger, 1943; Dirac, 1952).
- Refuted for $k \geq 7$ (Catlin, 1979).
- For $k = 5$ and $k = 6$, remains an open problem.
- Characterisation for graphs with no K_5-subdivision may lead to solving $k = 5$.

Progress towards solving TC(K_5)

- Kelmans-Seymour Conjecture:
  - 5-connected non-planar graph $\Rightarrow$ K_5-subdivision
  - (recent proof by He, Wang, Yu, 2015-16).
- 4-connected graph $\Rightarrow$ K_5 or $K_{2,2,2}$ as a minor (Halin & Jung, 1963) — but this doesn't necessarily imply a K_5-subdivision.
- Possible step along the way: solve for slightly simpler graph, K_5^-.



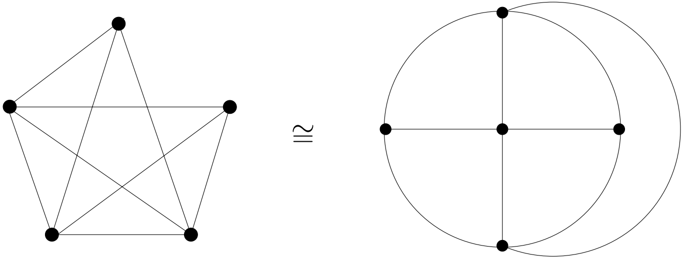
K_5



K_5^-

TC(K_5^-) in 4-connected graphs

- We show: **every 4-connected graph contains a K_5^--subdivision.**
  - a step in parallel to the Kelmans-Seymour Conjecture.
- Approach: start with a 'base' graph (W_4): subgraph of the pattern graph H (K_5^-), and good characterisation is already known.
- Look at all ways of enlarging base graph so conditions of hypothesis are met (in this case, 4-connectivity).
- For each enlarged graph, determine whether it contains an H-subdivision.



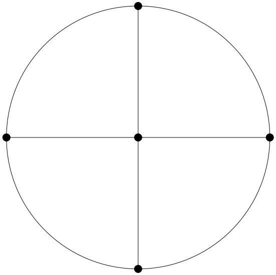
TC(K_5^-) in 4-connected graphs

Theorem

Let G be a 4-connected graph. G contains a K_5^--subdivision.

Proof — a summary

- Farr (1988) — If G is 3-connected, G contains a W_4-subdivision iff G has a vertex of degree ≥ 4.
- Since G is 4-connected, there exists a W_4-subdivision.
- Let H be a W_4-subdivision in G, chosen such that H is *short*.

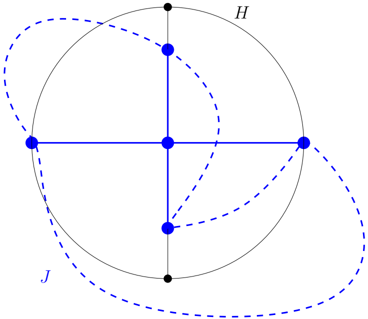


Definition

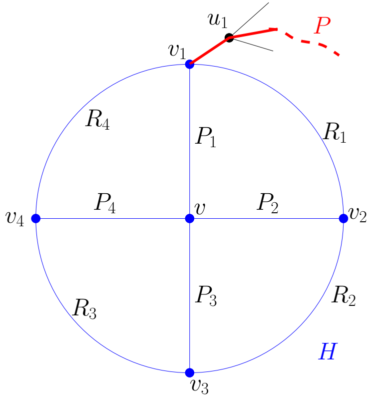
Let H be a W_n-subdivision in a graph G. We say that another W_n-subdivision J in G is **shorter than** *H if:*

- *the hubs of H and J are the same;*
- *the spokes of H and J are not all the same;*
- *each spoke of J is an initial segment of a spoke of H; and*
- *at least one spoke of J is a* **proper** *initial segment of a spoke of H.*

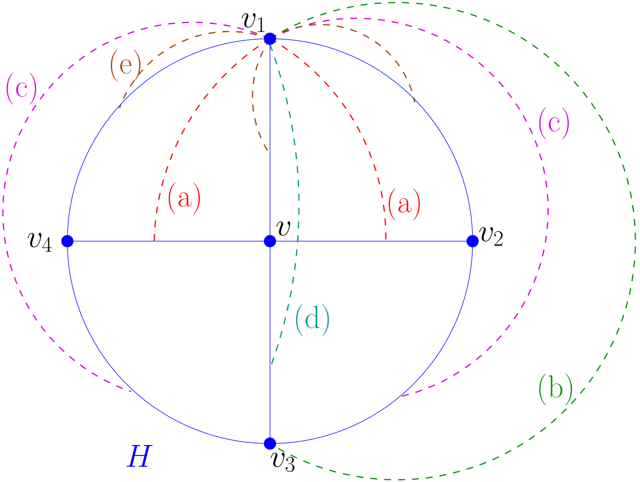
If no other W_n-subdivision in G is shorter than H, we call H **short**.



- By 4-connectivity, there is a fourth neighbour u_1 of v_1, where $u_1 \notin N_H(v_1)$.
- At least three paths from u_1 to $H - v_1$, disjoint except at u_1, that meet H only at their endpoints.
- Let $P = v_1 u_1 +$ one of these paths.

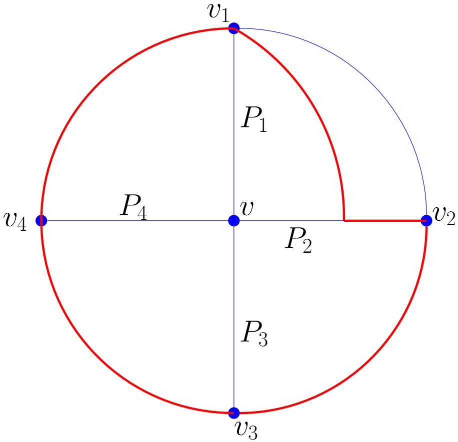


- Let p_1 be the vertex at which P meets H. There are five cases to consider...



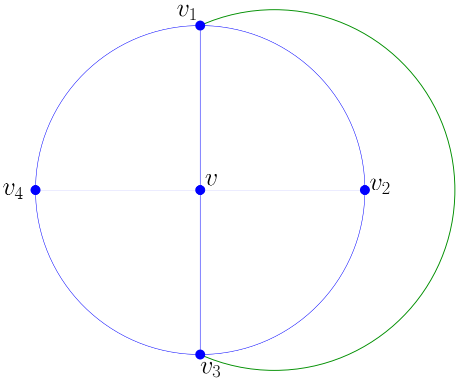
Case (a): p_1 is an internal vertex of P_2 or P_4

- Shortness of H is violated.



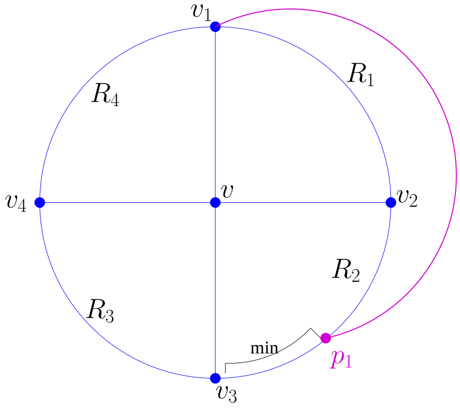
Case (b): $p_1 = v_3$

- $H + P$ forms a K_5^--subdivision.

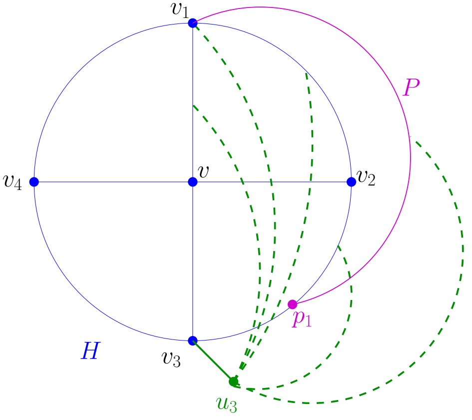


Case (c): p_1 is an internal vertex of R_2 or R_3

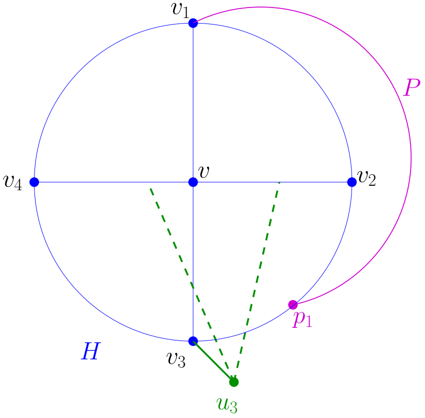
- Without loss of generality, assume p_1 is on R_2, and distance between p_1 and v_3 along R_2 is minimised.



- By 4-connectivity, there is a fourth neighbour u_3 of v_3.
- Let U_3 be the $(H \cup P)$-bridge of G containing the edge $v_3 u_3$.
- We consider the cases for where U_3's vertices of attachment can be.



K_5^--subdivision created

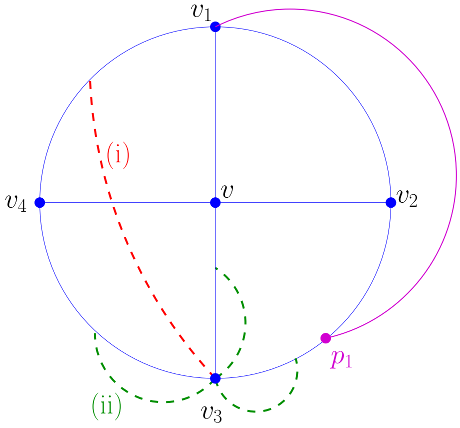


Shortness of H violated

Assume then U_3's vertices of attachment lie only on:

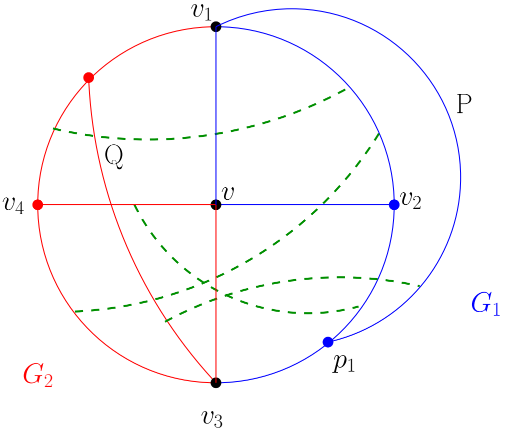
(i) R_4 (internally)

(ii) $p_1 R_2 v_3$, R_3, or P_3 (potentially at their endpoints)



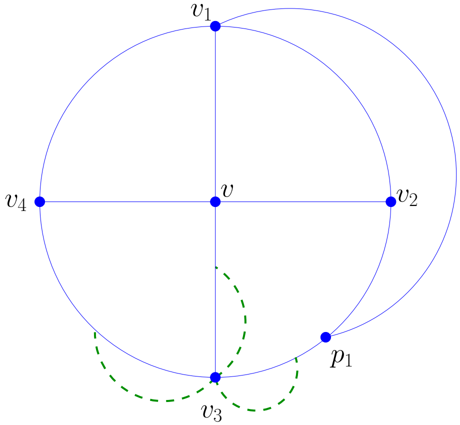
Case (c)(i): U_3 has a vertex of attachment internally on R_4

- By 4-connectivity: a path from G_1 to G_2, disjoint from $\{v_1, v, v_3\}$, which meets $G_1 \cup G_2$ only at endpoints.
- In each case, either shortness of H is violated, or a K_5^--subdivision is created. (Some cases require extra work to ensure 4-connectivity.)



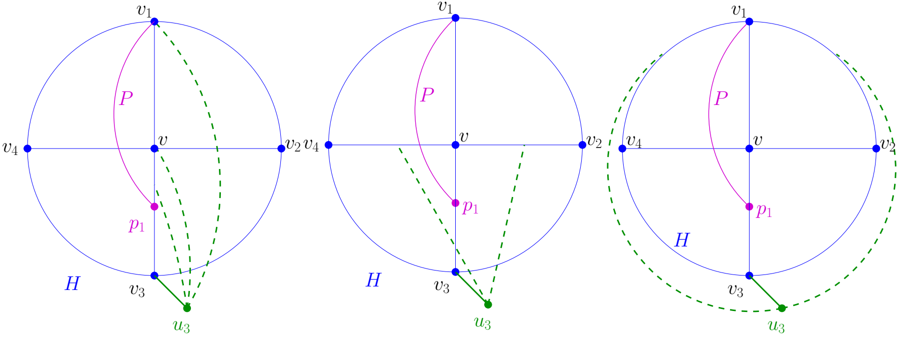
Case (c)(ii): U_3 only has vertices of attachment on $p_1R_2v_3$, R_3, or P_3

- We show that either 4-connectivity is violated, or there is some path that puts us in an earlier case.



Case (d): p_1 is an internal vertex of P_3

- Without loss of generality, choose P to minimise distance between p_1 and v_3 along P_3.
- By 4-connectivity, there is a fourth neighbour u_3 of v_3.
- Let U_3 be the $(H \cup P)$-bridge of G containing the edge $v_3 u_3$.
- Consider cases for where U_3's vertices of attachment can be...

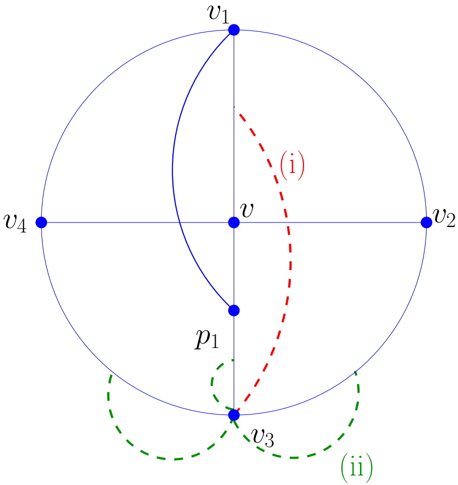


K_5^--subdivision created | Shortness of H violated | Symmetrically equivalent to Case (c)

Assume then that U_3's vertices of attachment lie only on:

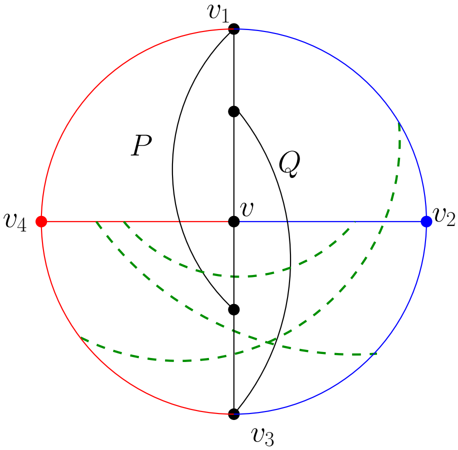
(i) P_1 (internally)

(ii) R_2, R_3, or $p_1 P_3 v_3$ (potentially at their endpoints)



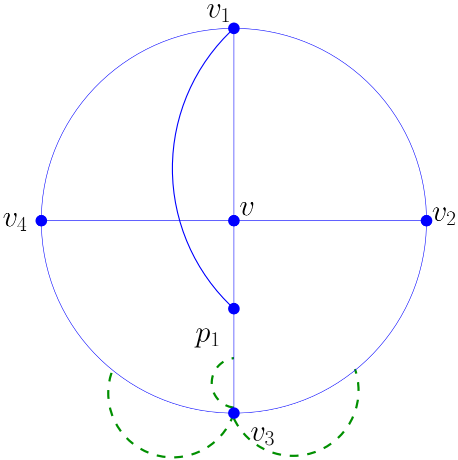
Case(d)(i): U_3 has some internal vertex of P_1 as a vertex of attachment

- By 4-connectivity: a path from G_1 to G_2, disjoint from $\{v_1, v, v_3\}$.
- In each case, either the shortness of H is violated, or a K_5^--subdivision is created, or 4-connectivity is violated.



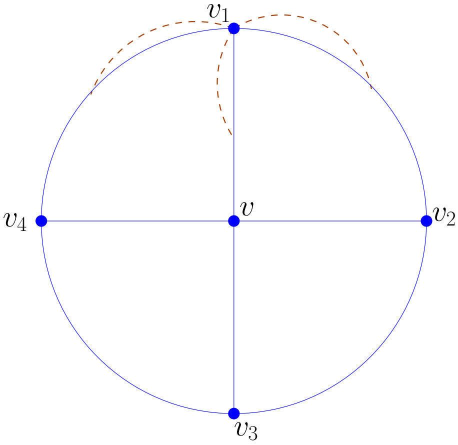
Case (d)(ii): U_3's vertices of attachment all lie on R_2, R_3, or $p_1 P_3 v_3$

- We show that either 4-connectivity is violated, or there is some path that puts the graph in an earlier case.



Case (e): p_1 **lies on** R_1**,** R_4**, or** P_1

- Preserving 4-connectivity requires a path that throws the graph back into a previous case.



Further work

- Obtain a complete characterisation for graphs with no K_5^--subdivision — need to consider graphs which are 3-connected but not 4-connected.
- Use this as a basis for a characterisation for graphs with no K_5-subdivision, using a similar systematic approach.