Kruskal’s Theorem

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Quasi-ordered set

A set Q together with a relation $\leq$ is *quasi-ordered* if $\leq$ is:

- reflexive ($a \leq a$); and
- transitive ($a \leq b \leq c \Rightarrow a \leq c$)

Good sequence

- An infinite sequence $q_1, q_2, \ldots$ of elements of Q, such that there exist positive integers i, j where $i < j$ and $q_i \leq q_j$.
- Example: $1, 2, 3, \ldots$

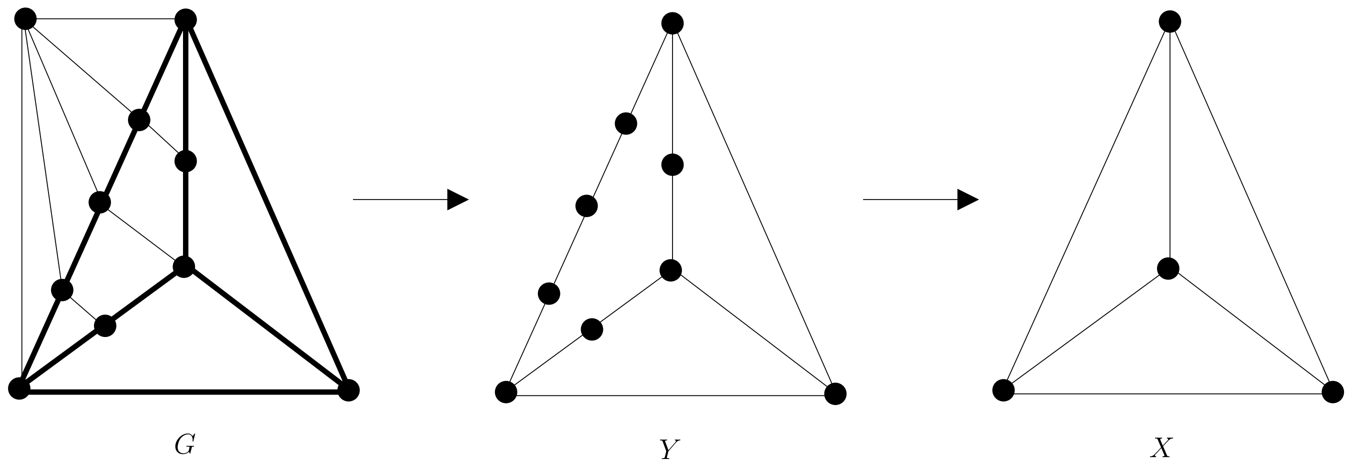
Bad sequence

- An infinite sequence of elements of Q that is not good.

Well-quasi-ordered (wqo) set

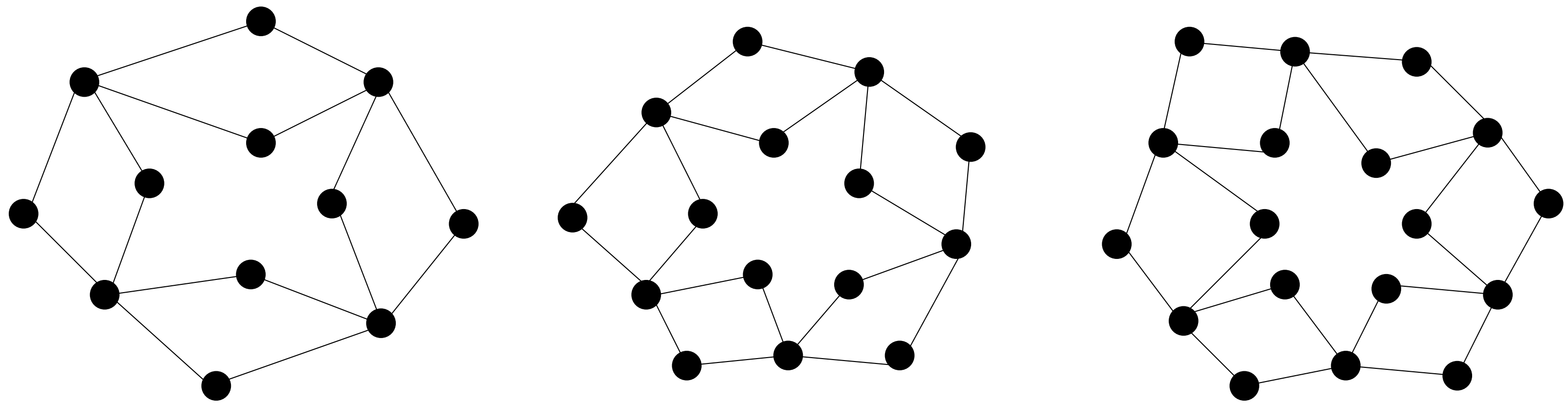
- A quasi-ordered set Q such that every infinite sequence in the set is good.
- $(\mathbb{N}, \leq)$, the set of natural numbers with standard ordering, is wqo
- $(\mathbb{Z}, \leq)$, the set of positive and negative integers with standard ordering, is *not* wqo, since it contains infinite strictly decreasing sequences.
- $(\mathbb{N}, |)$, the set of natural numbers ordered by divisibility, is *not* wqo, since the prime numbers form an *infinite antichain* (an infinite sequence in which any two elements are incomparable).

Topological containment



- Y is a subdivision of X; Y is the subgraph of another graph G
- G *topologically contains* X; there exists a homeomorphism of X in G.

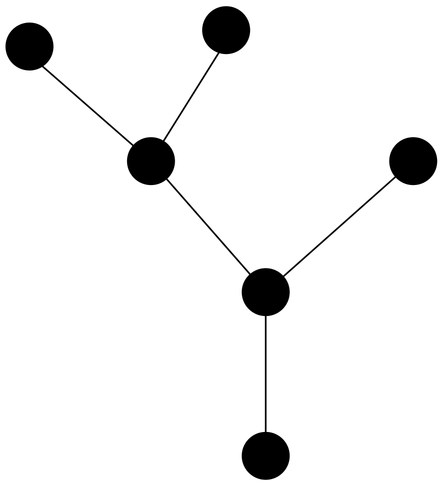
- The set of all graphs is not wqo over topological containment.



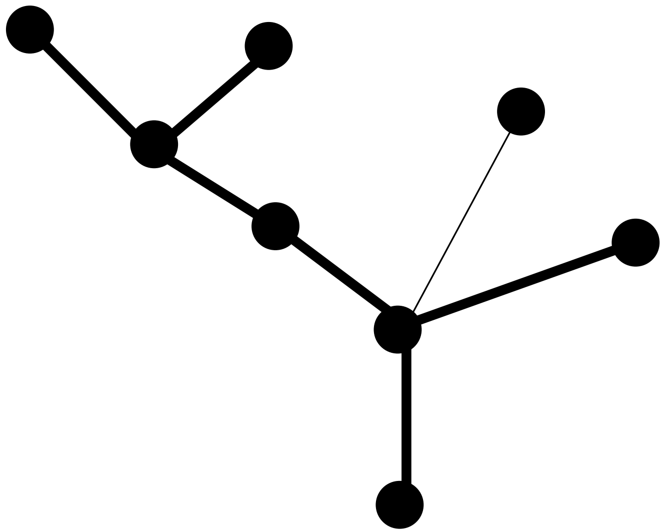
However . . .

Theorem (Kruskal, 1960): *The set of all trees is wqo over topological containment.*

- i.e. For every infinite sequence of trees $T_1, T_2, \ldots$ there exists some pair T_i, T_j where $i < j$ and T_i is topologically contained in T_j.



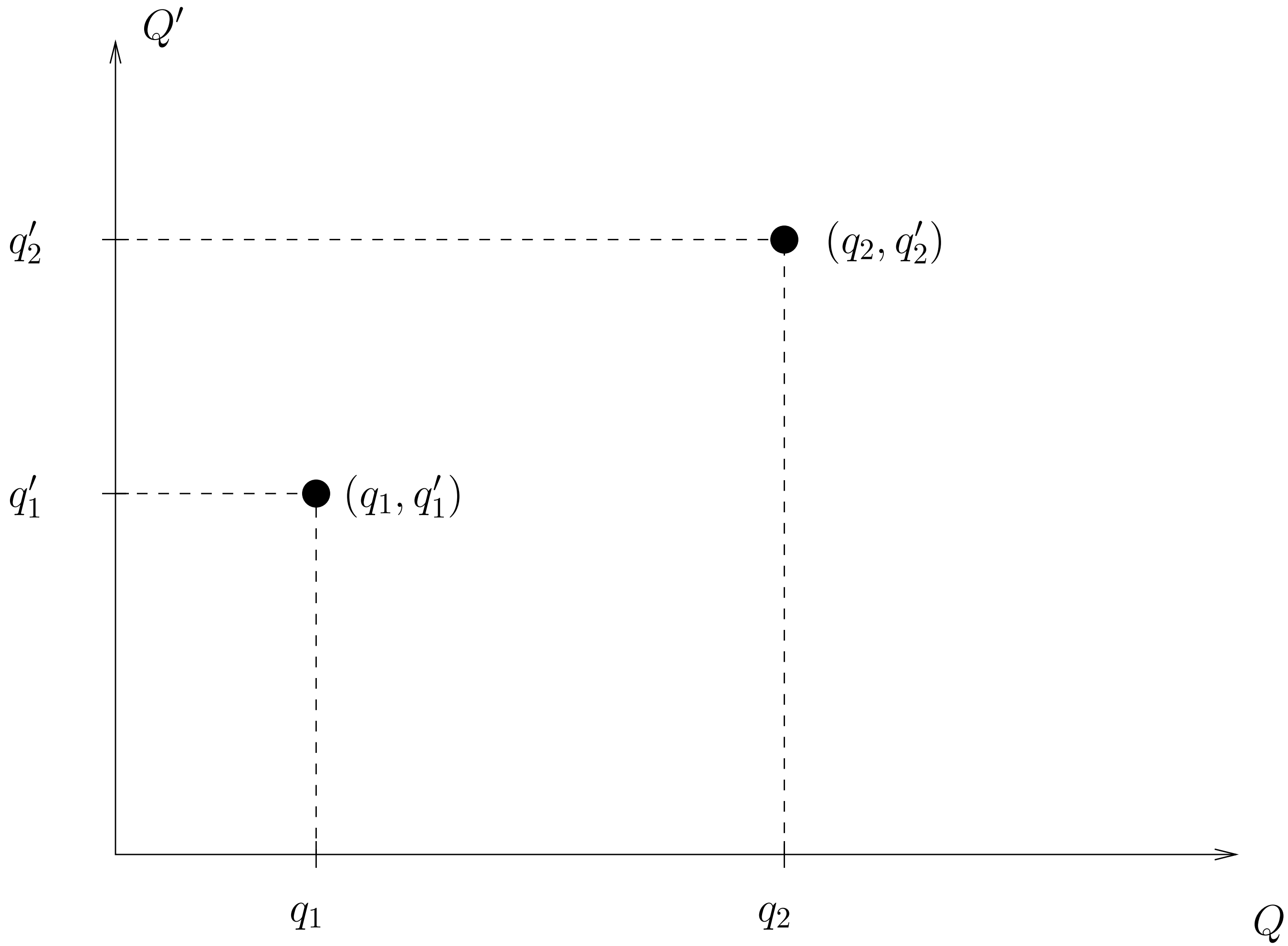
T_i



T_j

Proof. (Nash-Williams, 1963)

Cartesian product $Q \times Q'$: $(q_1, q_1') \leq (q_2, q_2')$ iff $q_1 \leq q_2$ and $q_1' \leq q_2'$



Lemma 1. *If* Q, Q' *are wqo, then* $Q \times Q'$ *is wqo.*

Proof.

Suppose Q and Q' are wqo.

- Must show that any infinite sequence $(q_1, q_1'), (q_2, q_2'), (q_3, q_3'), \ldots$ of elements of $Q \times Q'$ is good.

Call q_m terminal *if there is no* $n > m$ *such that* $q_m \leq q_n$*.*

In Q:

- There must be a finite number of terminal elements q_m, otherwise these elements would form a bad subsequence.

  $\Rightarrow$ there exists some N such that q_r is not terminal if $r > N$.

- Select $f(1) > N$ such that $q_{f(1)}$ is not terminal.
- Select $f(2) > f(1)$ such that $q_{f(2)} \leq q_{f(1)}$.
- Select $f(3) > f(2)$ such that $q_{f(3)} \leq q_{f(2)}$ ...etc.
- $q_{f(1)} \leq q_{f(2)} \leq q_{f(3)} \leq \dots$

In Q':

- There is some corresponding infinite sequence $q'_{f(1)}, q'_{f(2)}, q'_{f(3)}, \ldots$
- Since Q' is wqo, there exist i and j such that $i < j$ and $q'_{f(i)} \leq q'_{f(j)}$.
  $\Rightarrow (q_{f(i)}, q'_{f(i)}) \leq (q_{f(j)}, q'_{f(j)})$

- Define SQ as the class of finite subsets of Q.
- SQ is quasi-ordered by the rule that $A \leq B$ iff there exists a one-to-one non-descending mapping of A into B, where A and B are members of SQ.

Lemma 2. *If Q is wqo, then the class SQ of finite subsets of Q, SQ, is also wqo.*

Proof.

Let Q be wqo. Assume the hypothesis is false.

Define $A = A_1, A_2, A_3, \ldots$:

- a bad subsequence in SQ
- $|A_1|$ is chosen to be minimal
- Given A_1, $|A_2|$ is chosen to be minimal
- Given A_1 and A_2, $|A_3|$ is chosen to be minimal …etc.

No A_i is empty, or the sequence would be good.

$\Rightarrow$ Can select an element a_i from each A_i.

Let $B_i = A_i - \{a_i\}$.

Suppose some sequence:

$B_{f(1)}, B_{f(2)}, B_{f(3)} \ldots$

is bad, where $f(1) \leq f(i)$ for all i.

Then the sequence:

$A_1, A_2, \ldots, A_{f(1)-1}, B_{f(1)}, B_{f(2)}, \ldots$

must also be bad.

This contradicts the assumption that our original sequence A be of minimal size, since $B_{f(1)}$ is a smaller set than $A_{f(1)}$.

$\Rightarrow$ any sequence of B_i with $f(1) \leq f(i)$ must be good.

- Call $\mathcal{B}$ the class of sets B_i
- $\mathcal{B}$ must be wqo, since any bad sequence of sets B_i would have a bad infinite subsequence in which no suffix was less than the first.

- By Lemma 1, $Q \times \mathcal{B}$ is wqo.

  $\Rightarrow$ there exists i, j such that $i < j$ and $(a_i, B_i) \leq (a_j, B_j)$

  $\Rightarrow a_i \leq a_j$ and $B_i \leq B_j$

Since $a_i \cup B_i = A_i$ and $a_j \cup B_j = A_j$, this implies $A_i \leq A_j$.

This contradicts the assumption that our original sequence A is bad.

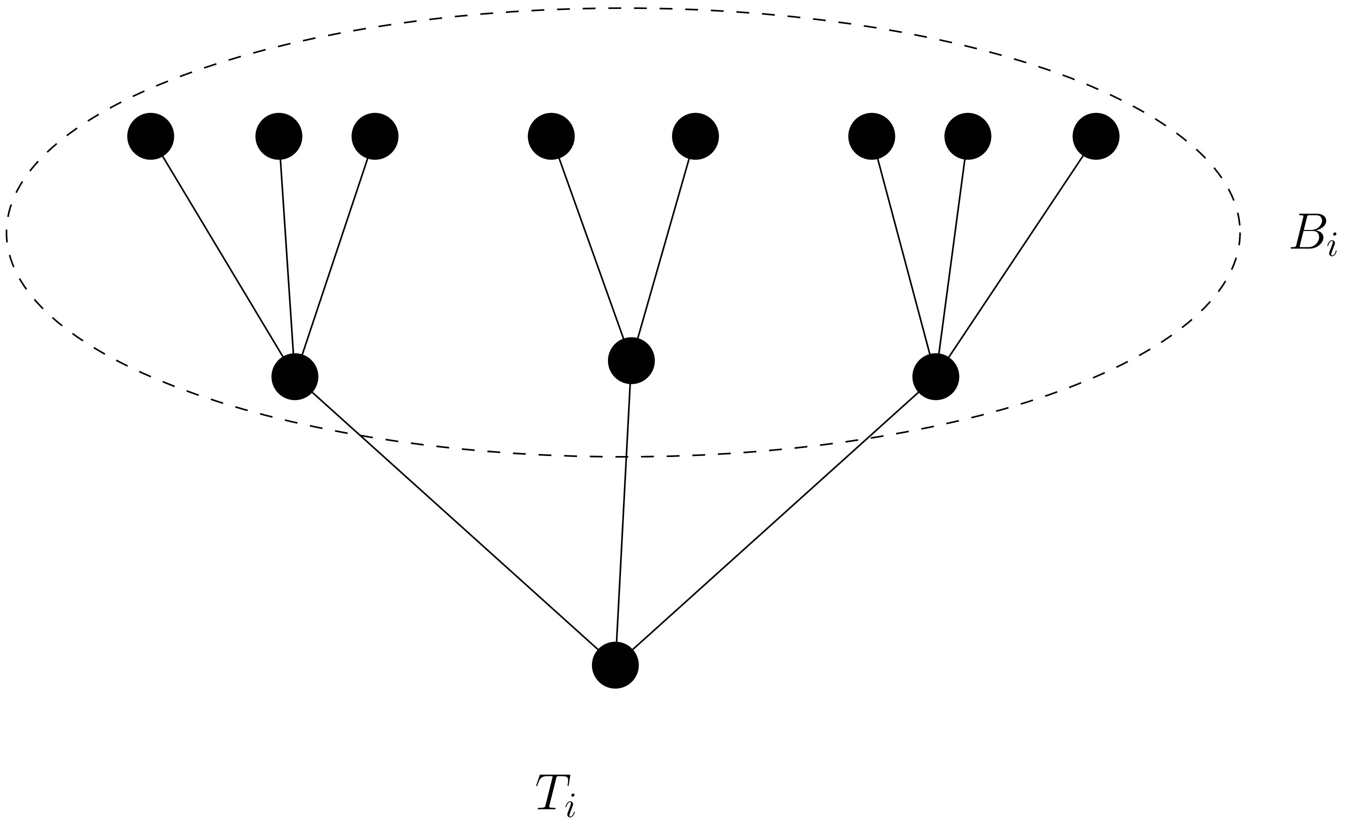
Theorem 1 (Kruskal's theorem). The set of all trees is wqo.

Proof.

Let $T = T_1, T_2, T_3, \ldots$ be an infinite sequence of trees, such that:

- T is bad.
- $|V(T_1)|$ is minimal, $|V(T_2)|$ is minimal with respect to T_1 ...etc.

Define B_i as the set of branches of T_i at the successors of its root.



$$B = B_1 \cup B_2 \cup B_3 \cup \ldots$$

Suppose there exists an infinite sequence $R_1, R_2, R_3, \ldots$ such that:

- $R_i \in B_{f(i)}$ and $f(1) \leq f(i)$ for all i, and
- the sequence is bad.

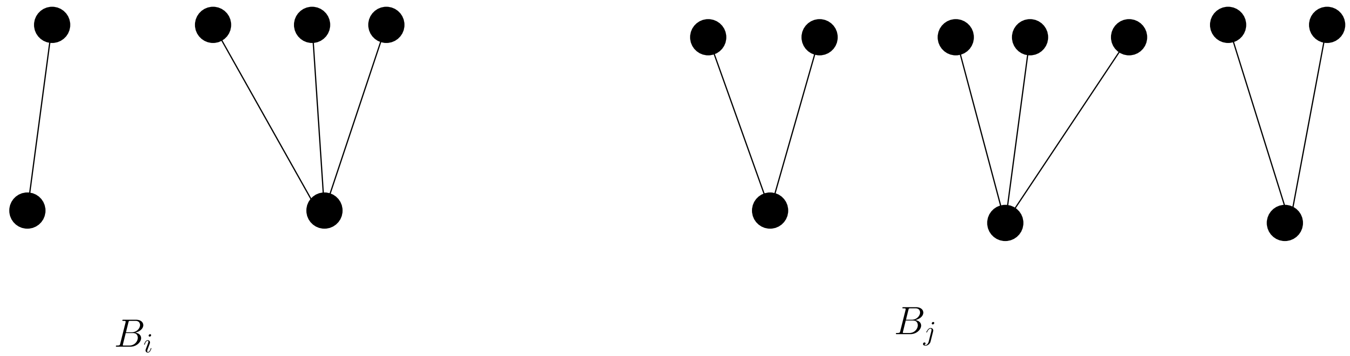
Then $T_1, T_2, \ldots, T_{f(1)-1}, R_1, R_2, \ldots$ is also a bad sequence, since if $T_i \leq R \in B_j$ then $T_i \leq T_j$ which contradicts the badness of T if $i < j$.

But if such a bad sequence exists, then T is no longer minimal.

- Thus, no such bad sequence $R_1, R_2, R_3, \ldots$ exists.
- This means no sequence of elements of B is bad, since any such sequence would have a bad subsequence where no suffix is less than the first.
- So B is wqo.

By Lemma 2, this means SB (the class of finite subsets of B) is also wqo:

- $B_i \leq B_j$ for some i, j such that $i < j$
- There exists a one-to-one non-descending mapping $\phi : B_i \rightarrow B_j$.



For each $R \in B_i$, $R \leq \phi(R)$.

$\Rightarrow$ there exists a homeomorphism h_R of R into $\phi(R)$.

We can thus define a homeomorphism h of T_i into T_j:

- identify the roots of T_i and T_j
- h coincides with h_R on the vertices of each $R \in B_i$.

$\Rightarrow T_i \leq T_j$, so T cannot be a bad sequence.