Graphs with no 7-wheel subdivision

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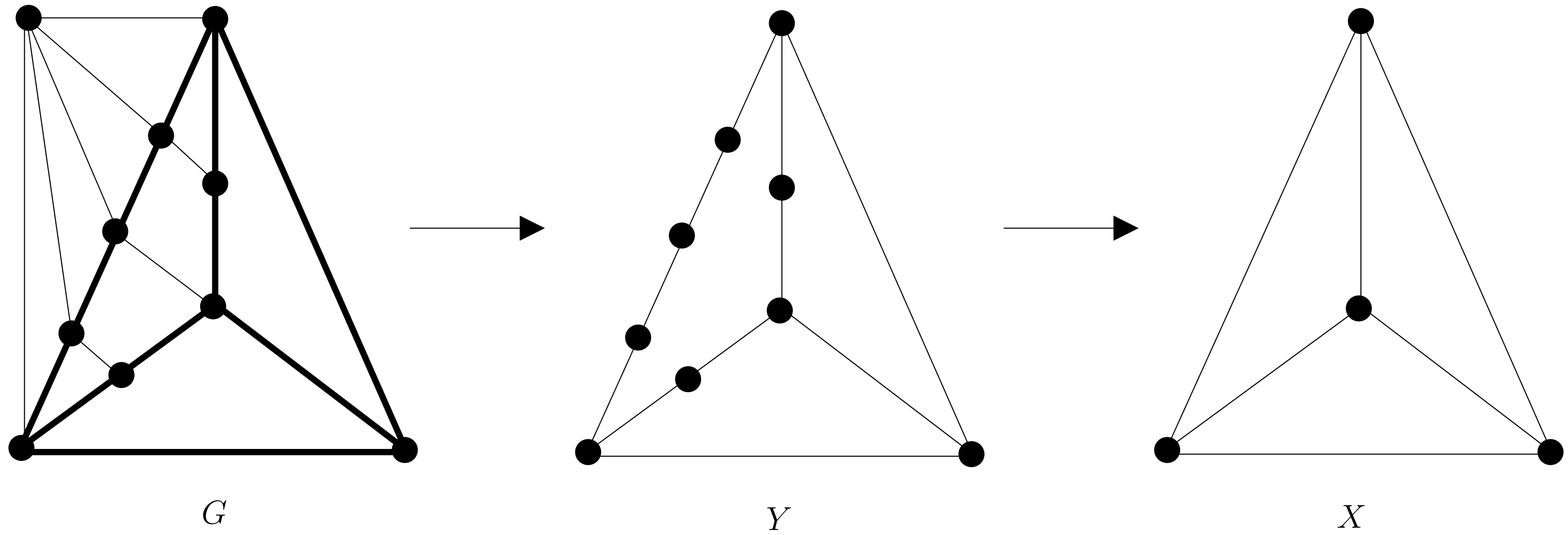
(joint work with Graham Farr)

1 Topological containment



- G *topologically contains* X
- G contains an X-*subdivision*

Applications of topological containment

- Forest — does not contain any K_3-subdivisions
- Planar graph — does not contain any K_5-subdivisions or $K_{3,3}$-subdivisions (Kuratowski, 1930)
- Series-parallel graph — does not contain any K_4-subdivisions (Duffin, 1965)

Problem of topological containment:

- For some fixed *pattern graph* H: given a graph G, does G contain an H-subdivision?

2 Robertson and Seymour results

DISJOINT PATHS (DP)

Input: Graph G; pairs $(s_1, t_1), ..., (s_k, t_k)$ of vertices of G.

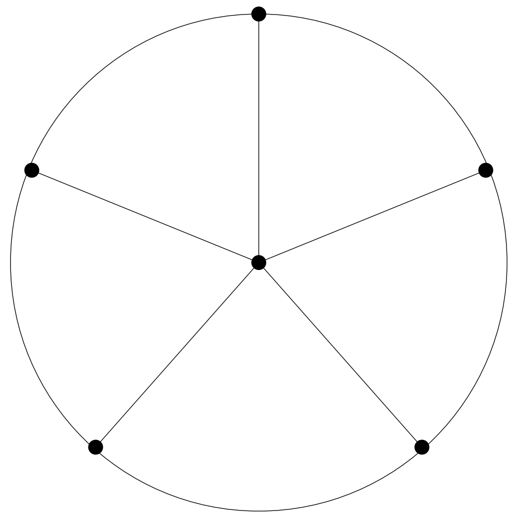
Question: Do there exist paths $P_1, ..., P_k$ of G, mutually vertex-disjoint, such that P_i joins s_i and t_i $(1 \leq i \leq k)$?

- DISJOINT PATHS is in P for any fixed k.
- This implies topological containment problem for fixed H is also in P — use DP repeatedly.
- We know p-time algorithms must exist for topological containment, but practical algorithms not given — huge constants.

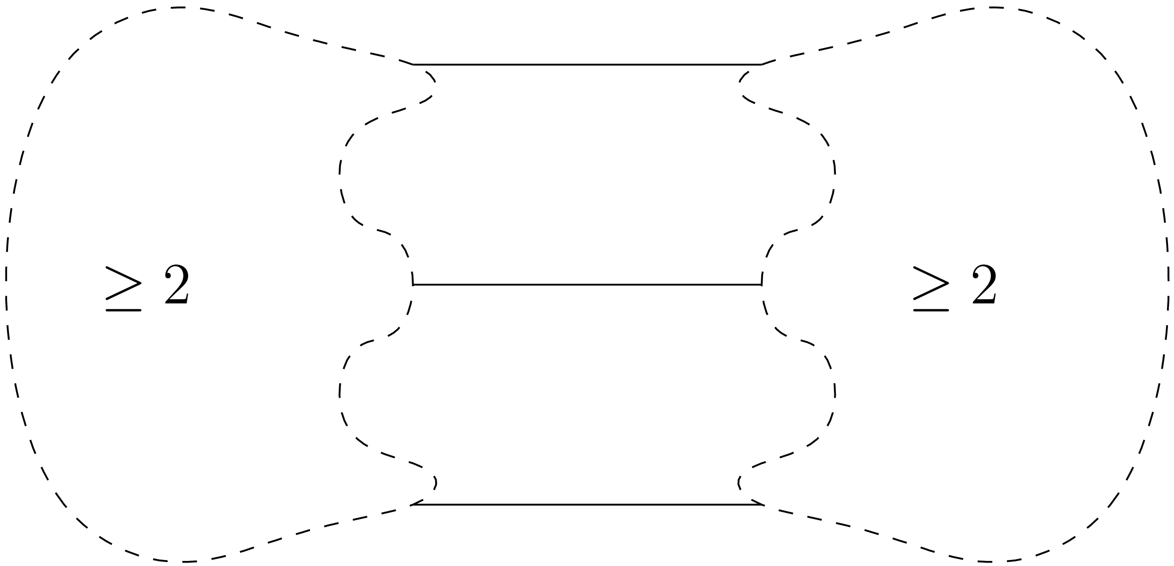
3 Previous results

Theorem (Farr, 88).

Let G be 3-connected, with no internal 3-edge-cutset. Then G has a W_5-subdivision if and only if G has a vertex v of degree at least 5 and a circuit of size at least 5 which does not contain v.



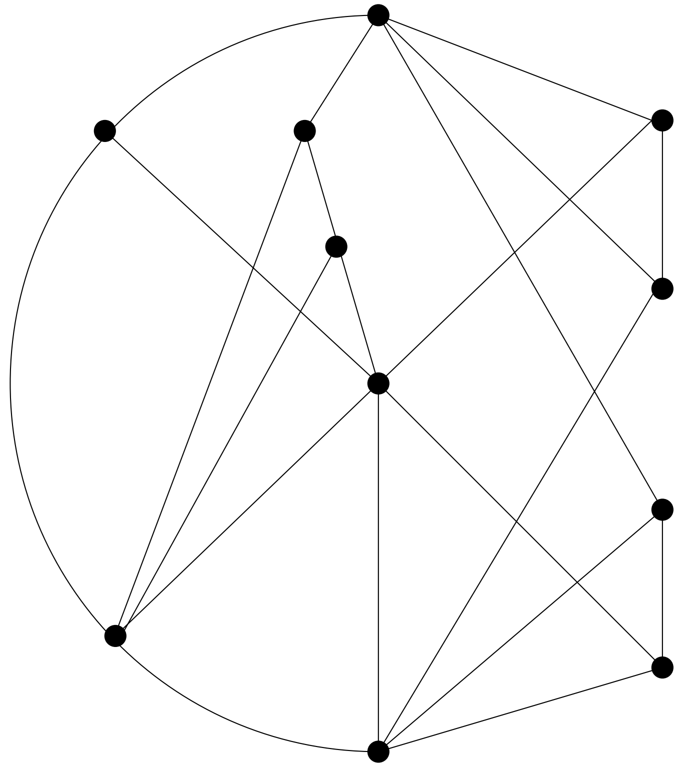
W_5: wheel with five spokes



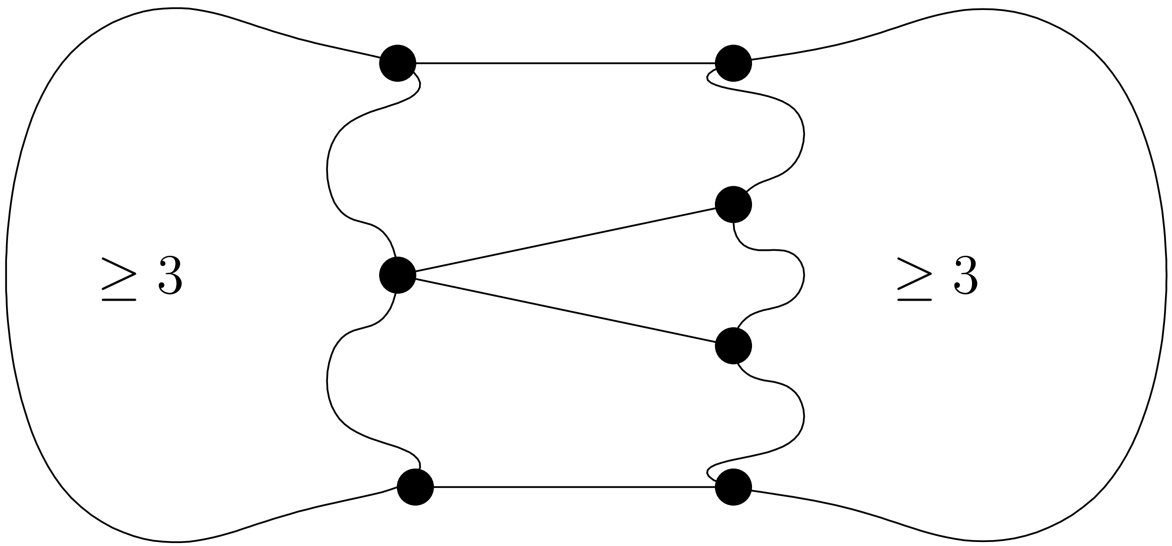
Internal 3-edge-cutset

Theorem (Robinson & Farr, 2008).

Let G be a 3-connected graph that is not topologically contained in the graph A. Suppose G has no internal 3-edge-cutsets, no internal 4-edge-cutsets, and is a graph on which neither Reduction 1A nor Reduction 2A can be performed, for $k = 6$. Then G has a W_6-subdivision if and only if G has a vertex v of degree at least 6.



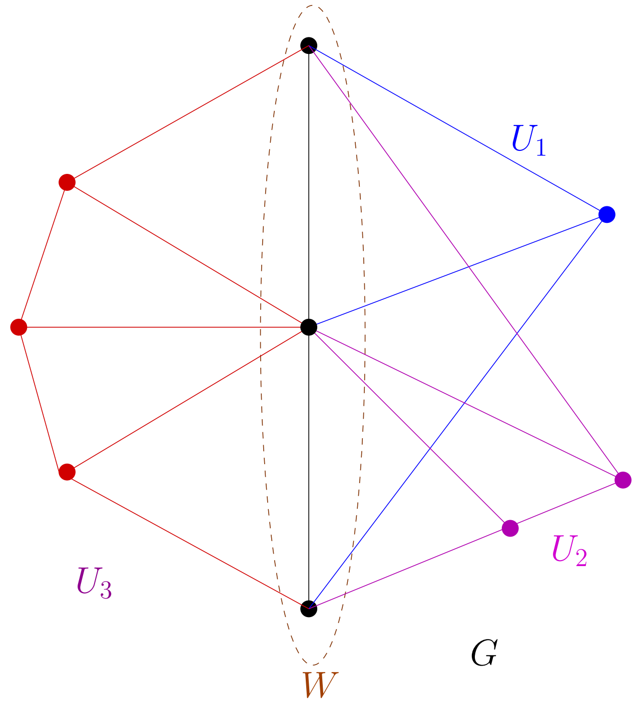
Graph A



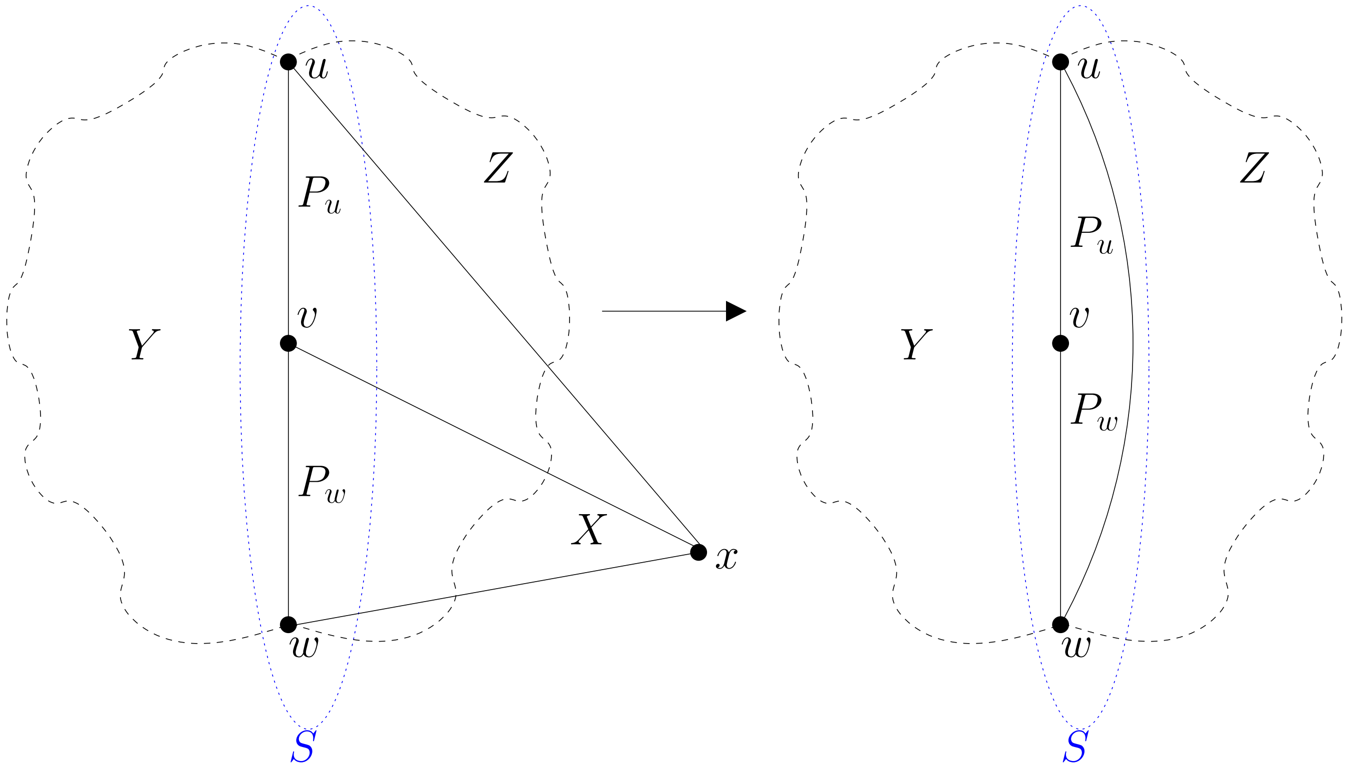
Internal 4-edge-cutset

Definition

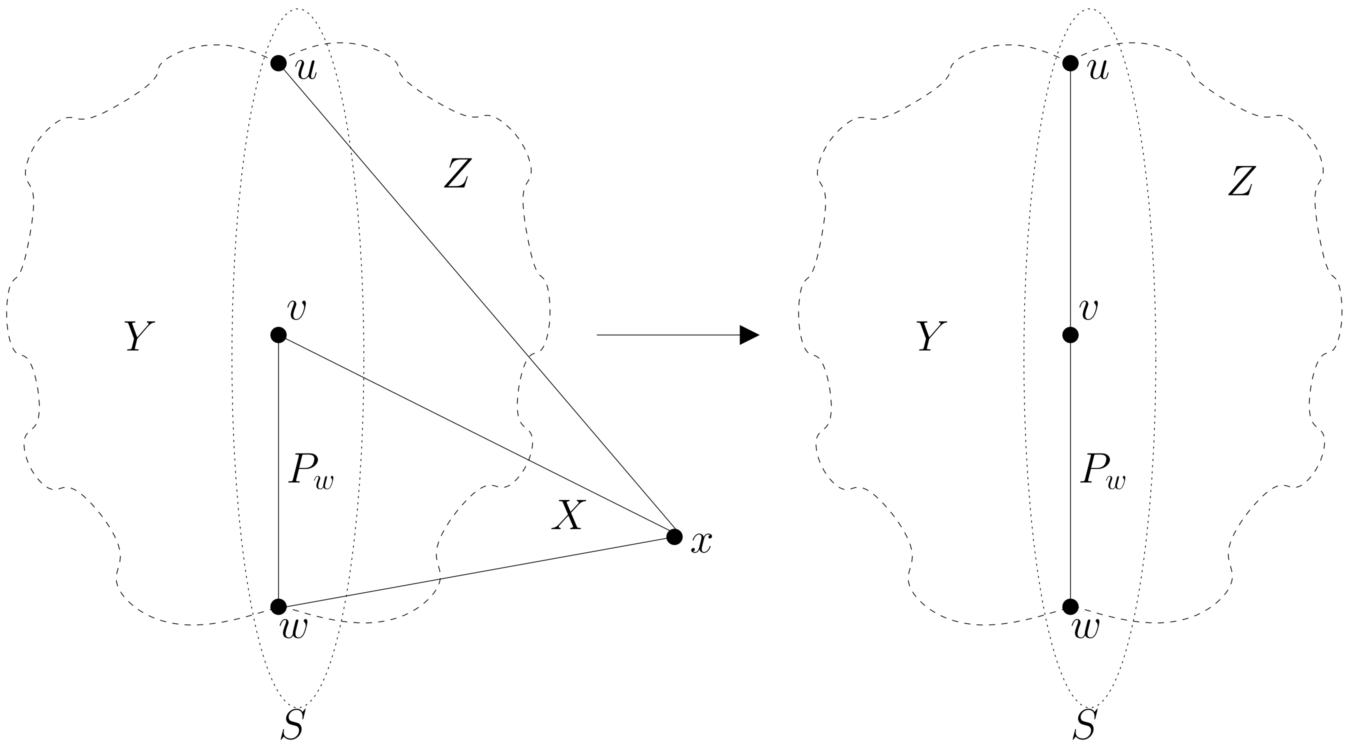
If W is a subset of graph G, then $G|W$ denotes the set of all maximal subsets U of $V(G)$ such that any two vertices of U are joined by a path in G with no internal vertex in W. Each element of $G|W$ is referred to as a *bridge* of $G|W$.



U_1, U_2, U_3 are all bridges of $G|W$.

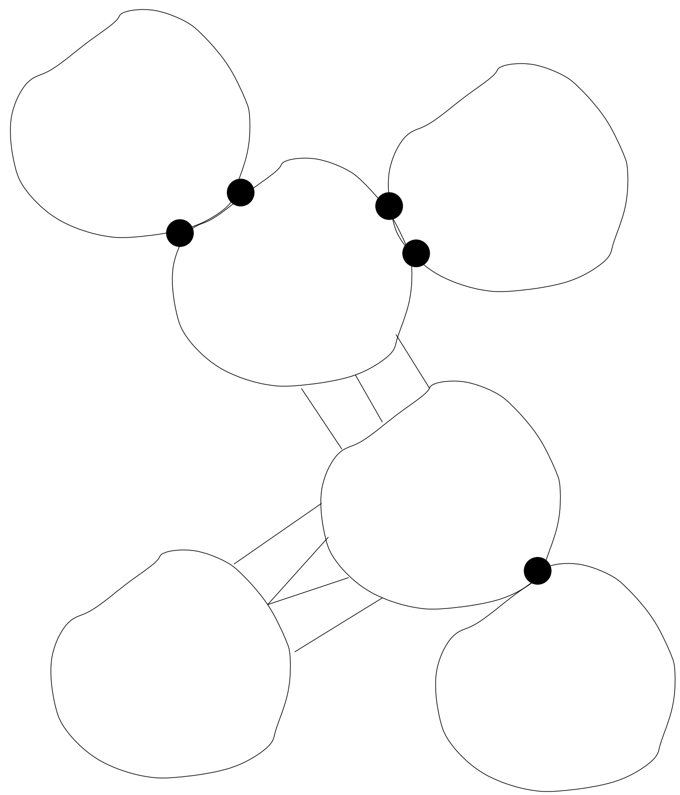


Reduction 1A



Reduction 2A

Structure of graphs with no W_6-subdivisions



4 New results based on W_6 theorem

Theorem.

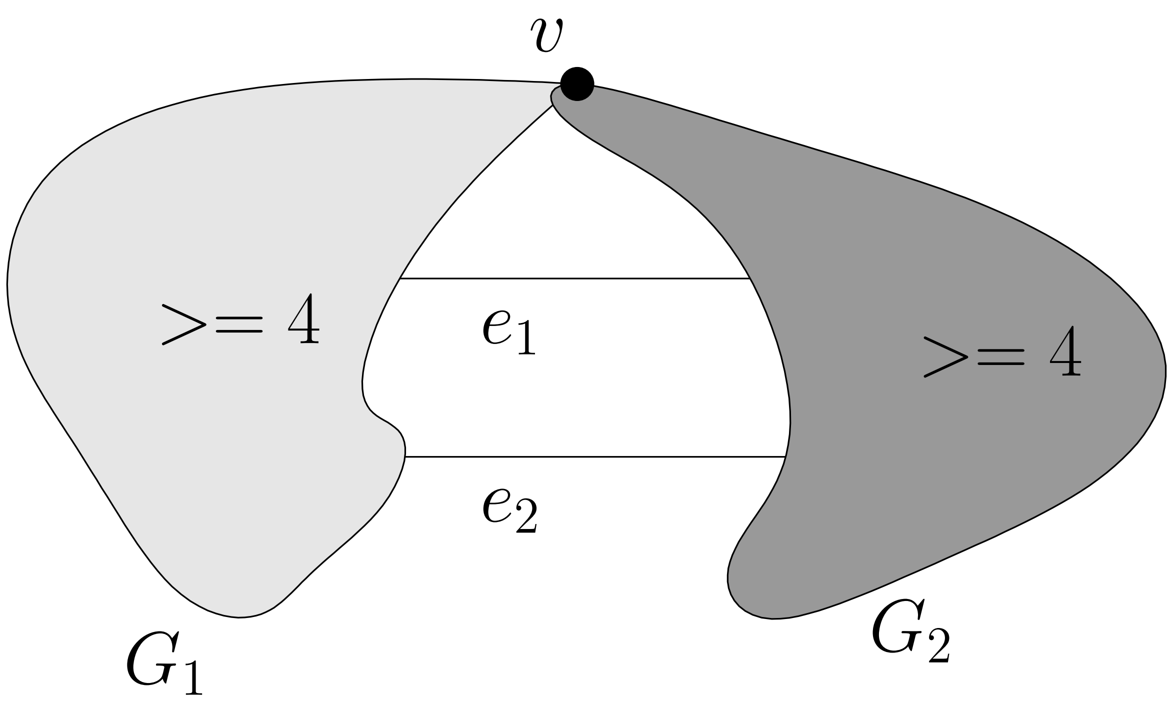
Let G be a 3-connected graph with at least 12 vertices. Suppose G has no internal 3-edge-cutsets, no internal 4-edge-cutsets, and is a graph on which neither Reduction 1A nor Reduction 2A can be performed, for $k = 6$.

Then G has a W_6-subdivision if and only if G contains some vertex v_0 of degree at least 6.

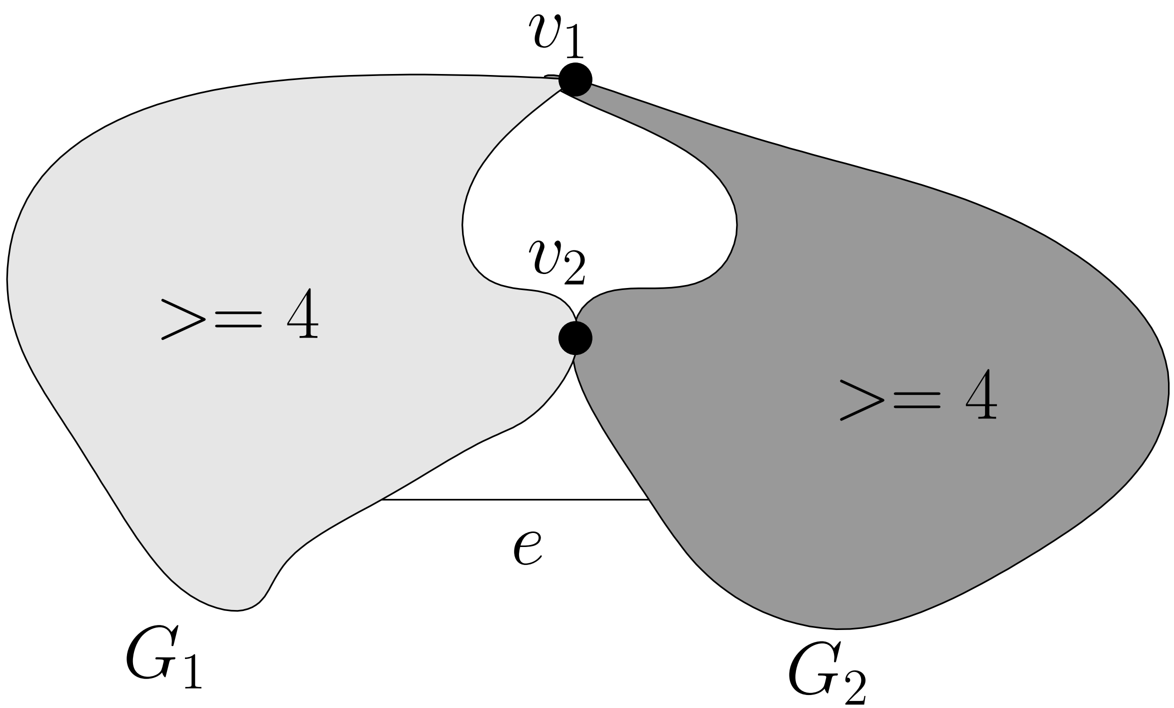
Theorem.

Let G be a 3-connected graph with at least 14 vertices. Suppose G has no type 1, 2, 3, or 4 edge-vertex-cutsets, and is a graph on which Reductions 1A, 1B, 2A, and 2B cannot be performed, for $k = 7$. Let v_0 be a vertex of degree ≥ 6 in G.

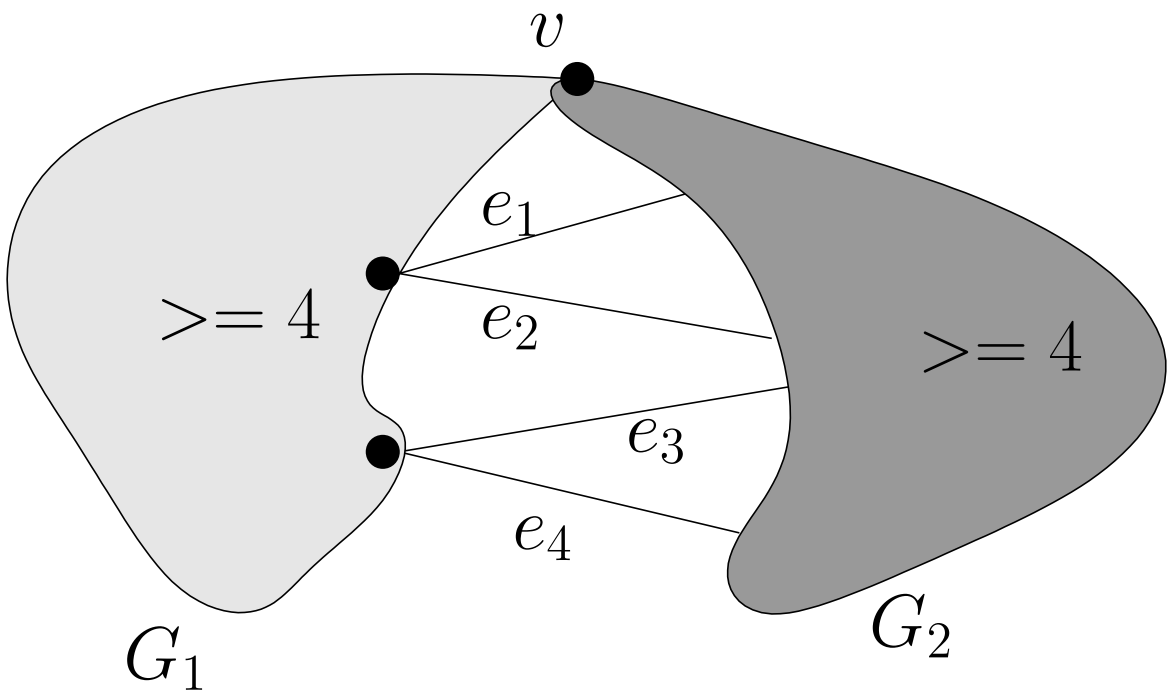
Then, either G has a W_6-subdivision centred on v_0, or G has a W_6-subdivision centred on some vertex v_1 of degree ≥ 7.



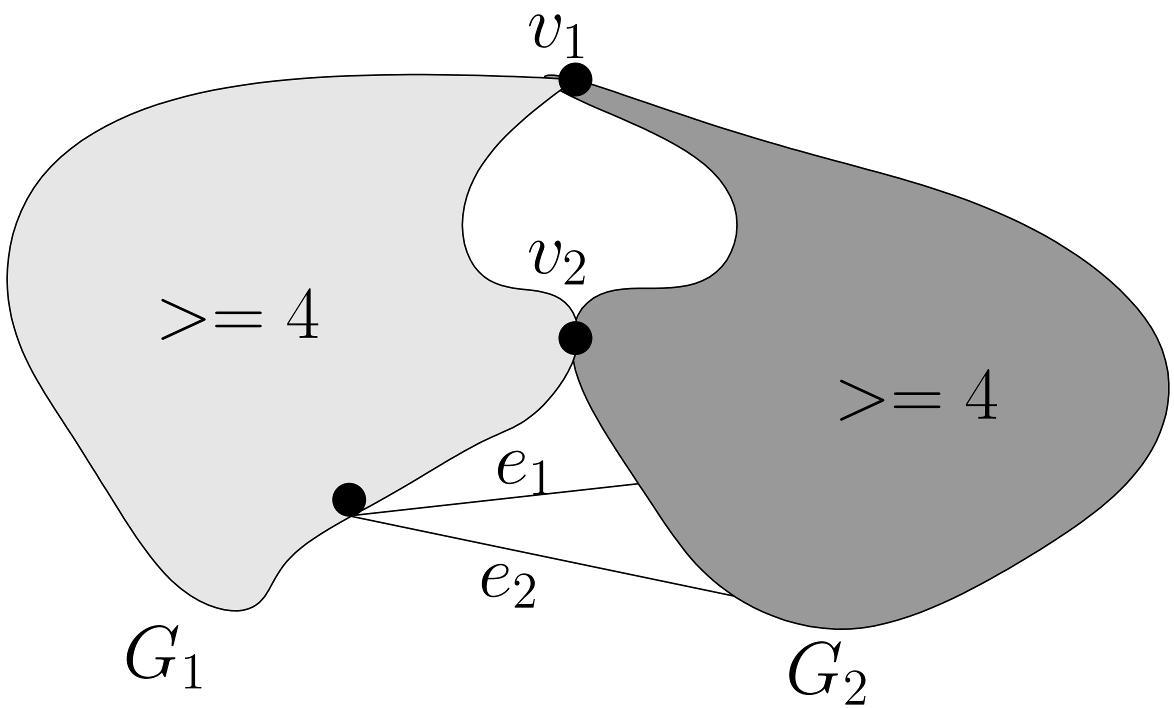
Type 1 edge-vertex-cutset



Type 2 edge-vertex-cutset



Type 3 edge-vertex-cutset



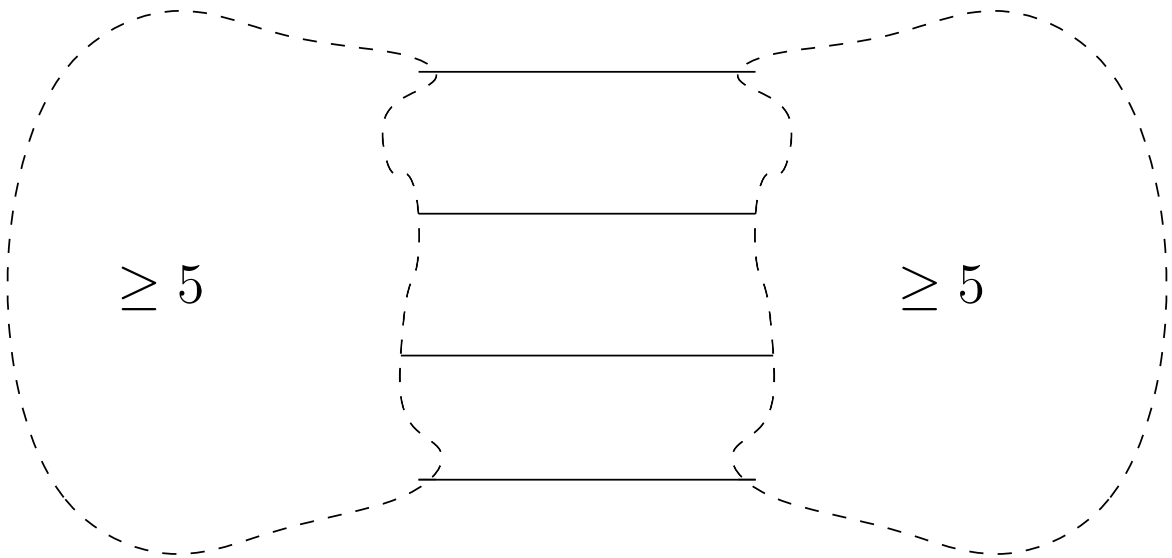
Type 4 edge-vertex-cutset

5 New result on W_7 graphs

- Characterization (up to bounded size pieces) of graphs that do not contain W_7-subdivisions
- Uses similar techniques to the W_5 and W_6 results

Theorem.

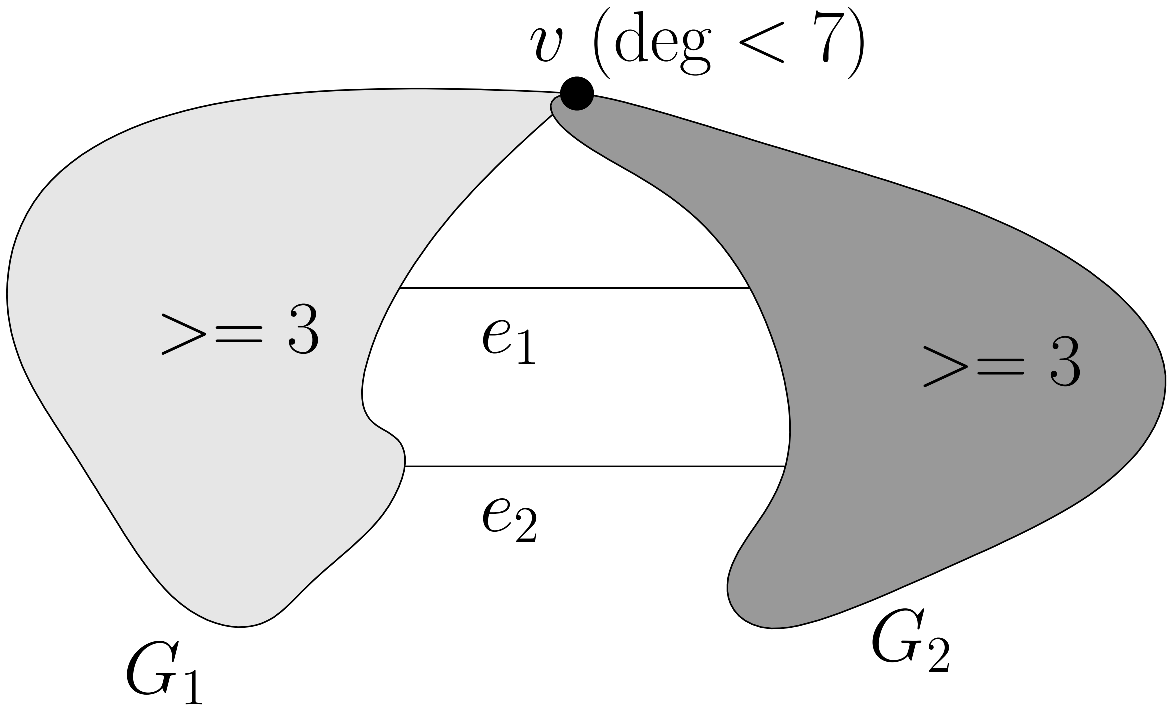
Let G be a 3-connected graph with at least 38 vertices. Suppose G has no internal 3 or 4-edge-cutsets, no internal $(1, 1, 1, 1)$-cutsets . . .



Internal $(1, 1, 1, 1)$-cutset

Theorem.

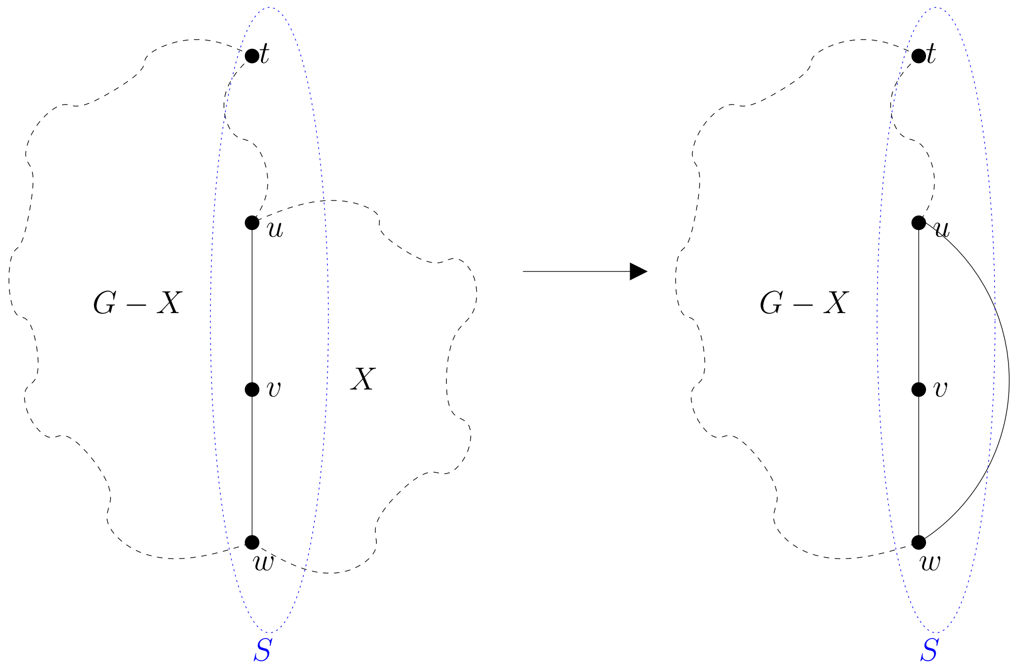
Let G be a 3-connected graph with at least 38 vertices. Suppose G has no internal 3 or 4-edge-cutsets, no internal $(1,1,1,1)$-cutsets, no type 1, 1a, 2, 2a, 3, 3a, 4, or 4a edge-vertex-cutsets . . .



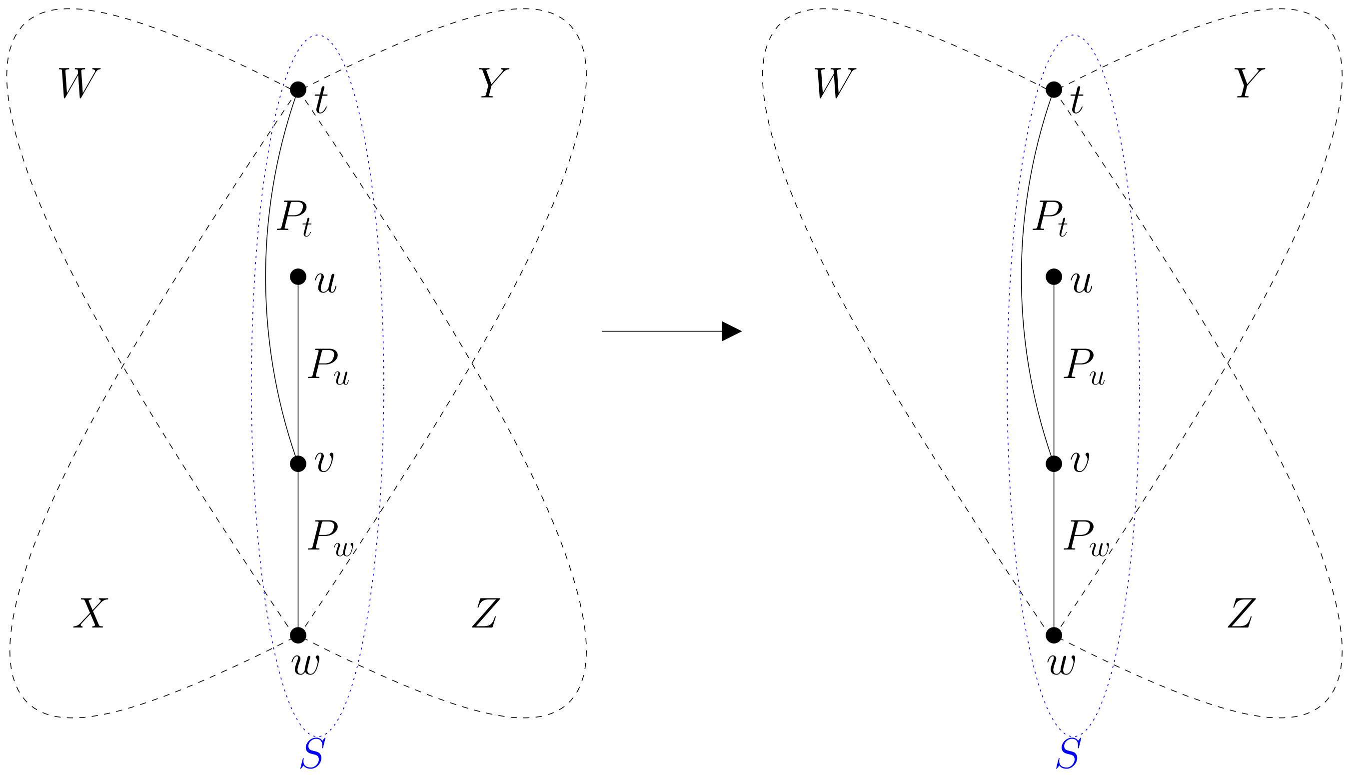
Type 1a edge-vertex-cutset

Theorem.

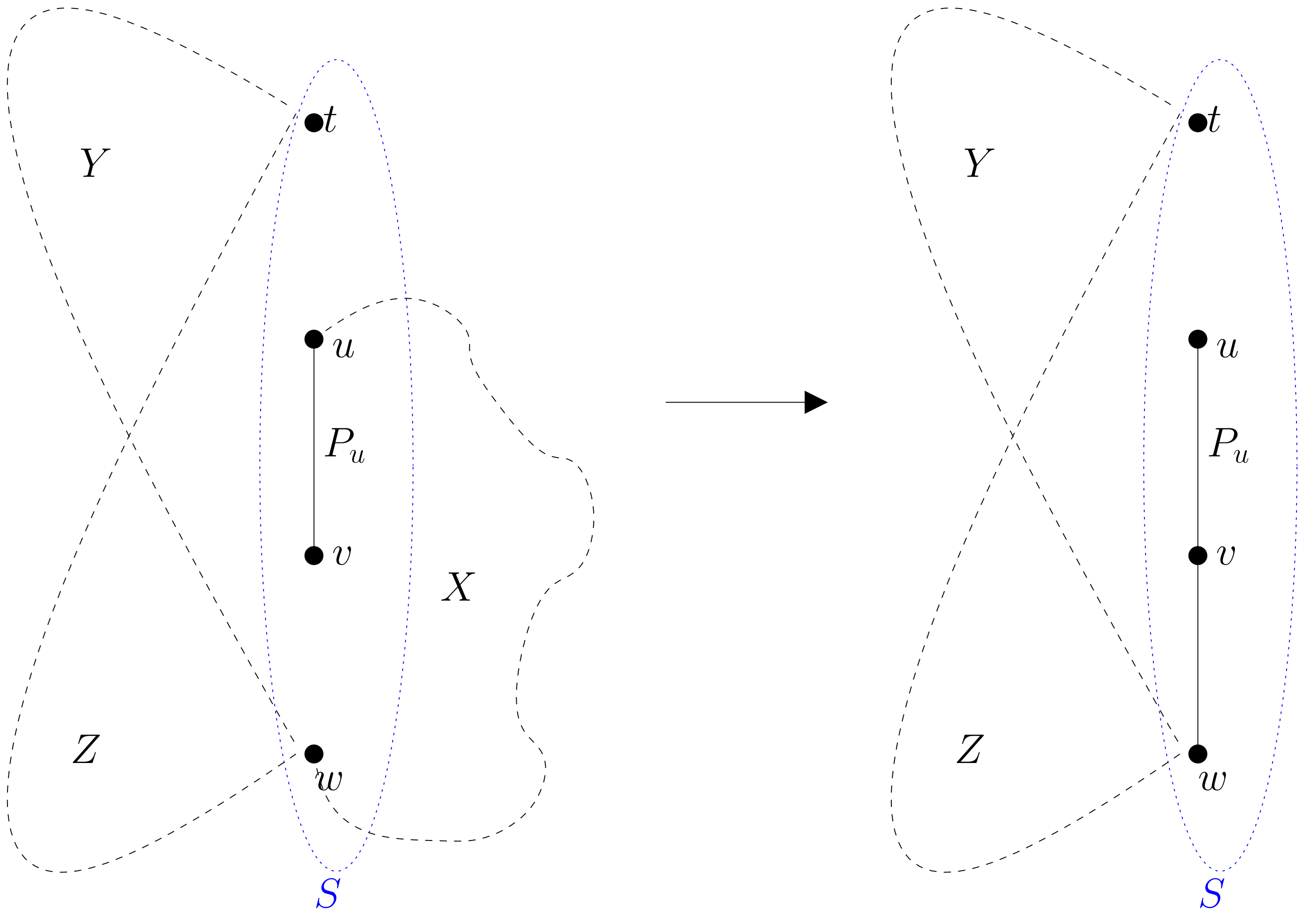
Let G be a 3-connected graph with at least 38 vertices. Suppose G has no internal 3 or 4-edge-cutsets, no internal $(1,1,1,1)$-cutsets, no type 1, 1a, 2, 2a, 3, 3a, 4, or 4a edge-vertex-cutsets, and is a graph on which Reductions 1A, 1B, 1C, 2A, 2B, 3, 4, 5, and 6 cannot be performed, for $k = 7$ ...



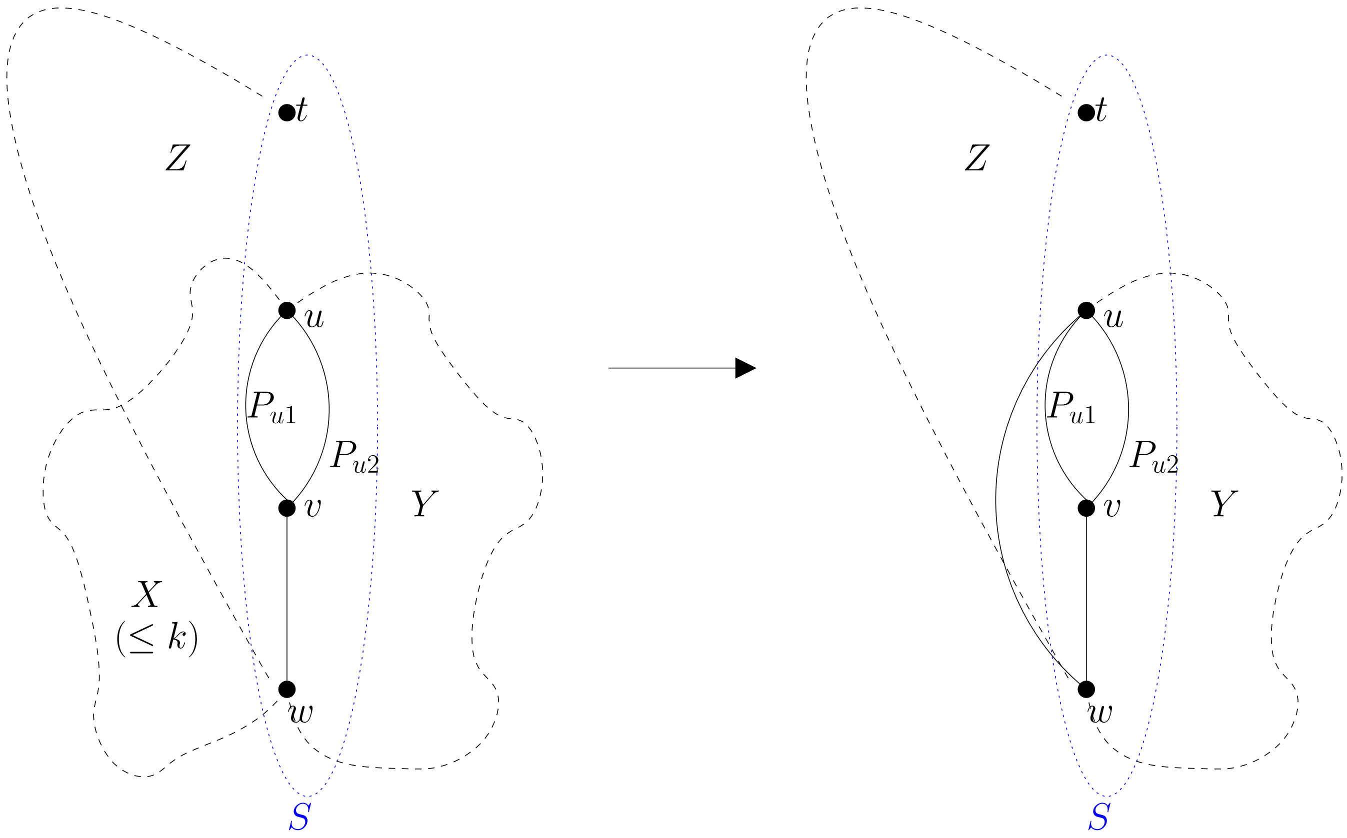
Reduction 3



Reduction 4



Reduction 5



Reduction 6

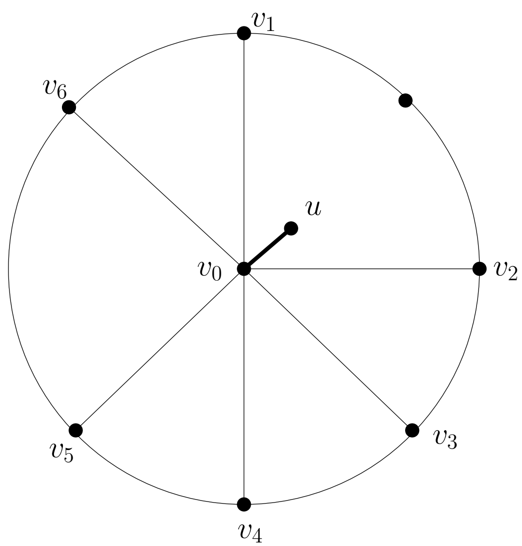
Theorem.

Let G be a 3-connected graph with at least 38 vertices. Suppose G has no internal 3 or 4-edge-cutsets, no internal $(1, 1, 1, 1)$-cutsets, no type 1, 1a, 2, 2a, 3, 3a, 4, or 4a edge-vertex-cutsets, and is a graph on which Reductions 1A, 1B, 1C, 2A, 2B, 3, 4, 5, and 6 cannot be performed, for $k = 7$.

Then G has a W_7-subdivision if and only if G contains some vertex v_0 of degree at least 7.

Proof — a summary.

- Suppose the conditions of the hypothesis hold for some graph G.
- By the strengthened W_6 result, there exists some vertex v_0 of degree ≥ 7 in G that has a W_6-subdivision H centred on it.



- How does u connect to the rest of H in order to preserve 3-connectivity?

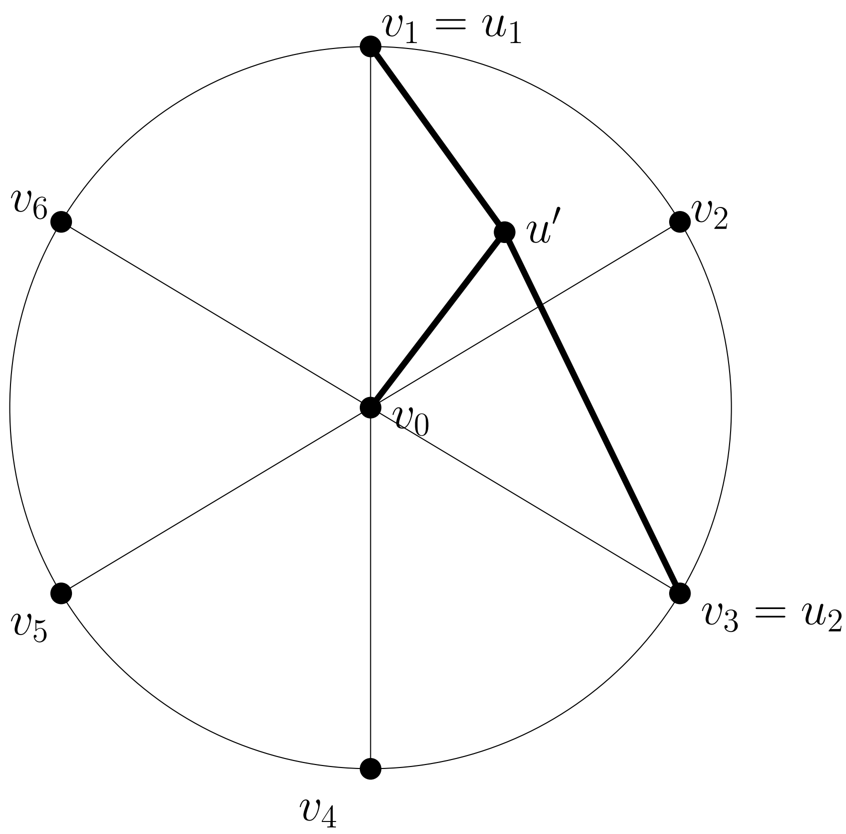
Three possibilities:

(a) Path from v_0 to some vertex u_1 on the rim of the W_6-subdivision, not meeting any spoke.

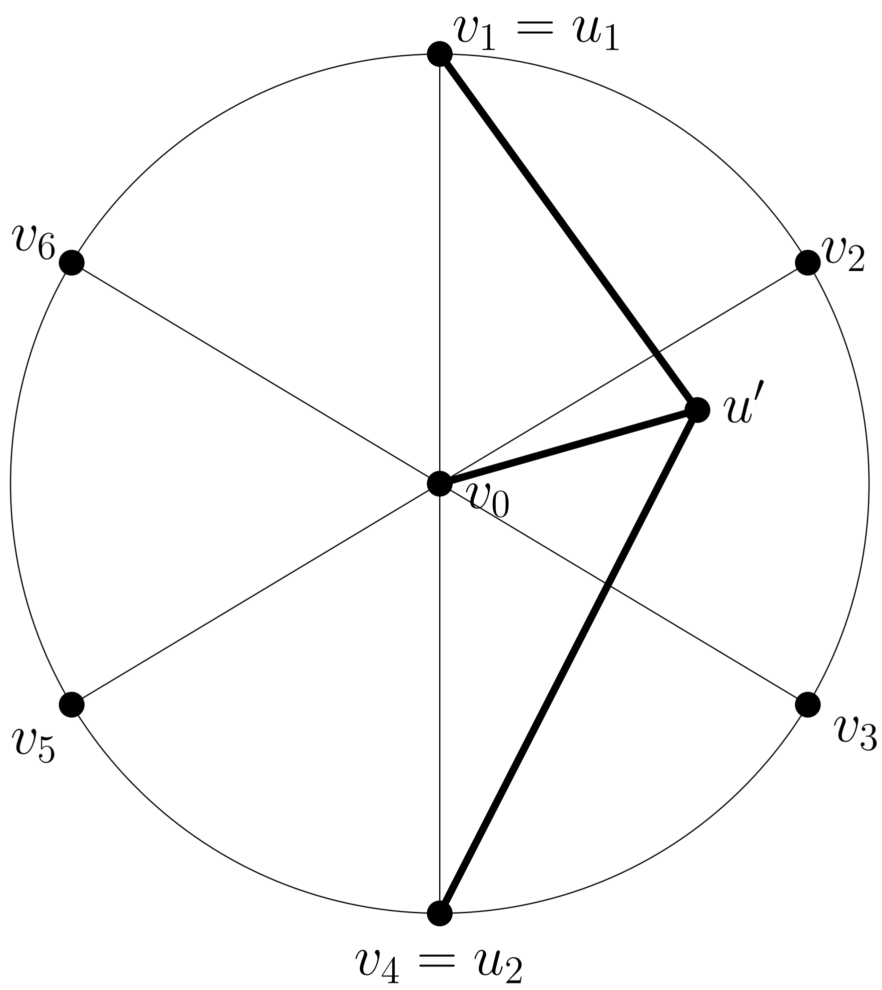
(b) Two paths from u to two separate spokes of H.

(c) Path from v_0 to some vertex u_1 on one of the spokes of the W_5-subdivision, such that this path that does not meet H except at its end points.

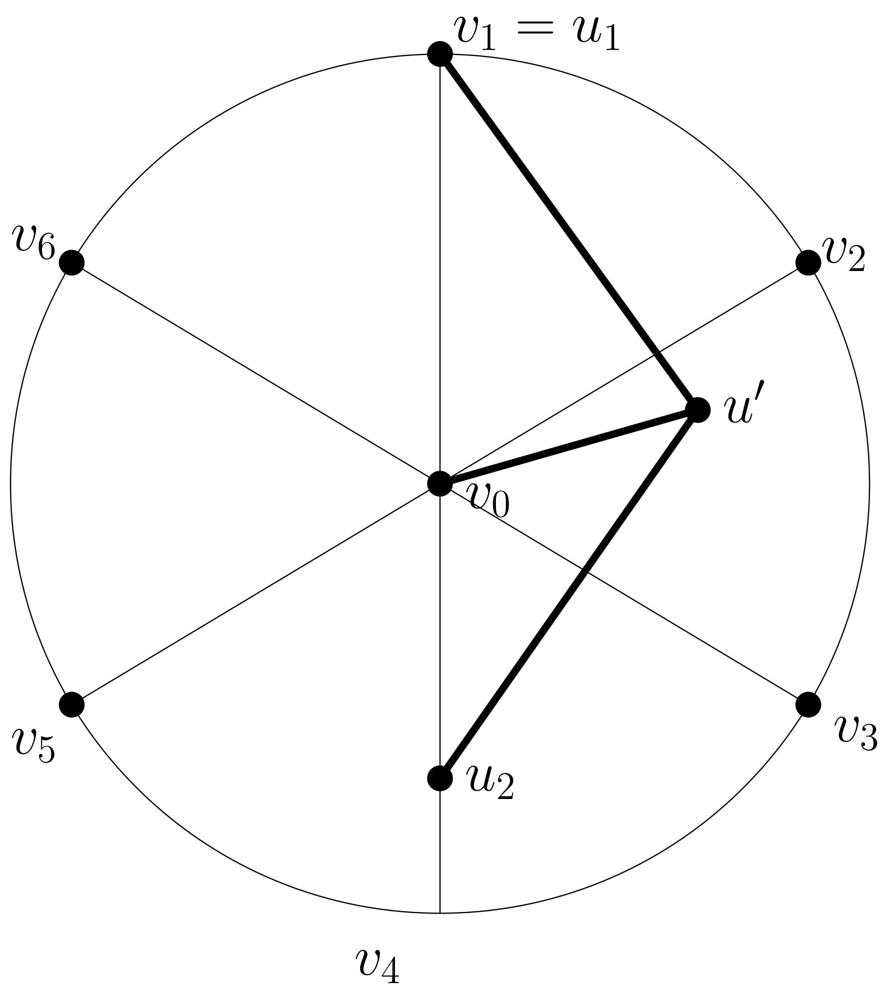
- Cases (a) and (c) are straightforward to deal with.
- Case (b) takes up most of the proof.
- All possible configurations in case (b) result in a W_7-subdivision except for three.



Case (b)(i)



Case (b)(ii)



Case (b)(iii)

- These graphs meet the 3-connectivity requirements, but not the other requirements of the hypothesis: eg. forbidden reductions.
- So there must be more structure to the graphs.
- More in-depth case analysis required, based on different ways of adding this structure.
- C program to automate parts of this analysis; many parts of the proof depend on results generated by this program.

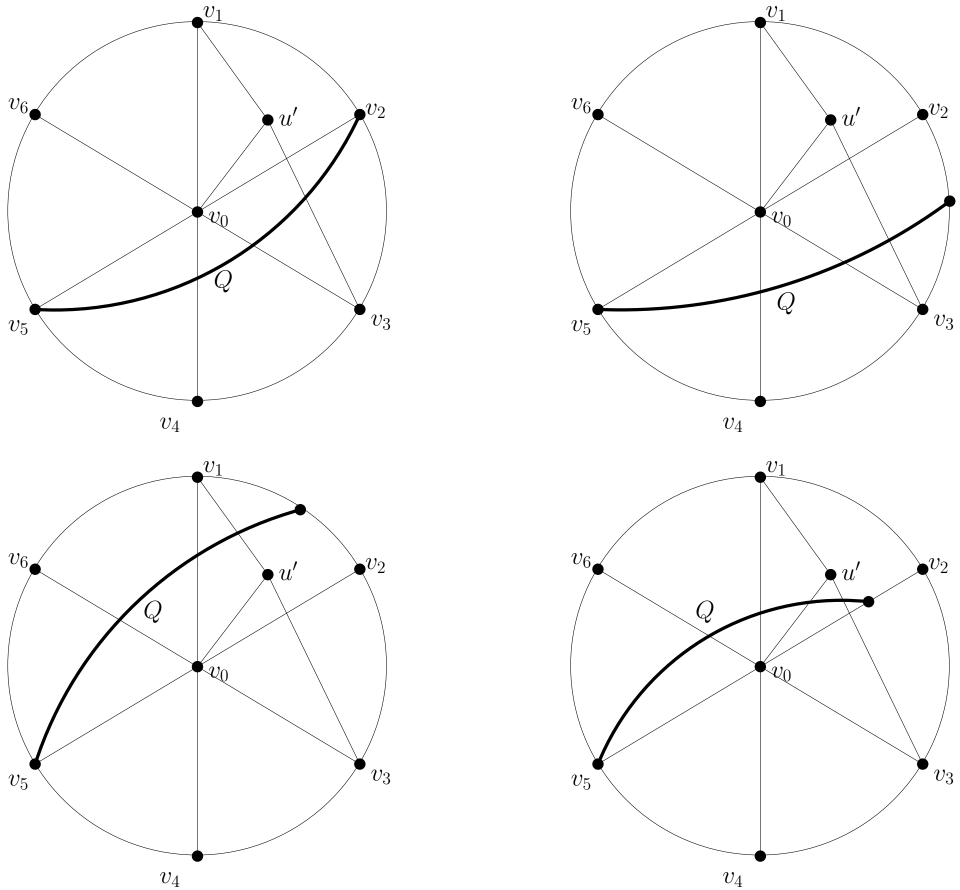
The program:

- constructs the various simple graphs that arise as cases in the proof, and
- tests each graph for the presence of a W_7-subdivision.

Case (b)(i): further detail

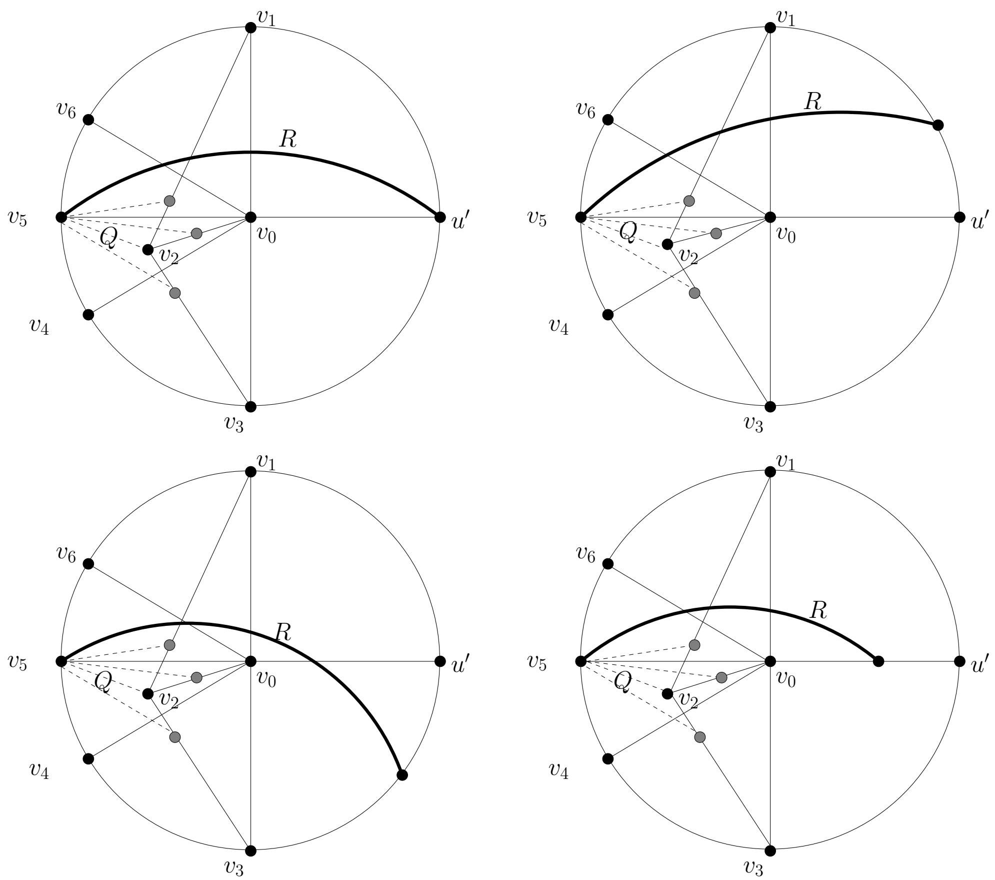
1. Path Q from H_2 to H_4

- Four cases with no W_7-subdivision



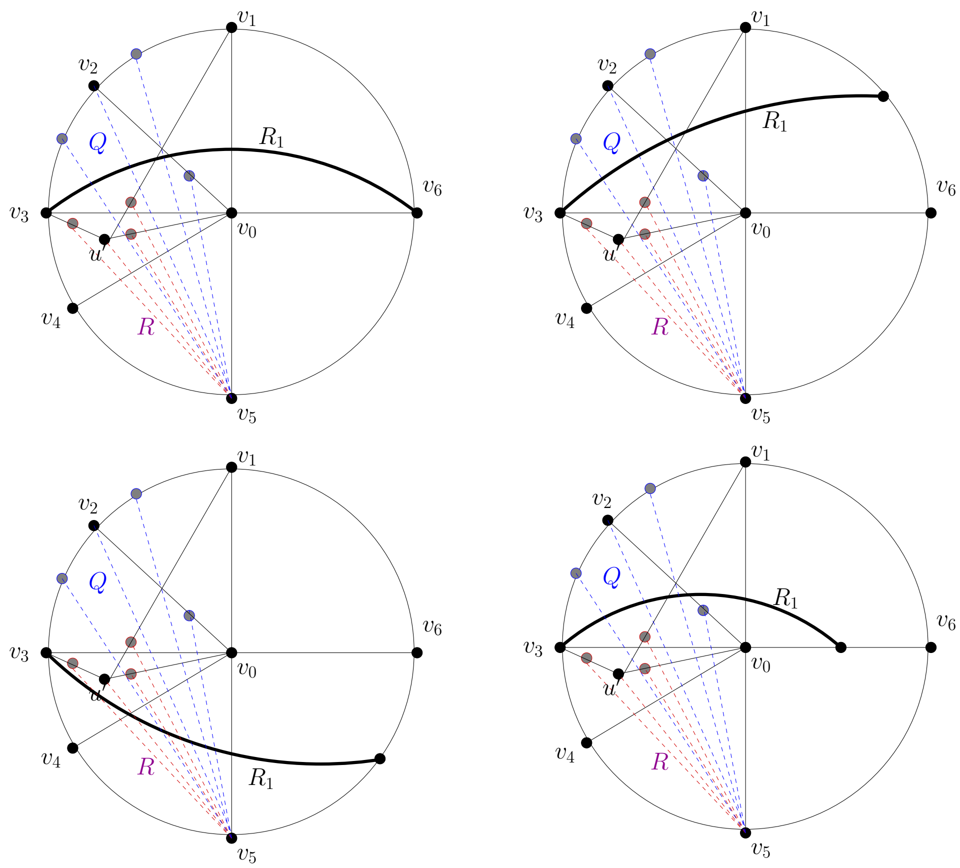
1.1. Path R from $U(u)$ to $H_2 \cup H_4$

- 16 cases with no W_7-subdivision



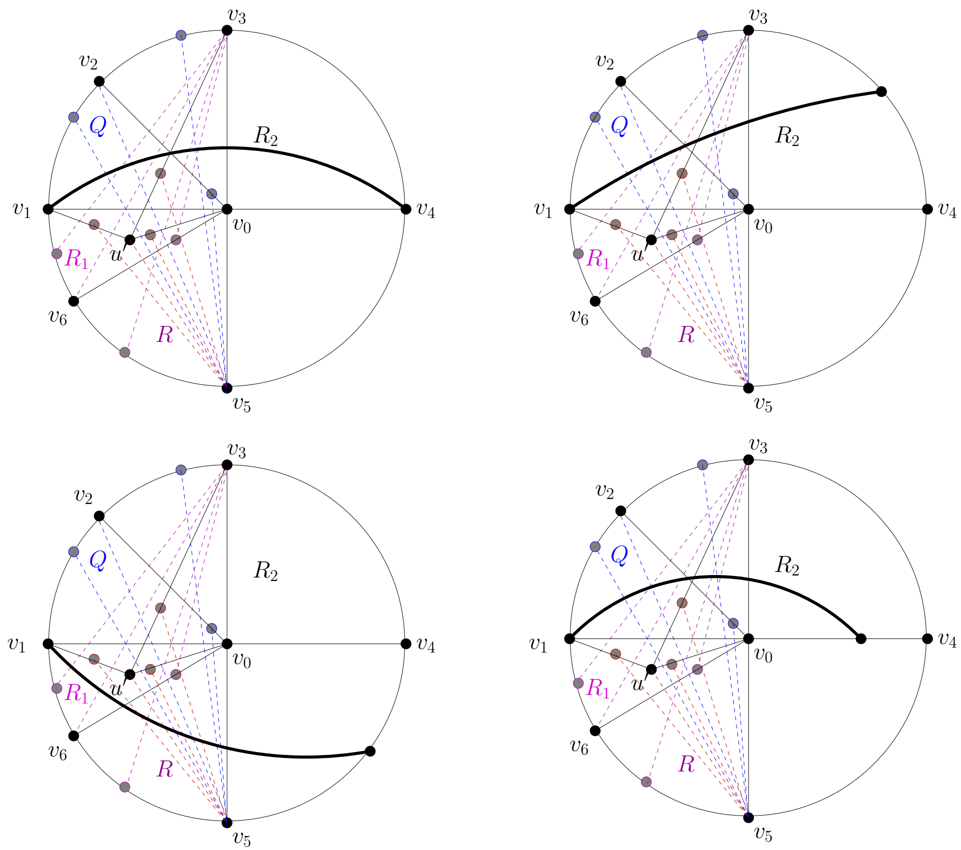
1.1.1. Path R_1 such that $S_1 = \{v_0, v_1, v_5\}$ is not a separating set

- 64 cases with no W_7-subdivision

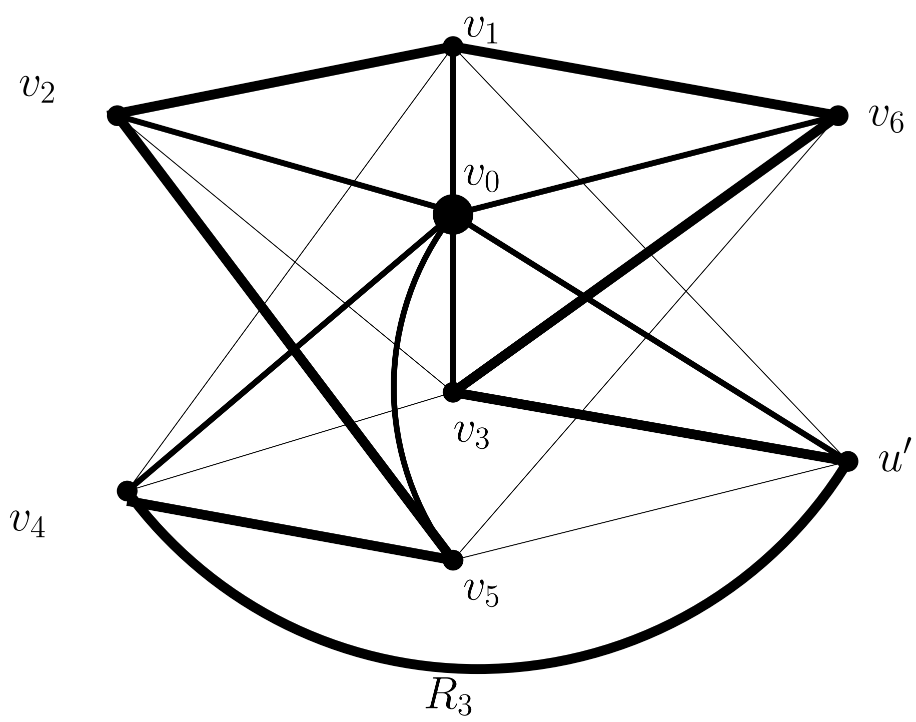


1.1.1.1. Path R_2 such that $S_2 = \{v_0, v_3, v_5\}$ is not a separating set

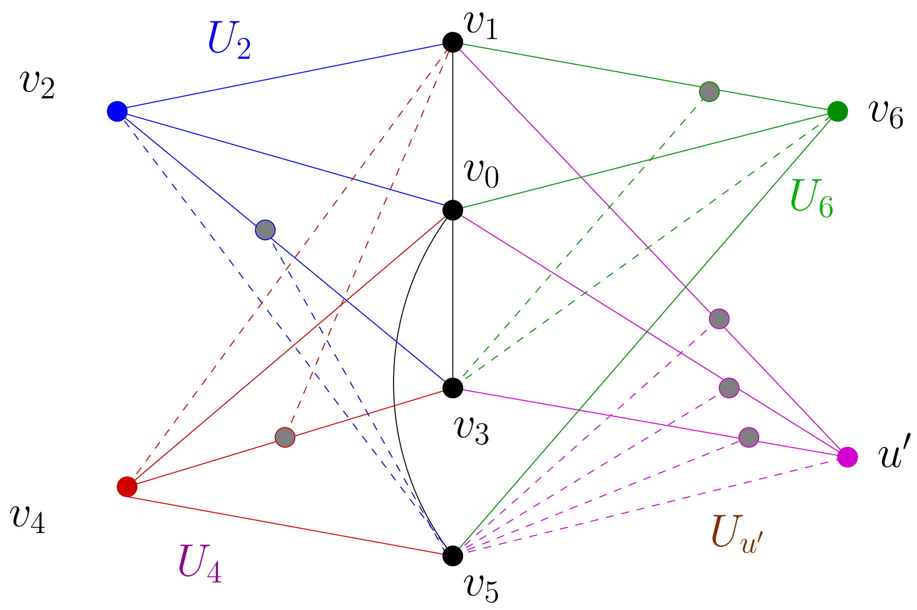
- 256 cases with no W_7-subdivision



1.1.1.1.1. Path R_3 such that $S_3 = \{v_0, v_1, v_3, v_5\}$ is not a separating set — always results in a W_7-subdivision

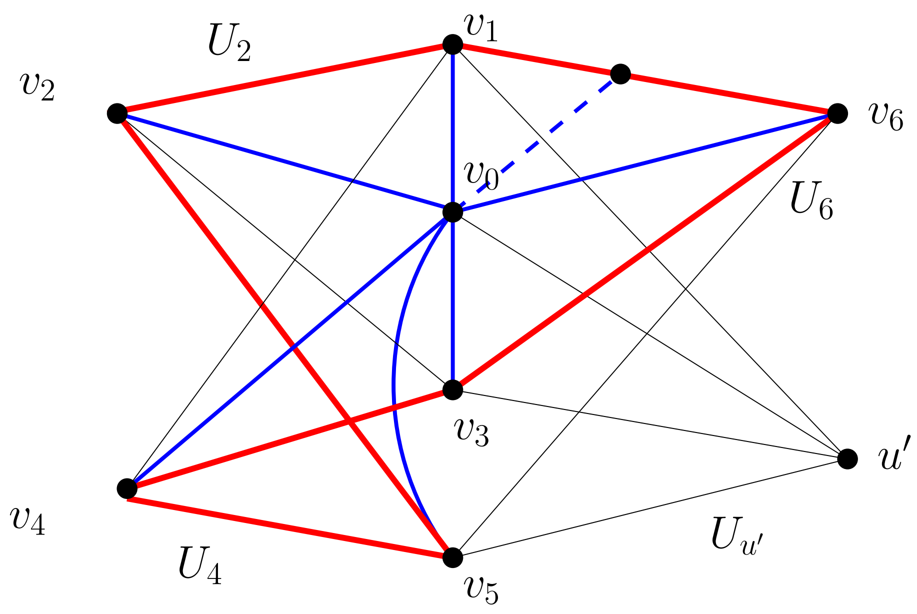


1.1.1.1.2. No such path: so S_3 forms a separating set.

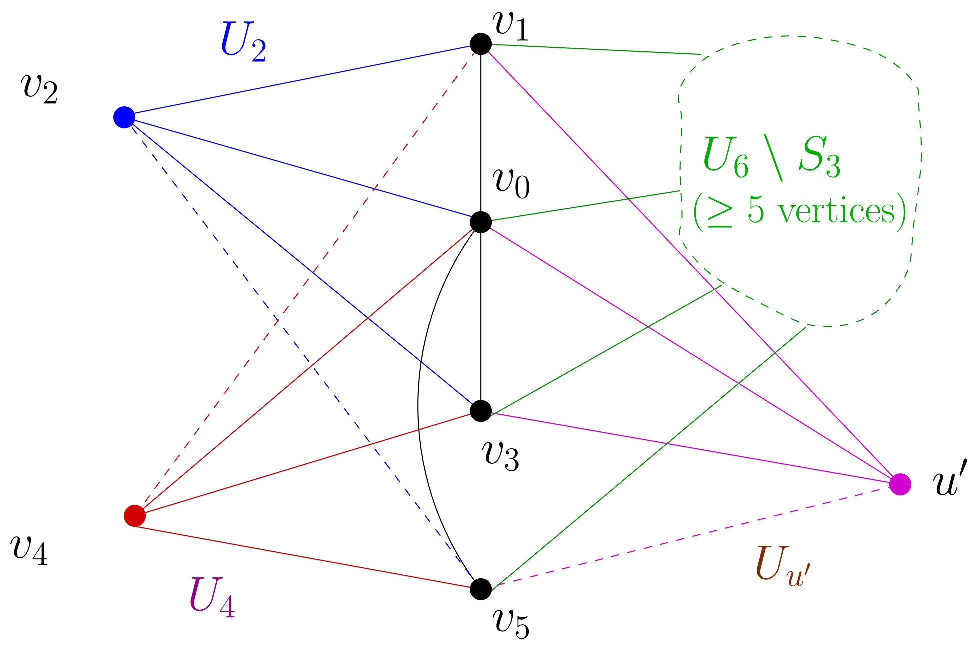


It can be shown that either:

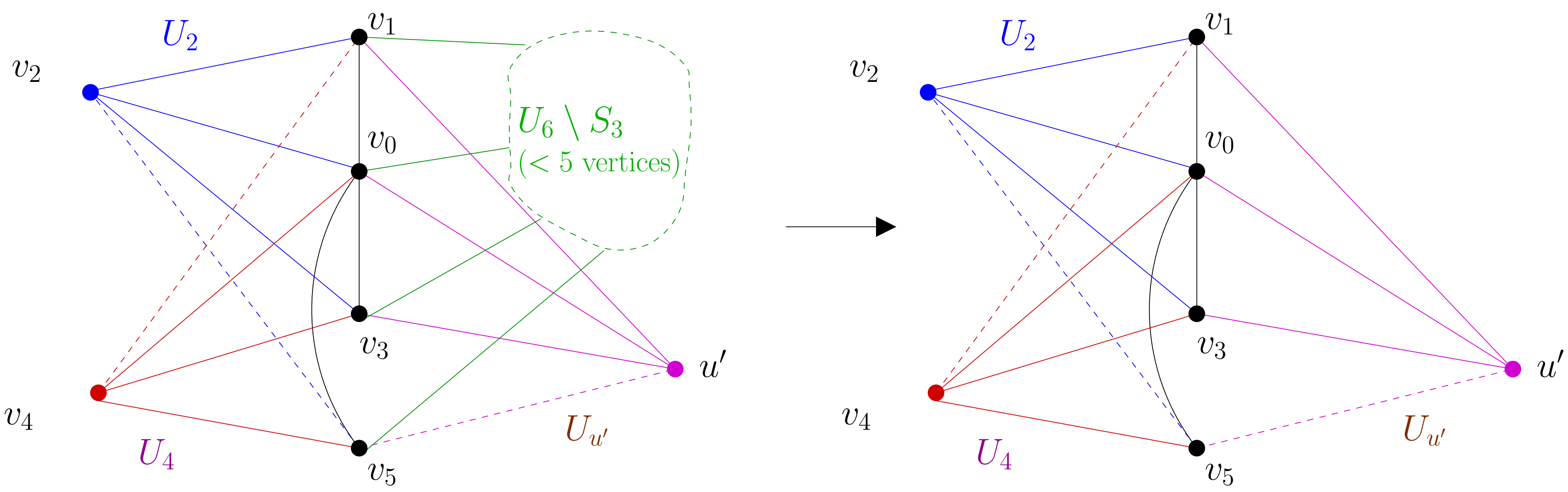
- a W_7-subdivision exists centred on some vertex in S_3



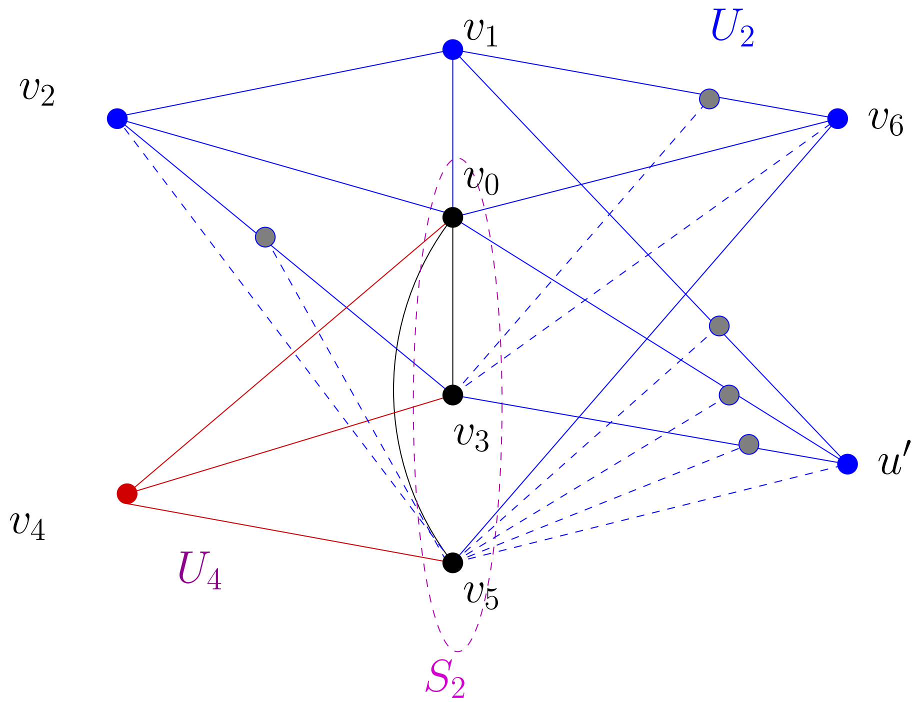
- an internal $(1,1,1,1)$-cutset exists in G



- or Reduction 4 can be performed on G



1.1.1.2. No path R_2 exists: S_2 forms a separating set.



Suppose there are *only* two bridges of $G|S_2$: U_2 and U_4.

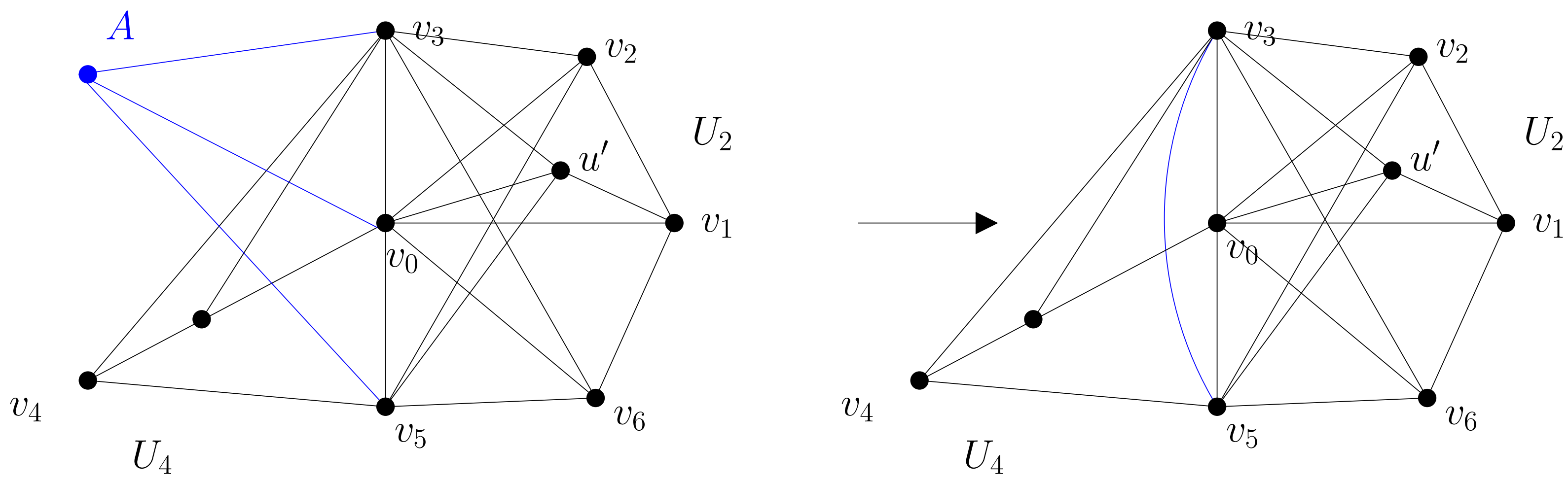
Lemma.

Let G be a 3-connected graph with at least 19 vertices. Suppose G has no internal 3 or 4-edge-cutsets, no type 1, 2, 2a, 3, 3a, or 4 edge-vertex-cutsets, and is a graph on which none of Reductions 1A, 1B, 1C, 2A, and 3 can be performed. Let $S = \{u, v, w\}$ be a separating set of vertices in G such that v is adjacent to both u and w, and such that there are exactly two bridges, X and Y, of $G|S$. Suppose that v has at least four neighbours in $X \setminus S$. Then G contains a W_7-subdivision.

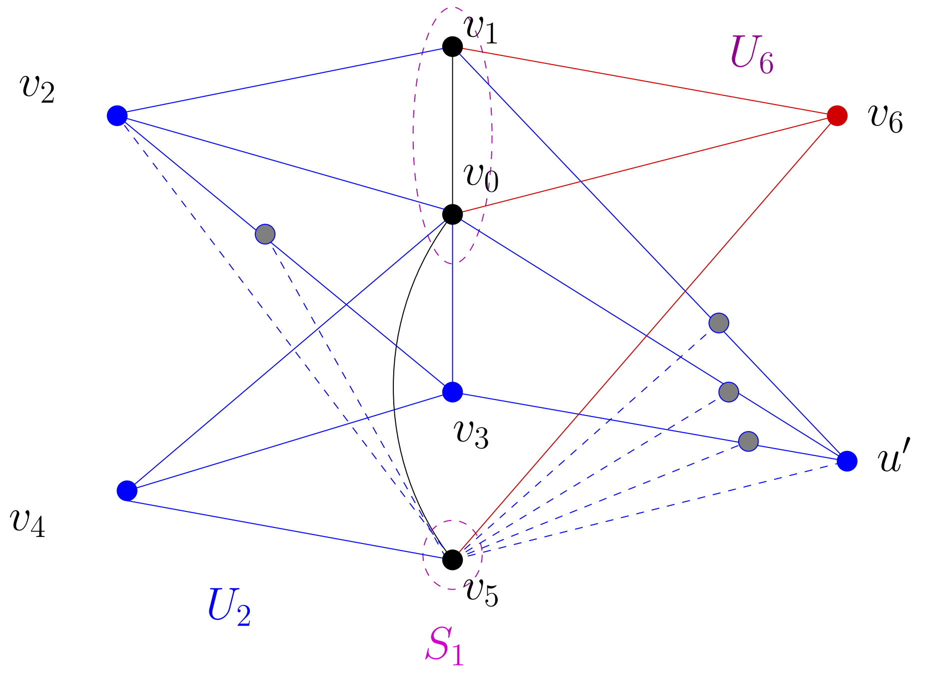
These conditions hold, so G must contain a W_7-subdivision.

Suppose there exists a third bridge A of $G|S_2$. Then either:

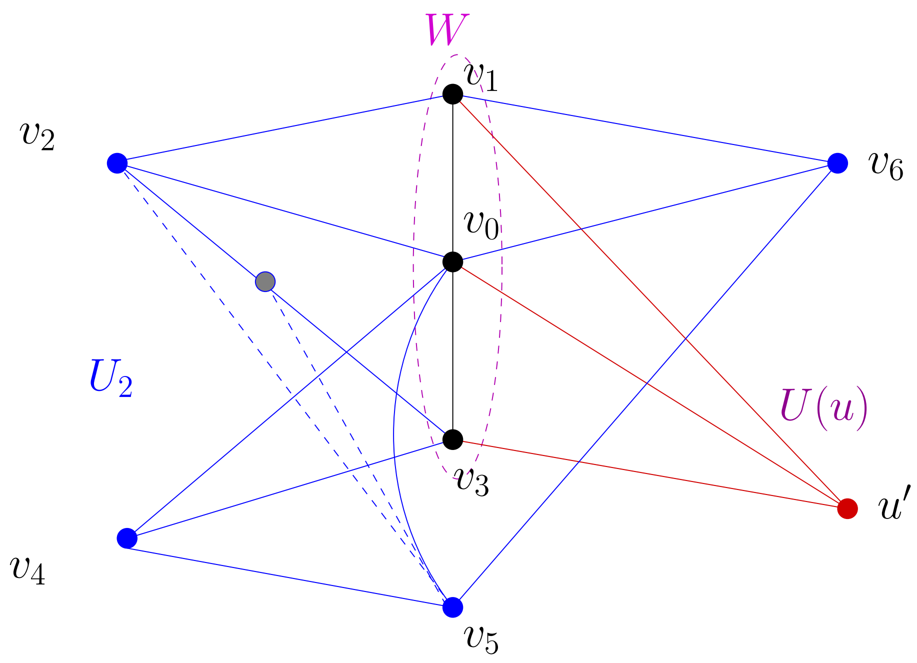
- a W_7-subdivision exists
- or one of the forbidden Reductions can be performed



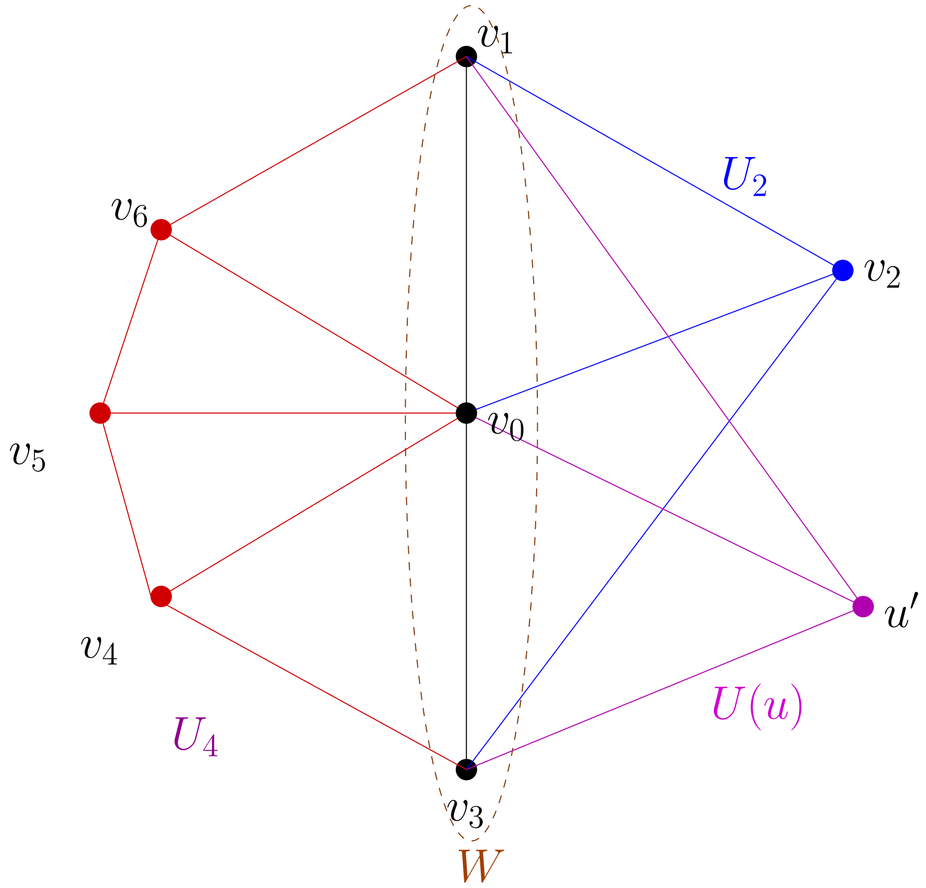
1.1.2. No path R_1 exists: S_1 forms a separating set.



1.2. No path R exists: W forms a separating set.



2. No path Q from H_2 to H_4.



- Again, either a W_7-subdivision exists, or a forbidden reduction can be performed.

Structure of graphs with no W_7-subdivisions

- First, must ‘reduce’ a graph as much as possible, using the six forbidden reductions.
- The resulting graph in its reduced form must be composed of ‘pieces’ that contain at least 38 vertices.
- Each piece must:
  - be 3-connected;
  - contain no internal 3- or 4-edge cutsets, or any of the other types of forbidden separating sets; and
  - contain no vertices of degree ≥ 7.

- Each of the pieces are joined together in a tree-like structure
- Each piece is joined to the rest of the graph so that either:
  - there exists a separating set of size ≤ 2, the removal of which disconnects one piece from another; or
  - there exists either an internal 3- or 4-edge cutset, or one of the forbidden separating sets, the removal of which disconnects one piece from another.