Topological containment of the 5-clique minus an edge in 4-connected graphs

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Formally: $G$ topologically contains $X$ iff $G$ contains some subgraph $Y$ such that $X$ can be obtained from $Y$ by performing a series of contractions limited to edges that have at least one endvertex of degree 2.

Also: $Y$ is an $X$-subdivision; $G$ contains an $X$-subdivision.
Problem of topological containment:

- \( TC(H) \): For some fixed pattern graph \( H \) — given a graph \( G \), does \( G \) contain an \( H \)-subdivision?
DISJOINT PATHS (DP)
Input: Graph $G$; pairs $(s_1, t_1), \ldots, (s_k, t_k)$ of vertices of $G$.
Question: Do there exist paths $P_1, \ldots, P_k$ of $G$, mutually vertex-disjoint, such that $P_i$ joins $s_i$ and $t_i$ ($1 \leq i \leq k$)?

- DISJOINT PATHS in $P$ for any fixed $k$ (Robertson & Seymour, 1995).
- $\implies TC(H)$ is also in $P$ — use DP repeatedly.
- Doesn’t give practical algorithms.
- We still want characterisations for particular pattern graphs.
Examples of good characterizations

- Trees — $K_3$
Kuratowski (1930) — $K_5$ or $K_{3,3}$ in non-planar graphs

Wagner (1937) and Hall (1943) strengthened this result to characterize graphs with no $K_{3,3}$-subdivisions
Duffin (1965) — $K_4$ in non-series-parallel graphs
Results for wheel graphs

- **$W_4$** (Farr, 1988) — If $G$ is 3-connected, $G$ contains a $W_4$-subdivision iff $G$ has a vertex of degree $\geq 4$

- **$W_5$** (Farr, 1988) — If $G$ is 3-connected with no internal 3-edge-cutsets, $G$ contains a $W_5$-subdivision iff $G$ has a vertex $v$ of degree $\geq 5$ and a circuit of length $\geq 5$ that does not contain $v$.

More recently, characterisations obtained for:

- graphs with no $W_6$-subdivision (Robinson & Farr, 2009); and
- graphs with no $W_7$-subdivision (Robinson & Farr, 2014).
Connections with Hajós’ Conjecture

- Of particular interest: solving $\text{TC}(K_5)$.
- Conjectured by Hajós, 1940s: no $K_k$-subdivision $\Rightarrow (k-1)$-colourable.
- Proved for $k \leq 4$ (Hadwiger, 1943; Dirac, 1952).
- Refuted for $k \geq 7$ (Catlin, 1979).
- For $k = 5$ and $k = 6$, remains an open problem.
- Characterisation for graphs with no $K_5$-subdivision may lead to solving $k = 5$. 
Progress towards solving $TC(K_5)$

- Kelmans-Seymour Conjecture:
  - 5-connected non-planar graph $\Rightarrow K_5$-subdivision
  - (recent proof by He, Wang, Yu, 2015-16).

- 4-connected graph $\Rightarrow K_5$ or $K_{2,2,2}$ as a minor (Halin & Jung, 1963) — but this doesn’t necessarily imply a $K_5$-subdivision.

- Possible step along the way: solve for slightly simpler graph, $K_5^-$.
We show: **every 4-connected graph contains a \( K_5^- \)-subdivision.**

- a step in parallel to the Kelmans-Seymour Conjecture.

**Approach:** start with a ‘base’ graph \((W_4)\): subgraph of the pattern graph \( H (K_5^-) \), and good characterisation is already known.

Look at all ways of enlarging base graph so conditions of hypothesis are met (in this case, 4-connectivity).

For each enlarged graph, determine whether it contains an \( H \)-subdivision.
Theorem

Let $G$ be a 4-connected graph. $G$ contains a $K_{5^{-}}$-subdivision.

Proof — a summary

- Farr (1988) — If $G$ is 3-connected, $G$ contains a $W_4$-subdivision iff $G$ has a vertex of degree $\geq 4$.
- Since $G$ is 4-connected, there exists a $W_4$-subdivision.
- Let $H$ be a $W_4$-subdivision in $G$, chosen such that $H$ is short.
Definition

Let $H$ be a $W_n$-subdivision in a graph $G$. We say that another $W_n$-subdivision $J$ in $G$ is shorter than $H$ if:

- the hubs of $H$ and $J$ are the same;
- the spokes of $H$ and $J$ are not all the same;
- each spoke of $J$ is an initial segment of a spoke of $H$; and
- at least one spoke of $J$ is a proper initial segment of a spoke of $H$.

If no other $W_n$-subdivision in $G$ is shorter than $H$, we call $H$ short.
By 4-connectivity, there is a fourth neighbour $u_1$ of $v_1$, where $u_1 \notin N_H(v_1)$.

At least three paths from $u_1$ to $H - v_1$, disjoint except at $u_1$, that meet $H$ only at their endpoints.

Let $P = v_1 u_1 +$ one of these paths.
Let $p_1$ be the vertex at which $P$ meets $H$. There are five cases to consider...
Case (a): $p_1$ is an internal vertex of $P_2$ or $P_4$

- Shortness of $H$ is violated.
Case (b): \( p_1 = v_3 \)

- \( H + P \) forms a \( K_5^- \)-subdivision.
Case (c): \( p_1 \) is an internal vertex of \( R_2 \) or \( R_3 \)

- Without loss of generality, assume \( p_1 \) is on \( R_2 \), and distance between \( p_1 \) and \( v_3 \) along \( R_2 \) is minimised.
By 4-connectivity, there is a fourth neighbour $u_3$ of $v_3$.
Let $U_3$ be the $(H \cup P)$-bridge of $G$ containing the edge $v_3u_3$.
We consider the cases for where $U_3$’s vertices of attachment can be.
Assume then $U_3$’s vertices of attachment lie only on:

(i) $R_4$ (internally)

(ii) $p_1 R_2 v_3$, $R_3$, or $P_3$ (potentially at their endpoints)
Case (c)(i): \( U_3 \) has a vertex of attachment internally on \( R_4 \)

- By 4-connectivity: a path from \( G_1 \) to \( G_2 \), disjoint from \( \{v_1, v, v_3\} \), which meets \( G_1 \cup G_2 \) only at endpoints.
- In each case, either shortness of \( H \) is violated, or a \( K_5^- \)-subdivision is created. (Some cases require extra work to ensure 4-connectivity.)
Case (c)(ii): $U_3$ only has vertices of attachment on $p_1 R_2 v_3$, $R_3$, or $P_3$

- We show that either 4-connectivity is violated, or there is some path that puts us in an earlier case.
Case (d): $p_1$ is an internal vertex of $P_3$

- Without loss of generality, choose $P$ to minimise distance between $p_1$ and $v_3$ along $P_3$.
- By 4-connectivity, there is a fourth neighbour $u_3$ of $v_3$.
- Let $U_3$ be the $(H \cup P)$-bridge of $G$ containing the edge $v_3 u_3$.
- Consider cases for where $U_3$’s vertices of attachment can be.

$K_5^-$-subdivision created  Shortness of $H$ violated  Symmetrically equivalent to Case (c)
Assume then that $U_3$’s vertices of attachment lie only on:

(i) $P_1$ (internally)

(ii) $R_2$, $R_3$, or $p_1 P_3 v_3$ (potentially at their endpoints)
Case(d)(i): \( U_3 \) has some internal vertex of \( P_1 \) as a vertex of attachment

- By 4-connectivity: a path from \( G_1 \) to \( G_2 \), disjoint from \( \{v_1, v, v_3\} \).
- In each case, either the shortness of \( H \) is violated, or a \( K_{5^-} \)-subdivision is created, or 4-connectivity is violated.
Case (d)(ii): $U_3$’s vertices of attachment all lie on $R_2$, $R_3$, or $p_1P_3v_3$

- We show that either 4-connectivity is violated, or there is some path that puts the graph in an earlier case.
Case (e): $p_1$ lies on $R_1$, $R_4$, or $P_1$

- Preserving 4-connectivity requires a path that throws the graph back into a previous case.
Further work

- Obtain a complete characterisation for graphs with no $K_5^-$-subdivision — need to consider graphs which are 3-connected but not 4-connected.

- Use this as a basis for a characterisation for graphs with no $K_5$-subdivision, using a similar systematic approach.