No-Idle, No-Wait: When Shop Scheduling Meets Dominoes, Eulerian and Hamiltonian Paths

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Flow Shop Scheduling

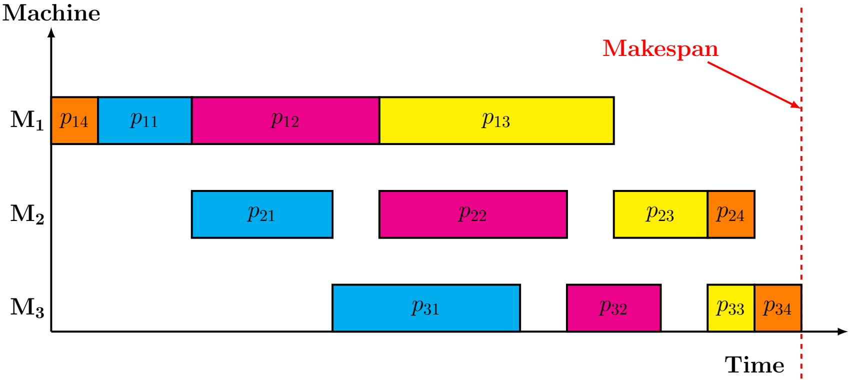
- There are m machines and n jobs.
- Each job contains exactly m operations.
- For each operation of each job a processing time is specified.
- No machine can perform more than one operation simultaneously.
- Operations cannot be interrupted (no preemption).

Flow Shop Scheduling

- Operations within one job must be performed in the specified order.
- The first operation gets executed on the first machine, then (as the first operation is finished) the second operation on the second machine, and so until the *m*-th operation.
- Jobs can be executed in any order.
- The problem is to determine an **optimal arrangement** of jobs.

Flow Shop Scheduling – Example

j	J_1	J_2	J_3	J_4
$p_{1,j}$	2	4	5	1
$p_{2,j}$	3	4	2	1
$p_{3,j}$	4	2	1	1



Problem Description

- We consider flow shop scheduling problems with **(machine) no-idle, (job) no-wait** constraints and makespan as objective.
- **Machine no-idle constraint**: use of very expensive equipment with the fee determined by the actual time consumption.
- **Job no-wait constraint**: in metal-processing industries (e.g., hot rolling) where delays between operations interfere with the technological process (e.g., cooling down).
- We focus on problem $F2|$ no-idle, no-wait $|C_{\max}$.

Literature

- Problem $F2||C_{\max}$ (Johnson rule $O(nlogn)$) $\in P$.
- Problem $F2|$ no-idle $|C_{\max}$ (trivially packing the jobs on the second machine to the right from Johnson's schedule) $\in P$.
- Problem $F2|$ no-wait $|C_{\max}$ (special case of Gilmore-Gomory TSP) $\in P$.
- Problem $F3||C_{\max}$ is NP-hard.

Literature

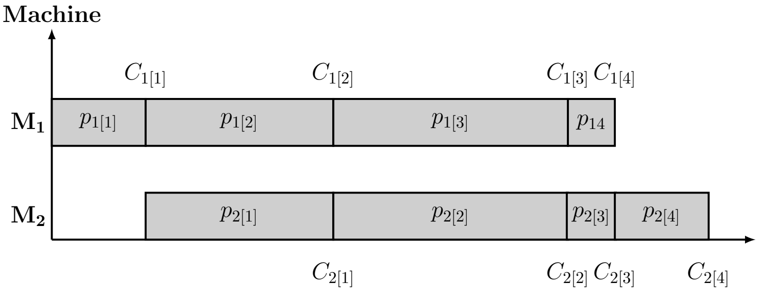
- Problem $F2|$ no-wait $|\mathcal{G}$ (minimizing the number of interruptions on the last machine in a 2-machine no-wait flow shop) is solvable in $O(n^2)$ time (Hohn et al. 2012).
- Problem $F3|$ no-wait $|\mathcal{G}$ is NP-hard. (Hohn et al. 2012).
- Problems $F2||\sum C_j$, $F2|$ no-wait $|\sum C_j$, $F2|$ no-idle $|\sum C_j$, $F2|$ no-idle, no-wait $|\sum C_j$ are NP-hard (Adiri and Pohoryles 1982).

$F2|$ no-idle, no-wait $|C_{max}$

- Two Machines M_1, M_2.
- Flow Shop environment.
- n jobs $1, 2, \ldots, n$ with processing times p_{1j} and p_{2j}.
- *No-idle* time is allowed (both machines M_1, M_2 must work continuously).
- *No-wait* discipline (no buffer — each job must start on M_2 right after its completion on M_1).
- Makespan (i.e. total time that elapses from the beginning to the end) objective.

$F2|$ no-idle, no-wait $|C_{\max}$

- The **no-idle**, **no-wait** constraint is a very strong requirement.



- $p_{i[j]}$ denotes the processing time of the j-th job of a sequence σ on machine M_i.
- $C_{i[j]}$ denotes the completion time of the j-th job of a sequence σ on machine M_i.

$F2|$ no-idle, no-wait $|C_{\max}$

Lemma (1)

(C1) *A necessary condition to have a feasible solution for problem $F2|no-idle, no-wait|C_{\max}$ is that there always exists an indexing of the jobs so that $p_{1,2}, \ldots p_{1,n}$ and $p_{2,1}, \ldots, p_{2,n-1}$ constitute different permutations of the same vector of elements.*

j	J_1	J_2	J_3	J_4	J_5
$p_{1,j}$	5	8	7	6	7
$p_{2,j}$	8	5	6	7	7

$F2|$ no-idle, no-wait $|C_{\max}$

Lemma (1)

(C1) *A necessary condition to have a feasible solution for problem $F2|no-idle, no-wait|C_{\max}$ is that there always exists an indexing of the jobs so that $p_{1,2}, ...p_{1,n}$ and $p_{2,1}, ..., p_{2,n-1}$ constitute different permutations of the same vector of elements.*

(C2) *When the above condition (C1) holds, then*

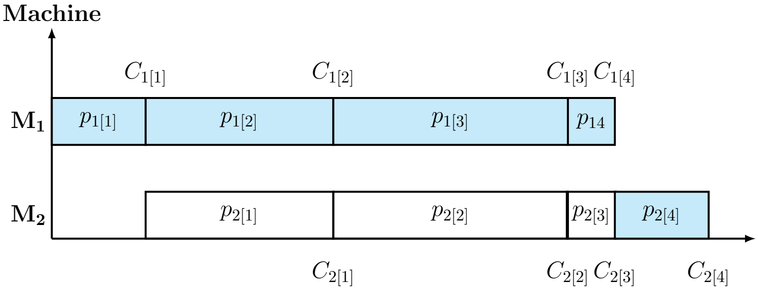
Case 1 *if $p_{1,1} \neq p_{2,n}$, every feasible sequence must have a job with processing time $p_{1,1}$ in first position and a job with processing time $p_{2,n}$ in last position.*

Case 2 *if $p_{1,1} = p_{2,n}$ and there exists a feasible sequence, then there do exist at least n feasible sequences each starting with a different job by simply rotating the starting sequence as in a cycle.*

$F2|$ no-idle, no-wait $|C_{\max}$

Lemma (2)

The makespan of any feasible sequence σ is given by the processing time of the **last** *(first) job on the* **second** *(first) machine plus the sum of jobs processing times on the* **first** *(second) machine.*



$F2|$ no-idle, no-wait $|C_{\max}$

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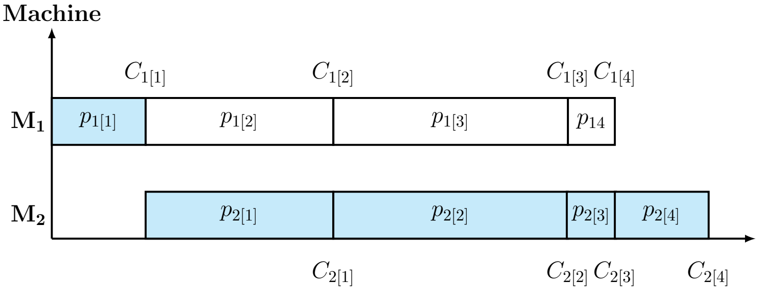
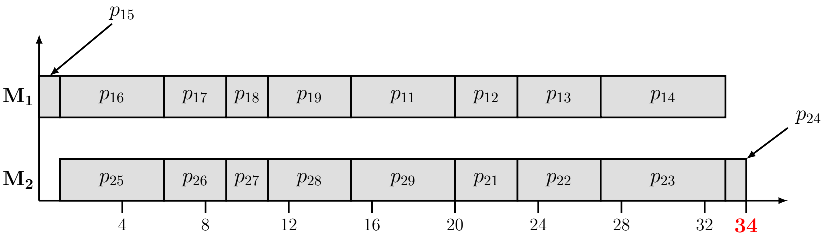


$F2|$ no-idle, no-wait $|C_{\max}$: an example

A 9-job instance of problem $F2|no-idle, no-wait|C_{\max}$.

j	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
$p_{1,j}$	5	3	4	6	1	5	3	2	4
$p_{2,j}$	3	4	6	1	5	3	2	4	5

and the corresponding optimal solution $C_{max} = 34$

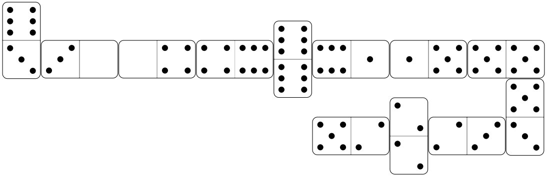


$F2|$ no-idle, no-wait $|C_{max}$

- Due to Lemma 1 and $F2|no-idle, no-wait|\mathcal{G}$ problem, the optimal solution can be calculated in $O(n^2)$ time...

...but...

- ...we decided to link the problem to the game of dominoes



Dominoes

- The **Single Player Domino (SPD) problem** (where a single player tries to lay down all dominoes in a chain with the numbers matching at each adjacency) is **polynomially solvable**: it can be seen as a **eulerian path problem on an undirected multigraph**.
- Here, we refer to the **oriented version of SPD called OSPD** where all dominoes have an orientation (given a tile with numbers i and j, only the orientation $i \to j$ is allowed but not viceversa).
- **Problem OSPD is polynomially solvable** (can be seen as a eulerian path problem on a **directed** multigraph).

Problem $F2|no-idle, no-wait|C_{\max}$ vs OSPD

Proposition

$F2|no-idle, no-wait|C_{\max} \propto OSPD$.

Proof.

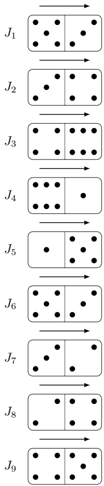
By generating for each job J_j a related domino tile $\{p_{1,j}, p_{2,j}\}$, any complete sequence of oriented dominoes in OSPD corresponds to a feasible sequence for $F2|no-idle, no-wait|C_{\max}$. Then, due to Lemma 1, the jobs processing times either respect case 1 or case 2 of condition C2. □

An example

A 9-job instance of problem $F2|no-idle, no-wait|C_{\max}$.

i	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
$p_{1,j}$	5	3	4	6	1	5	3	2	4
$p_{2,j}$	3	4	6	1	5	3	2	4	5

and the corresponding dominoes of the related OSPD problem

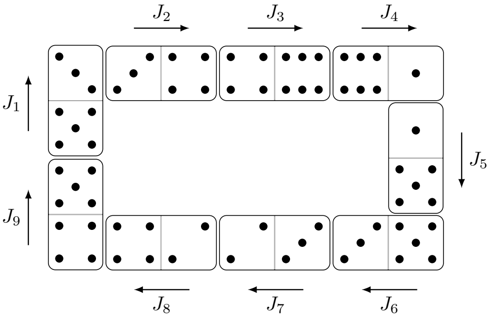


An example

A 9-job instance of problem $F2|no-idle, no-wait|C_{\max}$.

j	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
$p_{1,j}$	5	3	4	6	1	5	3	2	4
$p_{2,j}$	3	4	6	1	5	3	2	4	5

and the corresponding OSPD solution

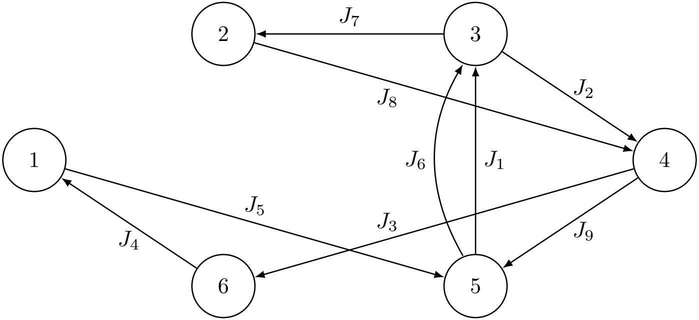


An example

A 9-job instance of problem $F2|no - idle, no - wait|C_{\max}$.

i	J_1	J_2	J_3	J_4	J_5	J_6	J_7	J_8	J_9
$p_{1,i}$	5	3	4	6	1	5	3	2	4
$p_{2,i}$	3	4	6	1	5	3	2	4	5

and the corresponding oriented multigraph



Complexity of $F2|$ no-idle, no-wait $|C_{\max}$

Proposition

Problem $F2|no-idle, no-wait|C_{\max}$ *can be solved in* $O(n)$ *time.*

Proof.

[Sketch]: The generation of the oriented multigraph can be done in linear time and the graph has $O(n)$ arcs. Besides, it is known (Fleischner 1991) that computing an Eulerian path in an oriented graph with n arcs can be done in $O(n)$ time. □

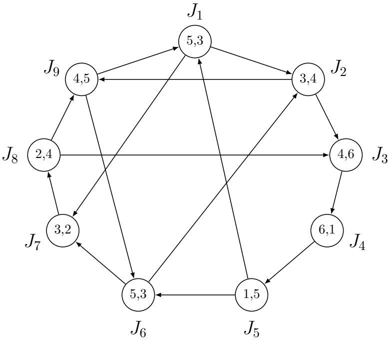
$F2|$ no-idle, no-wait $|C_{\max}$ vs the Hamiltonian Path problem

- Problem $F2|no-idle, no-wait|C_{\max}$ is also linked to a special case of the Hamiltonian Path problem on a connected digraph.
- Consider a digraph $G(V, A)$ that has the following property: $\forall v_i, v_j \in V$, either $S_i \cap S_j = \emptyset$, or $S_i = S_j$ where we denote by S_i the set of successors of vertex v_i.
- In other words, each pair of vertices either has no common successors or has all successors in common.
- We denote the Hamiltonian path problem in that graph as the Common/Distinct Successors Hamiltonian Oriented Path (CDSHOP*) problem.

$F2|$ no-idle, no-wait $|C_{max}$ vs the Hamiltonian Path problem

- $F2|no-idle, no-wait|C_{max} \propto$ CDSHOP easily holds.
- The CDSHOP problem corresponding to the considered $F2|$ no-idle, no-wait $|C_{max}$ instance.

i	$p_{1,i}$	$p_{2,i}$
J_1	5	3
J_2	3	4
J_3	4	6
J_4	6	1
J_5	1	5
J_6	5	3
J_7	3	2
J_8	2	4
J_9	4	5



Complexity of CDSHOP

Proposition

CDSHOP $\propto F2|no - idle, no - wait|C_{\max}$, *hence, CDSHOP* $\in P$.

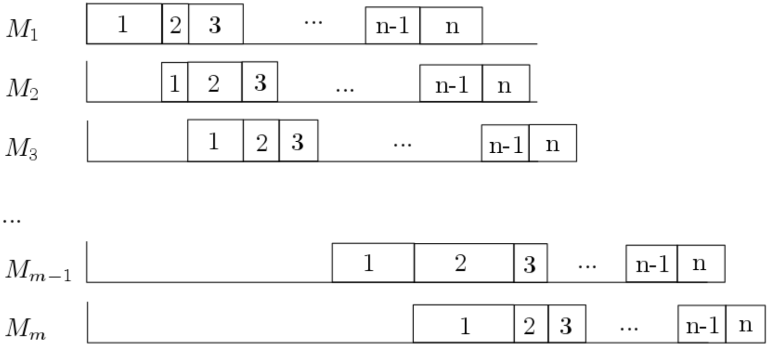
Proof.

[Sketch]:

- For any instance of CDSHOP with n vertices, we generate an instance of $F2|no - idle, no - wait|C_{\max}$ with n jobs where, if there is an arc from v_i to v_j, then, we have $p_{2,i} = p_{1,j}$.
- If a feasible sequence of $F2|no - idle, no - wait|C_{\max}$ exists, then, for each consecutive jobs J_i, J_j with $J_i \rightarrow J_j$, $p_{2,i} = p_{1,j}$ holds. Hence, there is an arc from v_i to v_j. Thus, the corresponding sequence of vertices in CDSHOP constitutes an hamiltonian directed path.
- Conversely, if a path exists for *CDSHOP*, the related sequence of jobs in $F2|no - idle, no - wait|C_{\max}$ is also feasible.

Problem $F|$ no-idle, no-wait $|C_{max}$

The **no-idle, no-wait** constraint on m machines.



Problem $F|$ no-idle, no-wait $|C_{\max}$

Lemma (3)

(C3) *A necessary condition to have a feasible solution for problem $F|no-idle, no-wait|C_{\max}$ is that there always exists an indexing of the jobs so that $p_{j+1,1}, ..., p_{j+1,n-1}$ and $p_{j,2}, ..., p_{j,n}$, for $j = 1, ...m-1$, constitute different permutations of the same vector of elements.*

(C4) *When the above condition (C3) holds, then*

Case 1 *if $(p_{1,1} \neq p_{2,n}$ or $p_{2,1} \neq p_{3,n}$ or ... or $p_{m-1,1} \neq p_{m,n})$, every feasible sequence must have a job with processing times $(p_{1,1}, ..., p_{m-1,1})$ on machines 1 to $(m-1)$ in first position and a job with processing time $(p_{2,n}, ..., p_{m,n})$ on machines 2 to m in last position.*

Case 2 *if $(p_{1,1} = p_{2,n}$ and $p_{2,1} = p_{3,n}$ and ... and $p_{m-1,1} = p_{m,n})$ and there exists a feasible sequence, then there do exist at least n feasible sequences each starting with a different job by simply rotating the starting sequence as in a cycle.*

Problem $F|$ no-idle, no-wait $|C_{\max}$

- We can evince that in an optimal sequence, if job J_i immediately precedes job J_k, we have that $p_{j+1,i} = p_{j,k}$, $\forall j = 1, \ldots, m-1$ holds.

- Then, for a feasible 3-job subsequence (ℓ, i, k) we must have:
  1. $[p_{2,\ell}; \ldots; p_{m,\ell}] = [p_{1,i}; \ldots; p_{m-1,i}]$ and,
  2. $[p_{2,i}; \ldots; p_{m,i}] = [p_{1,k}; \ldots; p_{m-1,k}]$.

This can be represented in terms of vectorial dominoes as follows.

$[p_{1,\ell}; \ldots; p_{m-1,\ell}]$	$[p_{2,\ell}; \ldots; p_{m,\ell}]$	$[p_{1,i}; \ldots; p_{m-1,i}]$	$[p_{2,i}; \ldots; p_{m,i}]$	$[p_{1,k}; \ldots; p_{m-1,k}]$	$[p_{2,k}; \ldots; p_{m,k}]$
domino 1		domino 2		domino 3	

Problem $F|$ no-idle, no-wait $|C_{\max}$: an example

As an example, a 3-job instance on 4 machines of problem $F|no-idle, no-wait|C_{\max}$.

i	J_1	J_2	J_3
$p_{1,i}$	5	3	4
$p_{2,i}$	3	4	9
$p_{3,i}$	4	9	7
$p_{4,i}$	9	7	2

induces the following vectorial dominoes

domino 1 $\rightarrow$

5	3	4	3	4	9

domino 2 $\rightarrow$

3	4	9	4	9	7

domino 3 $\rightarrow$

4	9	7	9	7	2

Problem $F|$ no-idle, no-wait $|C_{\max}$

Proposition

Problem $F|no-idle, no-wait|C_{\max}$ can be solved to optimality in $O(mnlog(n))$ time.

Proof.

[Sketch]:

The result can be proved by showing that any instance of the $F|no-idle, no-wait|C_{\max}$ problem can be reduced in polynomial time to a vectorial OSPD that is always solved by computing an Eulerian path in an oriented graph with n arcs. □

Complexity of problems $(J2, O2)|$ no-idle, no-wait $|C_{\max}$

- Job-shop (J) problem: operations of a job totally ordered
- Open-shop (O) problem: no ordering constraints on operations

Proposition

Problems $J2|no-idle, no-wait|C_{\max}$ *and* $O2|no-idle, no-wait|C_{\max}$ *are NP-hard in the strong sense.*

Proof.

[Sketch of proof for problem $J2|no-idle, no-wait|C_{\max}$]:
We show that NMTS (Numerical Matching with Target Sums) reduces to $J2|no-idle, no-wait|C_{\max}$. □

Thank You.