Structure and recognition of graphs with no 6-wheel subdivision

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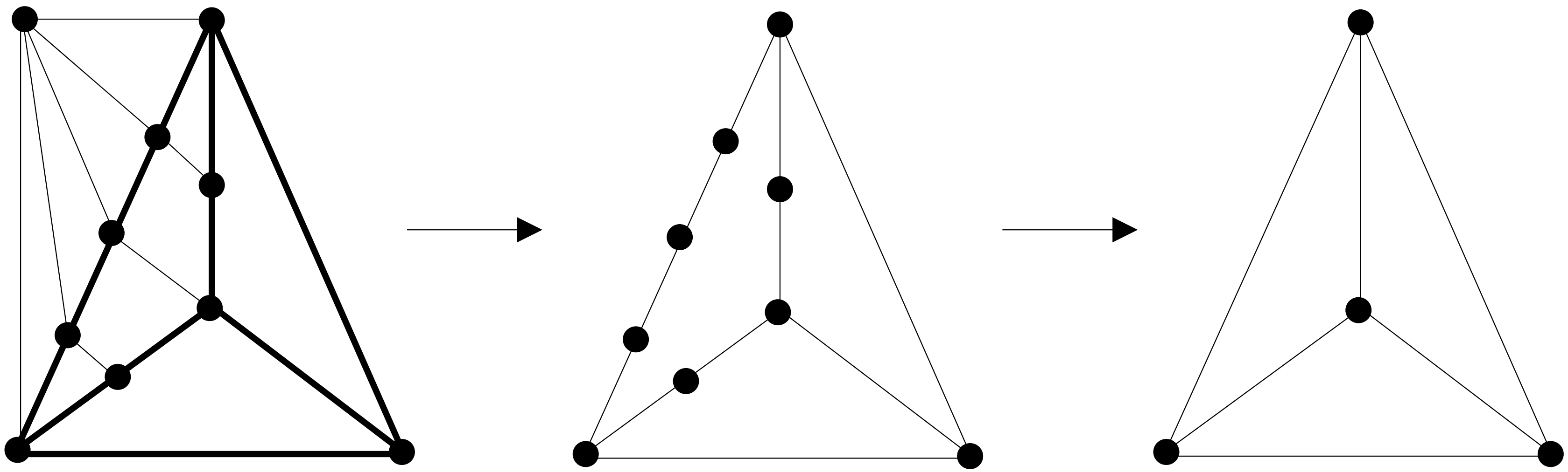
(joint work with Graham Farr)

1 Topological containment



G Y X

2 Applications of topological containment

- Forest — does not topologically contain K_3
- Planar graph — does not topologically contain K_5 or $K_{3,3}$ (Kuratowski, 1930)
- Series-parallel graph — does not topologically contain K_4 (Duffin, 1965)

3 The Subgraph Homeomorphism Problem

SHP(H)

Instance: Graph G.

Question: Does G topologically contain H?

4 Robertson and Seymour results

DISJOINT PATHS (DP)

Input: Graph G; pairs $(s_1, t_1), ..., (s_k, t_k)$ of vertices of G.

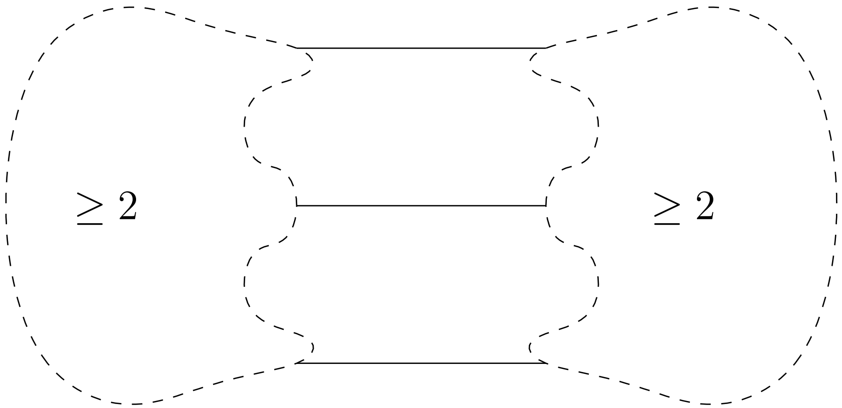
Question: Do there exist paths $P_1, ..., P_k$ of G, mutually vertex-disjoint, such that P_i joins s_i and t_i $(1 \leq i \leq k)$?

- DISJOINT PATHS is in P for any fixed k.
- This implies SHP(H) is also in P — use DP repeatedly.
- We know p-time algorithms must exist for SHP(H), but practical algorithms not given — huge constants.

5 Characterizations of wheel graphs

Theorem (Farr, 88).

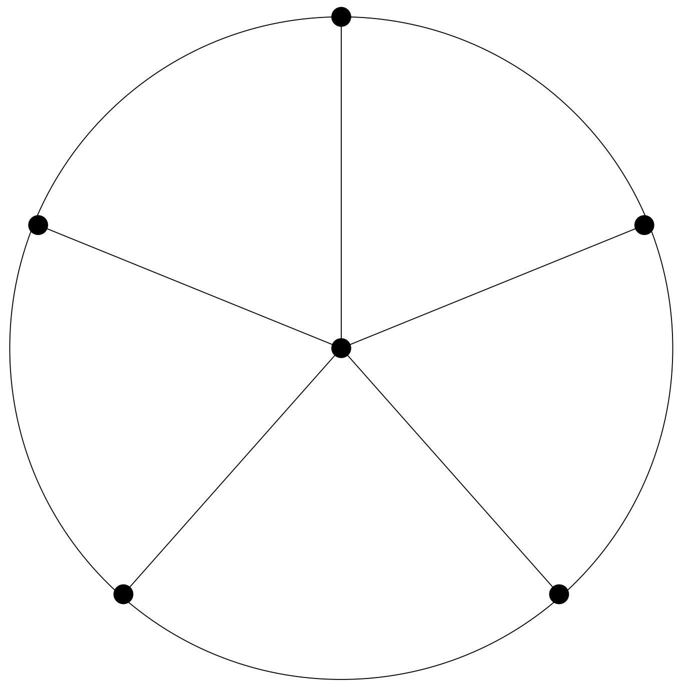
Let G be 3-connected, with no ***internal 3-edge-cutset*** *...*



Internal 3-edge-cutset

Theorem (Farr, 88).

Let G be 3-connected, with no internal 3-edge-cutset. Then G has a W_5-subdivision if and only if G has a vertex v of degree at least 5 and a circuit of size at least 5 which does not contain v.



W_5: wheel with five spokes

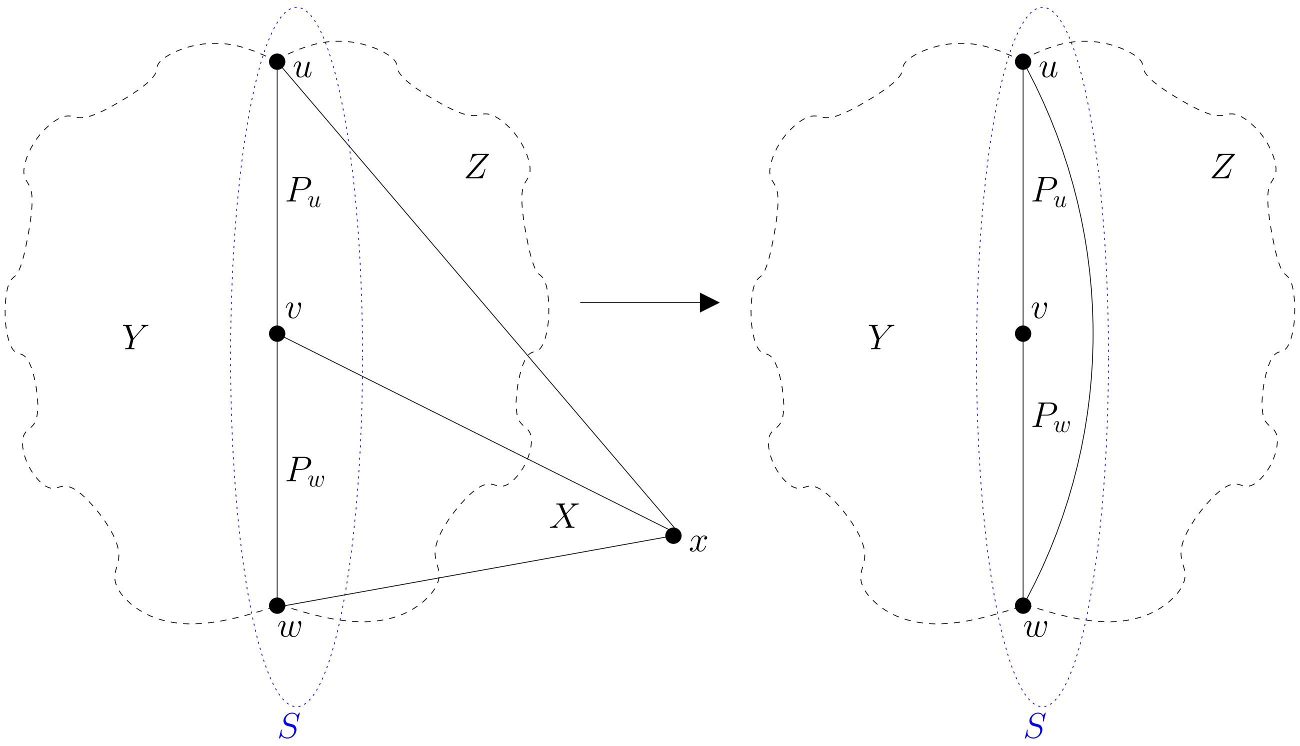
This work (R & F, 2006):

- Characterization of graphs not containing W_6-subdivisions, using a strengthening of this W_5 result.

5.1 Strengthened W_5 result

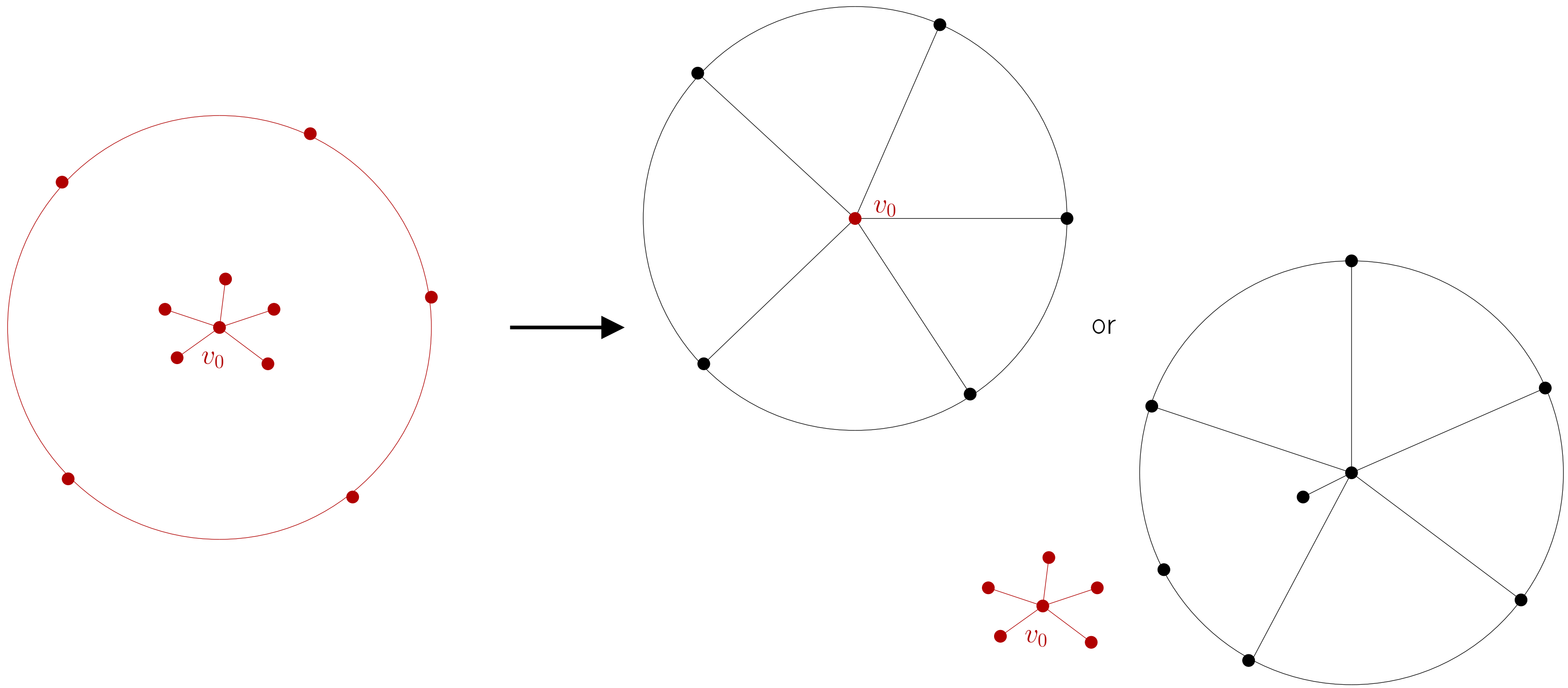
Theorem.

Let G be a 3-connected graph, with no internal 3-edge-cutset, such that ***Reduction 1*** *cannot be performed on G . . .*



Theorem.

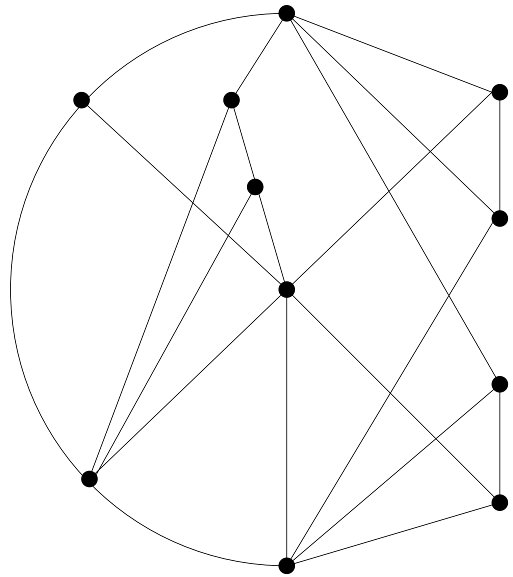
Let G be a 3-connected graph, with no internal 3-edge-cutset, such that Reduction 1 cannot be performed on G. Let v_0 be a vertex of degree ≥ 5 in G. Suppose there is a cycle of size at least 5 in G which does not contain v_0. Then either G has a W_5-subdivision centred on v_0, or G has a W_5-subdivision centred on some vertex v_1 of degree ≥ 6, with a rim of size at least 6.



5.2 Characterization of graphs that do not contain a W_6-subdivision

Theorem.

Let G be a 3-connected graph that is not topologically contained in the ***graph A*** *. . .*

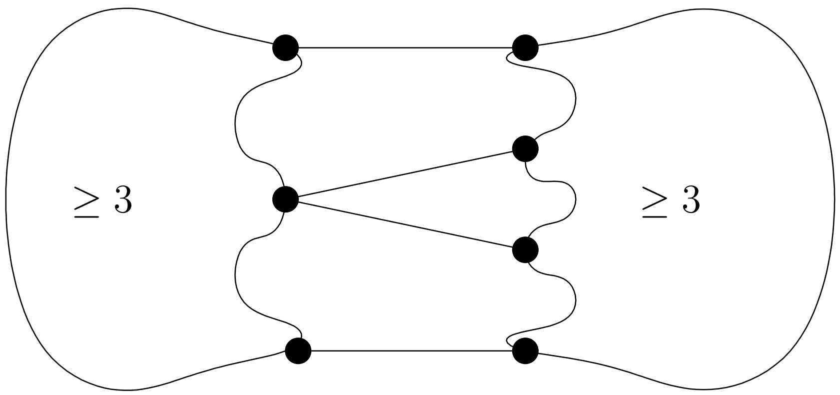


Graph A

Theorem.

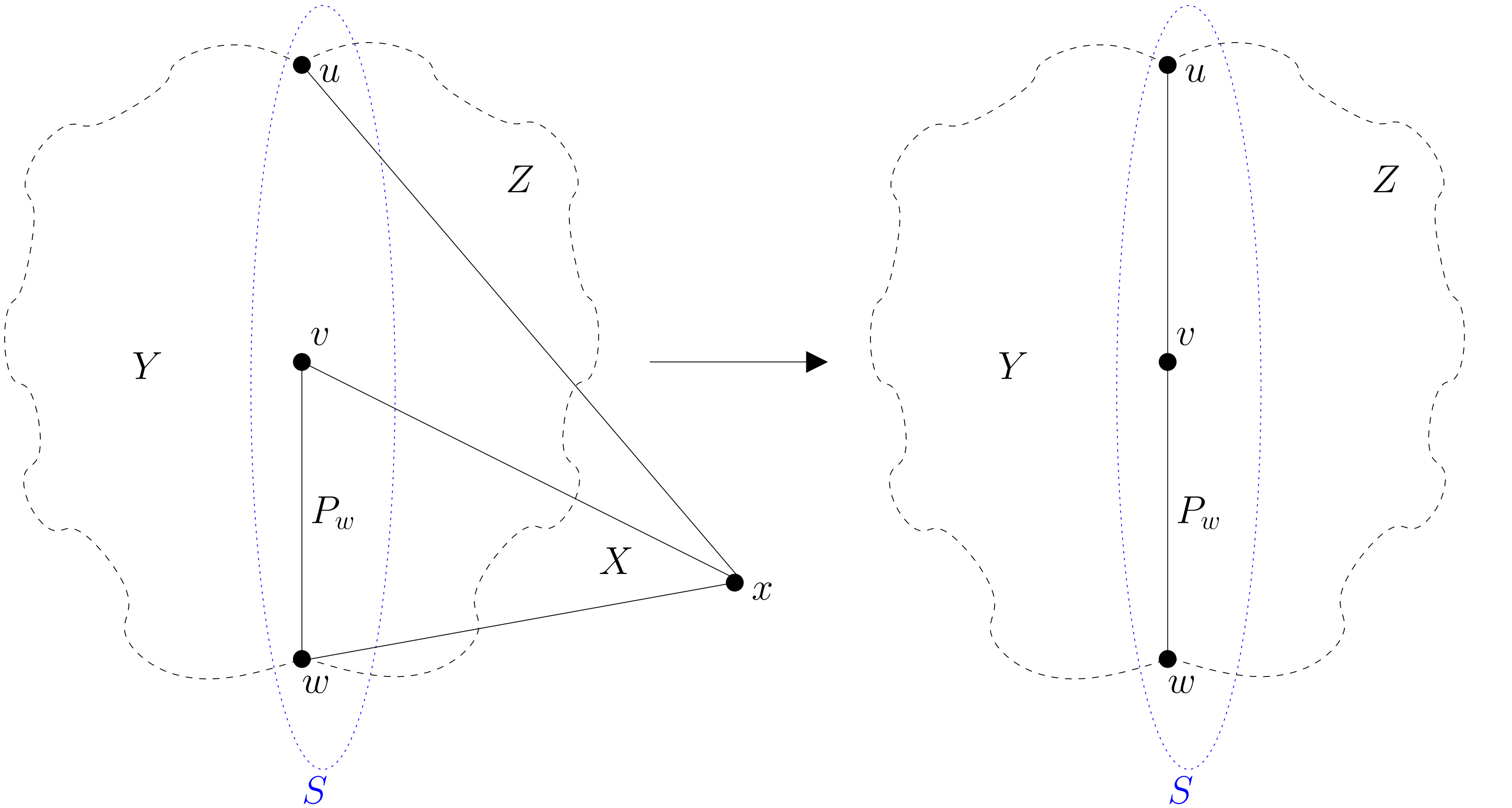
Let G be a 3-connected graph that is not topologically contained in the graph A.

*Suppose G has no internal 3-edge-cutsets, no **internal 4-edge-cutsets** . . .*



Theorem.

*Let G be a 3-connected graph that is not topologically contained in the graph A. Suppose G has no internal 3-edge-cutsets, no internal 4-edge-cutsets, and is a graph on which neither Reduction 1 nor **Reduction 2** can be performed . . .*



Theorem.

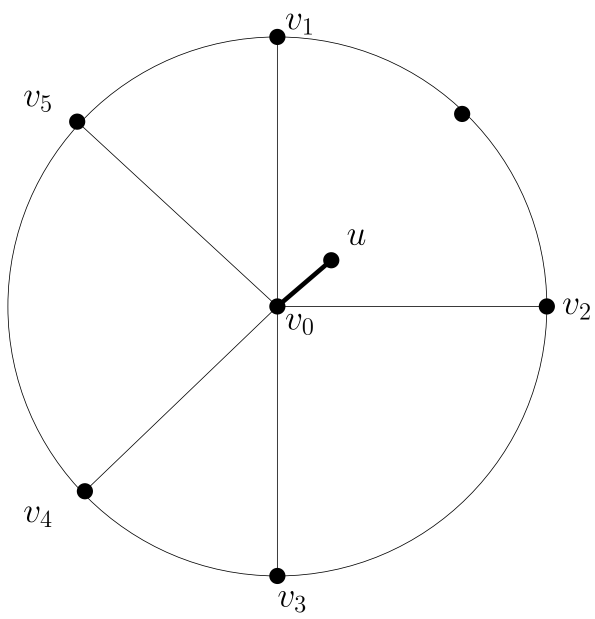
Let G be a 3-connected graph that is not topologically contained in the graph A. Suppose G has no internal 3-edge-cutsets, no internal 4-edge-cutsets, and is a graph on which neither Reduction 1 nor Reduction 2 can be performed.

Then G has a W_6-subdivision if and only if the following is true:

- *G contains some vertex v of degree at least 6, and*
- *G contains some cycle C, where $|C| \geq 6$ and C is disjoint from v.*

Proof — a summary.

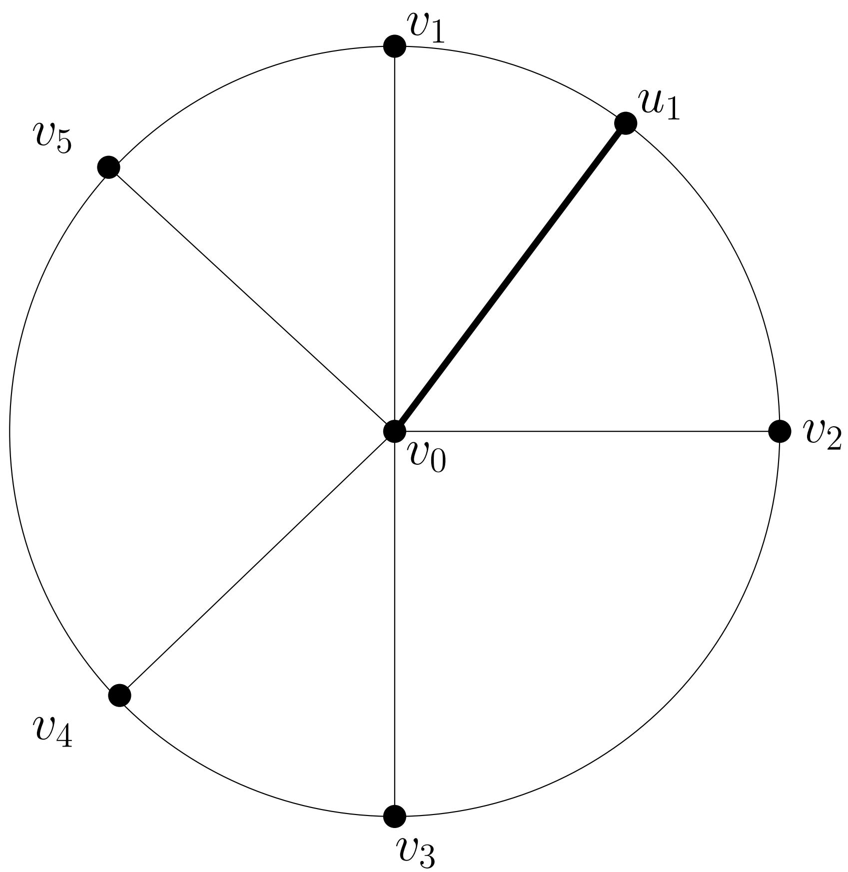
- Suppose the conditions of the hypothesis hold for some graph G.
- By the strengthened W_5 result above, there exists some vertex v_0 of degree ≥ 6 in G that has a W_5-subdivision H centred on it, such that H has a rim of length at least 6.



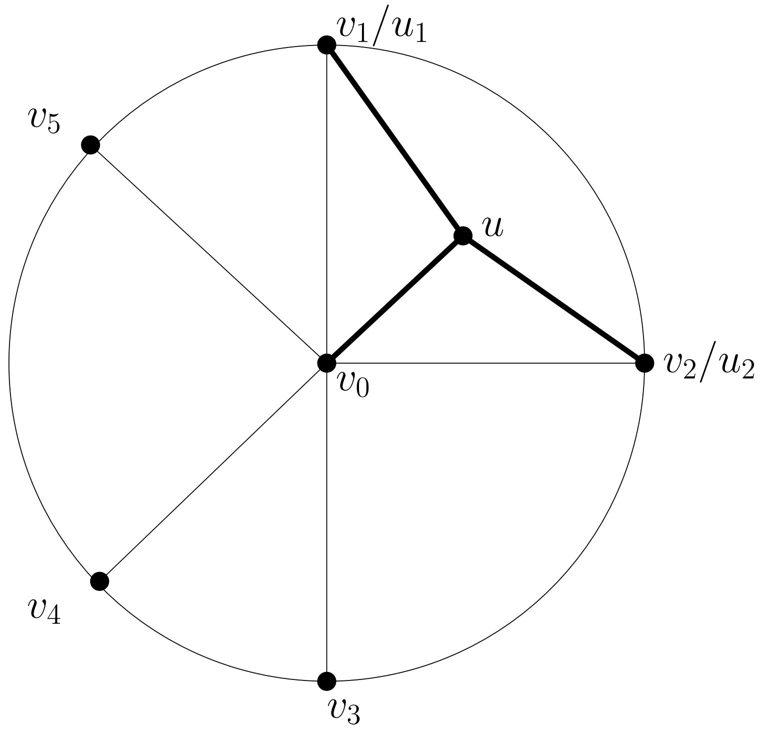
- How does u connect to the rest of H in order to preserve 3-connectivity?

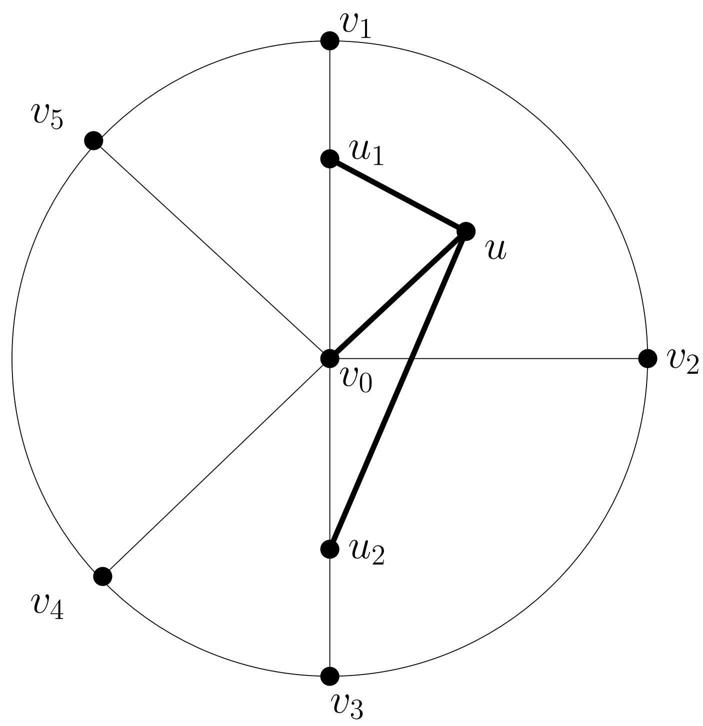
Three possibilities:

(a) Path from v_0 to some vertex u_1 on the rim of the W_5-subdivision, not meeting any spoke.

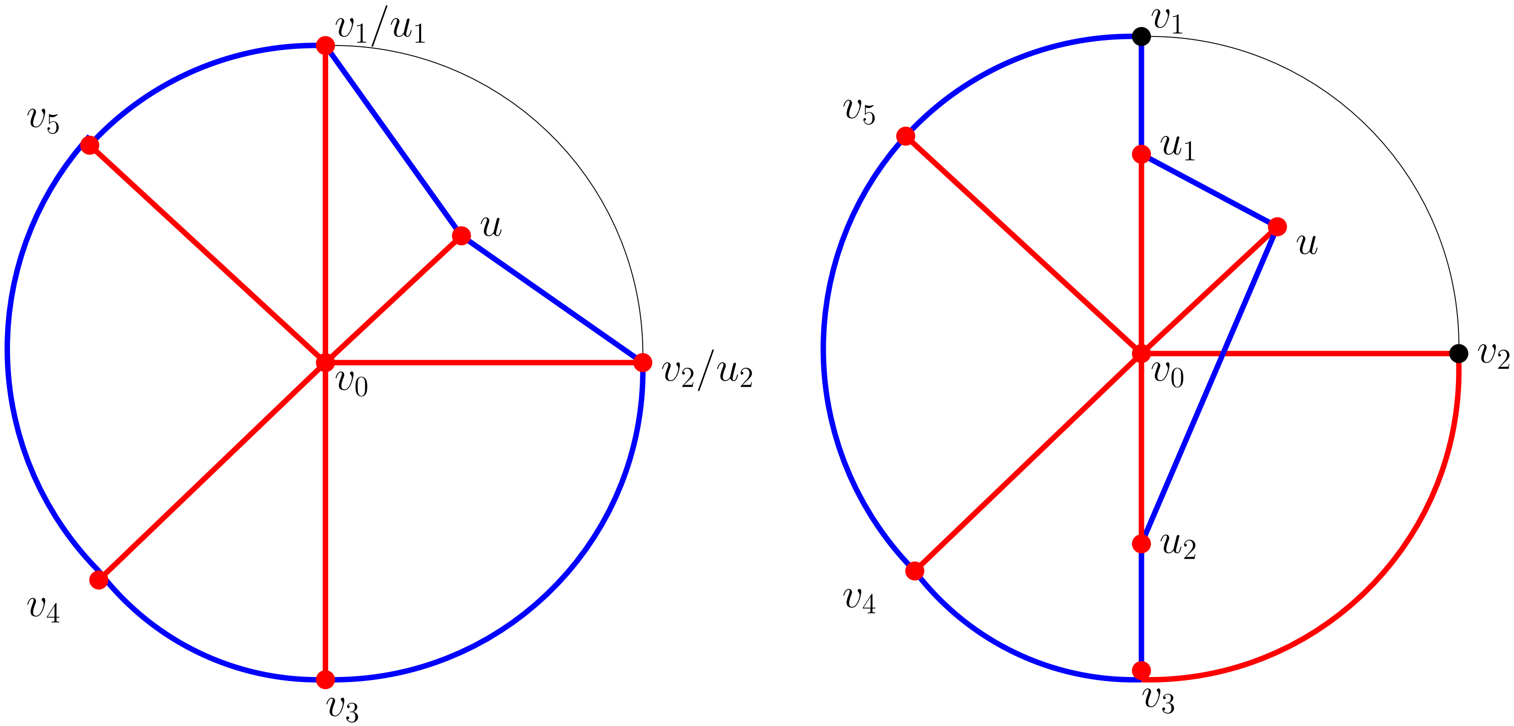


(b) Two paths from u to two separate spokes of H.

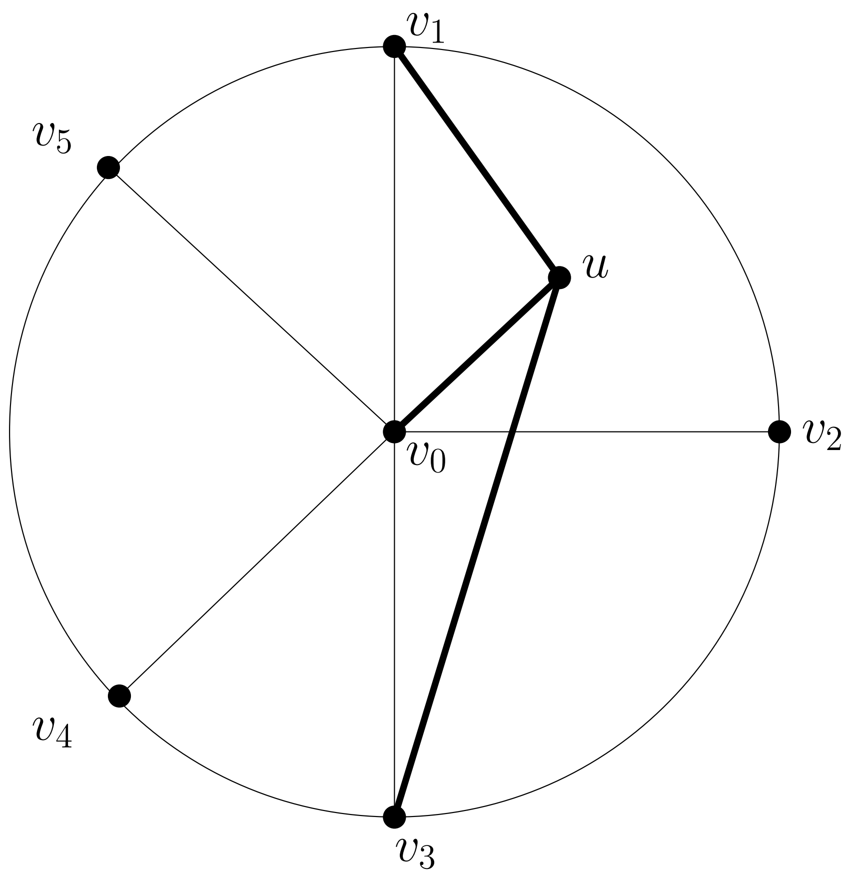




(b) Two paths from u to two separate spokes of H.



- Dealing with one particular case takes up the majority of the proof:

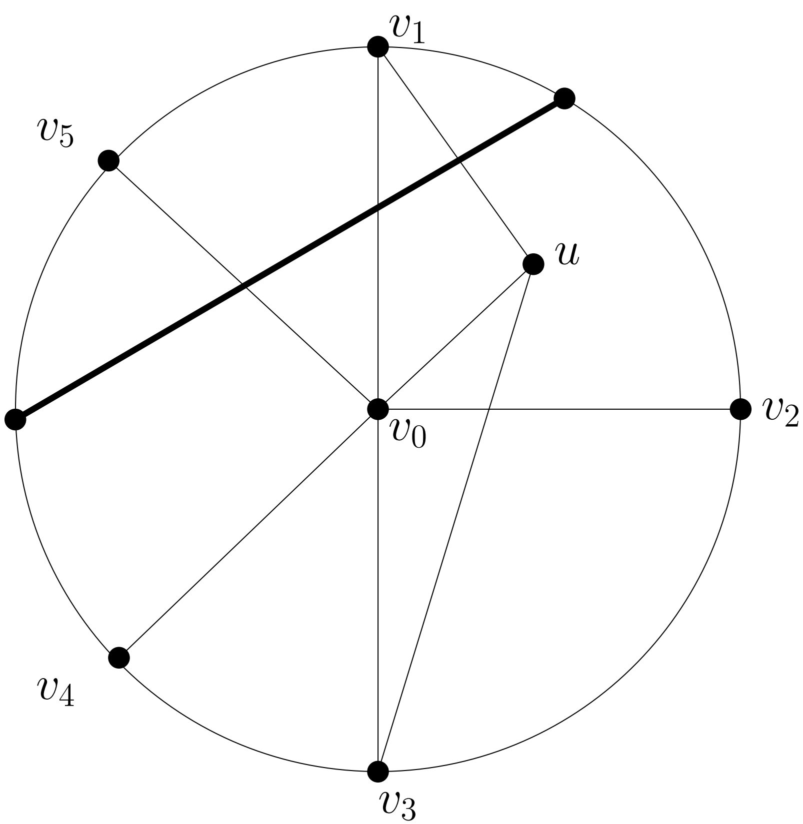


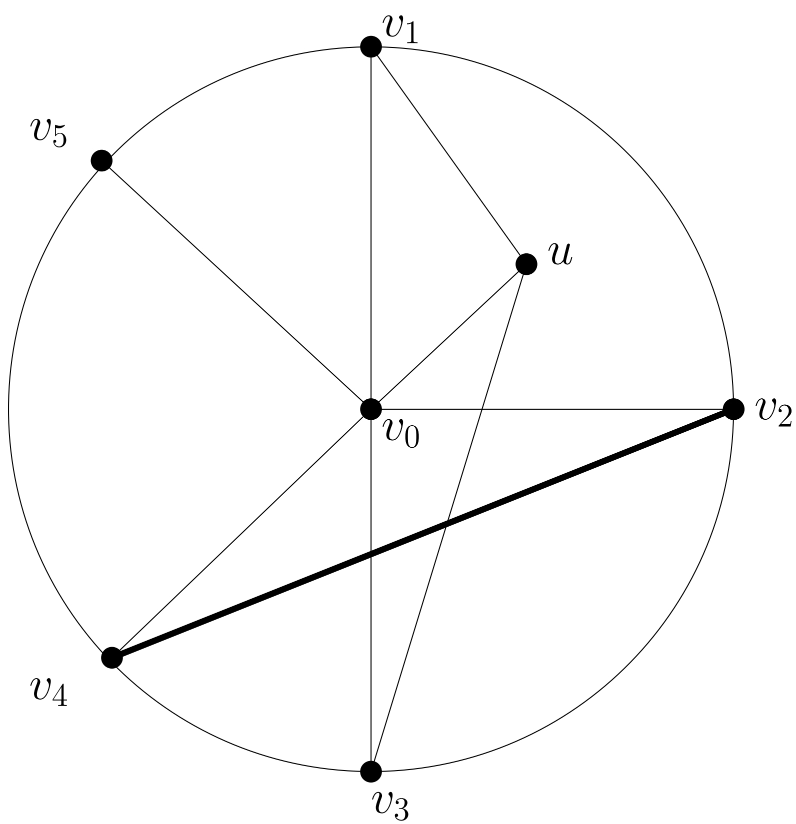
- This graph meets 3-connectivity requirements, but Reduction 1 can be performed on it.
- So there must be more structure to the graph.
- More in-depth case analysis required, based on different ways of adding this structure.
- Program developed in C to automate parts of this analysis; parts of proof depend on results generated by this program.

The program:

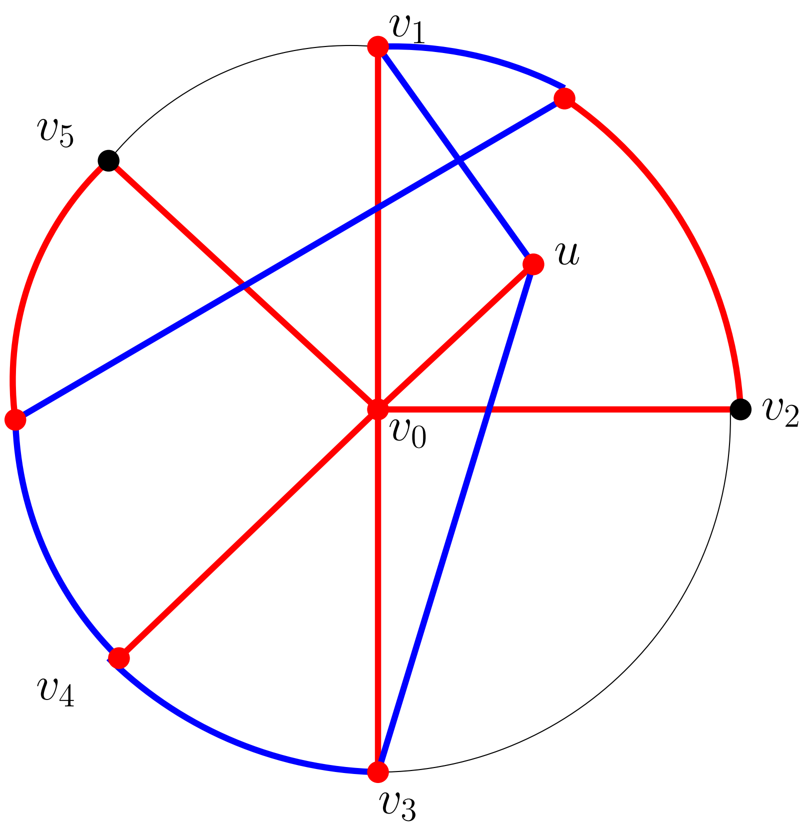
- constructs the various simple graphs that arise as cases in the proof, and
- tests each graph for the presence of a W_6-subdivision.

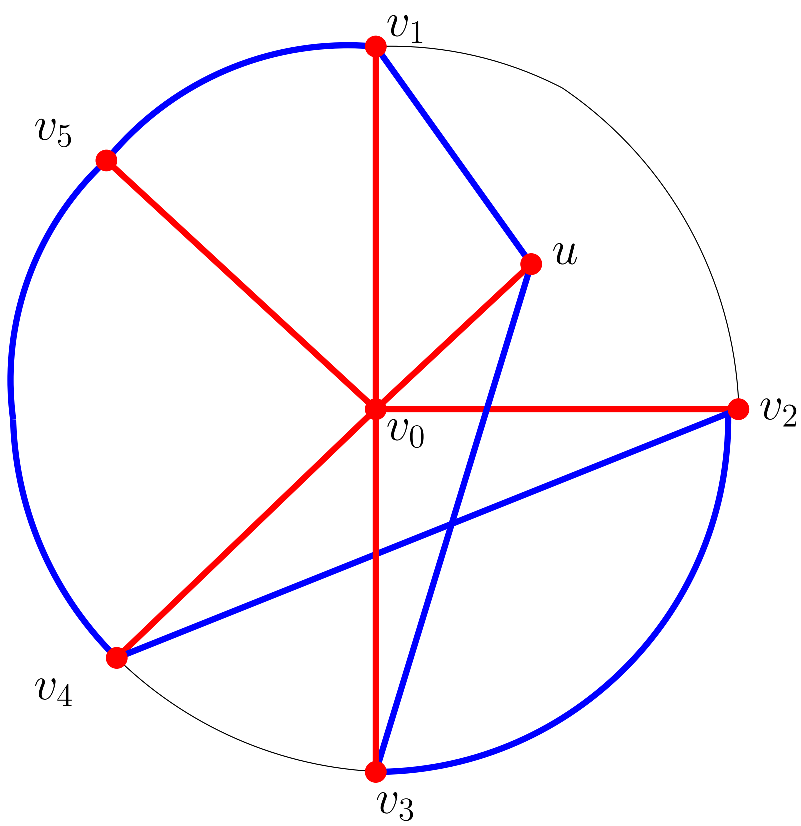
Examples:





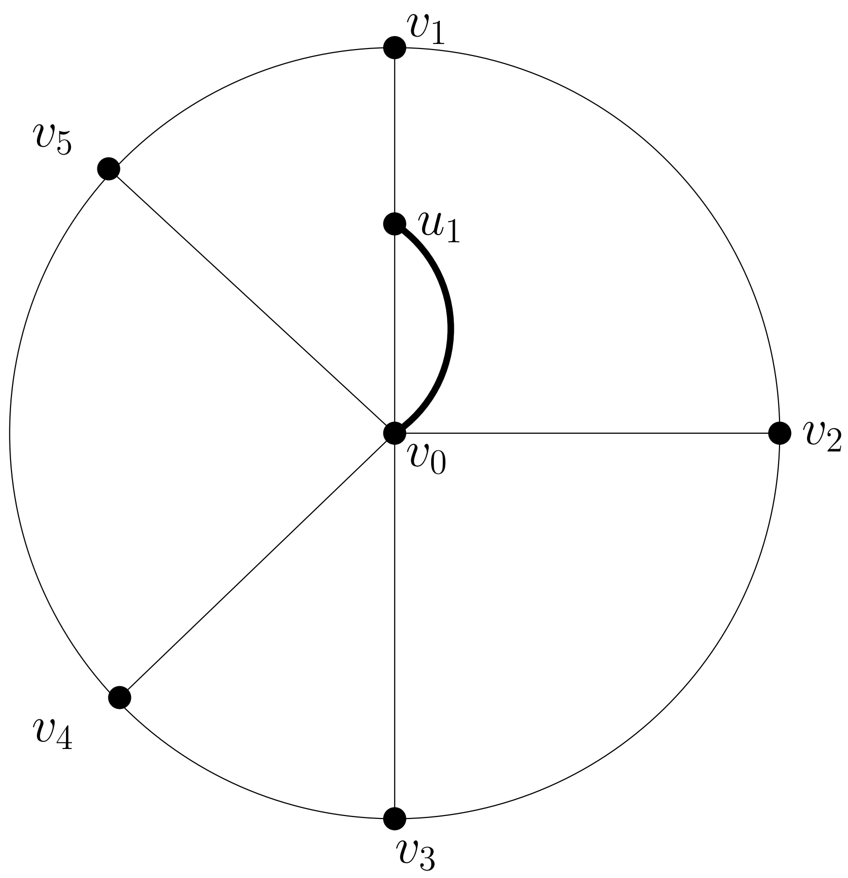
Examples:





- The program determines if a W_6-subdivision is present by recursively testing all subgraphs obtained by removing a single edge from the input graph.
- Base cases are W_6-subdivisions or graphs that have too few vertices or edges to contain such a subdivision.
- Naive algorithm; takes exponential time, but is sufficient for the small input graphs that arise as cases in the proof.
- Once the possibility of performing reductions and the presence of internal 3- and 4-edge-cutsets is eliminated, all resulting graphs are found to either:
  - contain a W_6-subdivision; or
  - be topologically contained in Graph A.

(c) Path from v_0 to some vertex u_1 on one of the spokes of the W_5-subdivision, such that this path that does not meet H except at its end points.



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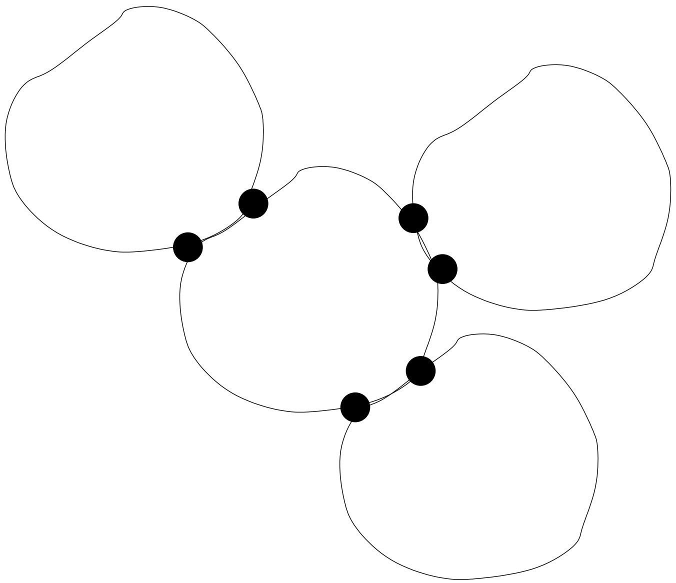
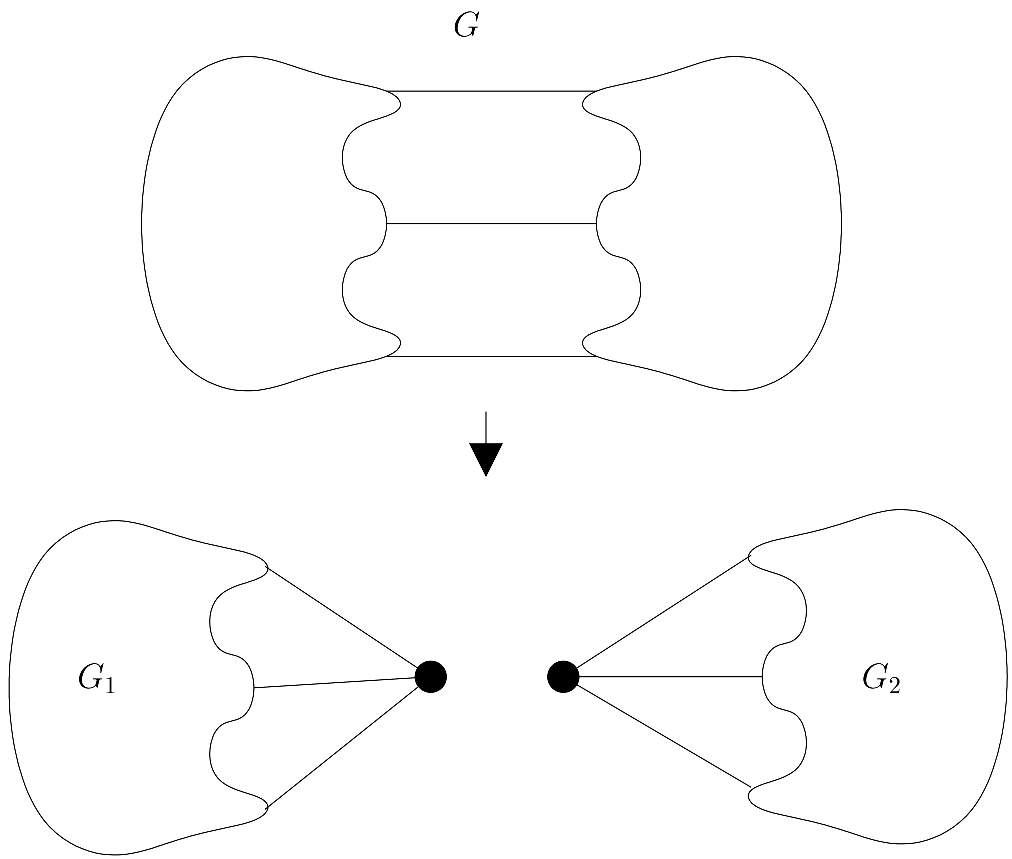
6 Using characterization to solve SHP(W_6)

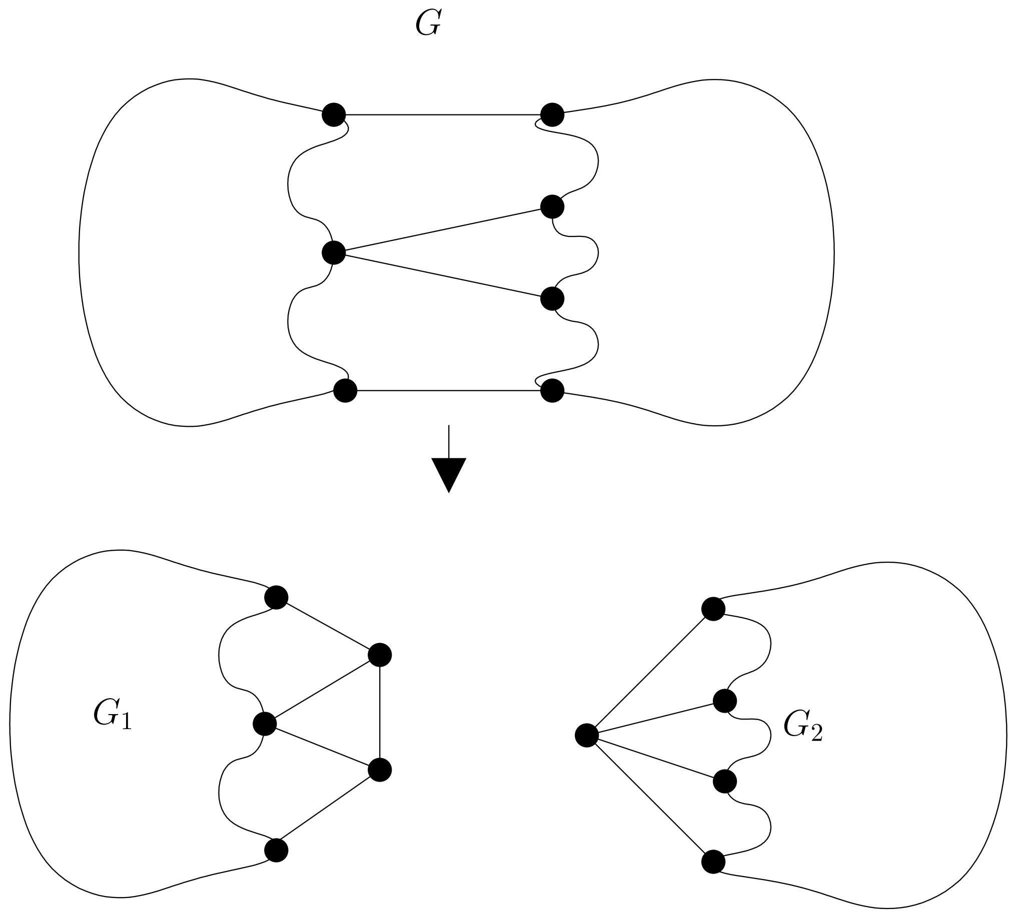
- Find 3-connected components of G.



- Separate G into components along its 3-edge cutsets.



- Separate G into components along its 4-edge-cutsets.



- If G is topologically contained in Graph A, G has no W_6-subdivision.
- If some reduction R (either Reduction 1 or 2) can be performed on G, let $G' = R(G)$. G contains a W_6-subdivision iff G' does.
- If G has no vertex of degree at least 6, G has no W_6-subdivision.
- For each vertex v of G of degree at least 6, determine whether $G - v$ has a circuit of length at least 6. If no $G - v$ has such a circuit, G has no W_6-subdivision.