Brownian Motion Area with Generatingfunctionology

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Some continuous time processes...

A *Brownian Motion* of duration 1 is a stochastic process $\mathcal{B}(t)$, $t \in [0,1]$ such that

- $t \mapsto \mathcal{B}(t)$ is a.s. continuous, $\mathcal{B}(0) = 0$,
- for $s < t$, $\mathcal{B}(t) - \mathcal{B}(s) \sim \mathcal{N}(0, t-s)$ and
- increments are independent.

A *Brownian Meander* $\mathcal{M}(t)$, $t \in [0,1]$ is a BM $\mathcal{B}(t)$ conditioned on $\mathcal{B}(s) \geq 0$, $s \in]0,1]$.

A *Brownian Excursion* $\mathcal{E}(t)$, $t \in [0,1]$ is $\mathcal{M}(t)$ conditioned on $\mathcal{M}(1) = 0$ (quick and dirty def.).

... and their discrete counterparts

The *Bernoulli Random Walk* $\mathcal{B}_n(k)$ on $\mathbb{Z}$, $k \in \{0, 1, \ldots, n\}$, with

- $\mathcal{B}_n(0) = 0$,
- $\mathcal{B}_n(k+1) - \mathcal{B}_n(k) \in \{-1, 1\}$, each with prob. $1/2$.

The *Bernoulli Meander* $\mathcal{M}_n(k)$, $k \in \{0, \ldots, n\}$ on $\mathbb{Z}_{\geq 0}$ is $\mathcal{B}_n(k)$ conditioned to stay non-negative.

The *Bernoulli Excursion* $\mathcal{E}_{2n}(k)$, $k \in \{0, \ldots, 2n\}$ on $\mathbb{Z}_{\geq 0}$ is $\mathcal{M}_{2n}(k)$ conditioned on $\mathcal{M}_{2n}(2n) = 0$.

Scaling limits

For $n \longrightarrow \infty$ we have the weak limits

- $\left\{ \frac{1}{\sqrt{n}} \mathcal{B}_n(\lfloor nt \rfloor),\ t \in [0,1] \right\} \longrightarrow \{\mathcal{B}(t),\ t \in [0,1]\}$,
- $\left\{ \frac{1}{\sqrt{n}} \mathcal{M}_n(\lfloor nt \rfloor),\ t \in [0,1] \right\} \longrightarrow \{\mathcal{M}(t),\ t \in [0,1]\}$,
- $\left\{ \frac{1}{\sqrt{2n}} \mathcal{E}_{2n}(\lfloor 2nt \rfloor),\ t \in [0,1] \right\} \longrightarrow \{\mathcal{E}(t),\ t \in [0,1]\}$.

Drmota (2003): Weak limits imply *moment convergence* for certain functionals. E.g. for area (i.e. integrals)

$$\mathbb{E}\left[\left(\int_0^1 \frac{1}{\sqrt{2n}}\mathcal{E}_{2n}(\lfloor 2nt \rfloor)dt\right)^r\right] \longrightarrow \mathbb{E}\left[\mathcal{E}\mathcal{A}^r\right],$$
$$\mathbb{E}\left[\left(\int_0^1 \frac{1}{\sqrt{n}}\mathcal{M}_{n}(\lfloor nt \rfloor)dt\right)^r\right] \longrightarrow \mathbb{E}\left[\mathcal{M}\mathcal{A}^r\right],$$

for $n \longrightarrow \infty$, where

$$\mathcal{E}\mathcal{A} := \int_0^1 \mathcal{E}(t)dt, \quad \mathcal{M}\mathcal{A} := \int_0^1 \mathcal{M}(t)dt.$$

So studying functionals on $\mathcal{E}$ or $\mathcal{M}$ amounts to studying the discrete models!

Particularly $\mathcal{EA}$ appears in a number of *discrete* contexts, e.g.

- Construction costs of hash tables,
- cost of breadth first search traversal of a random tree,
- path lengths in random trees,
- area of polyominoes,
- enumeration of connected graphs.

Many of the discrete results rely on recursions for the moments of $\mathcal{EA}$ and $\mathcal{MA}$ found by Takács (1991,1995) studying $\mathcal{E}_{2n}$ and $\mathcal{M}_n$.

Results

We choose a different combinatorial approach and obtain

- new formulae for $\mathbb{E}\left(\mathcal{E}\mathcal{A}^r\right)$ and $\mathbb{E}\left(\mathcal{M}\mathcal{A}^r\right)$,
- the joint distribution of $(\mathcal{M}\mathcal{A}, \mathcal{M}(1))$ in terms of the joint moments $\mathbb{E}\left(\mathcal{M}\mathcal{A}^r\mathcal{M}(1)^s\right)$,
- the joint distribution of (signed) areas and endpoint of $\mathcal{B}$,

and as an application of these

- area of discrete meanders with arbitrary finite step sets,
- area distribution of column convex polyominoes.

In the discrete world, we can write the joint distribution of the random variables

$$A_n = \sum_{k=0}^{n} \mathcal{M}_n(k) \text{ and } H_n = \mathcal{M}_n(n)$$

as

$$\mathbb{P}(A_n = k,\ H_n = l) = \frac{p_{n,k,l}}{\sum_{r,s} p_{n,r,s}},$$

where $p_{n,k,l}$ is the number of meanders of length n, area k and final height l.

The generating function of the class of meanders is the formal power series

$$M(z,q,u) = \sum_n \left(\sum_{k,l} p_{n,k,l} q^k u^l \right) z^n,$$

The above probabilities can be rewritten as

$$\begin{aligned}\mathbb{P}(A_n = k,\ H_n = l) &= \frac{p_{n,k,l}}{\sum_{r,s} p_{n,r,s}} \\ &= \frac{\left[z^n q^k u^l\right] M(z,q,u)}{[z^n]\, M(z,1,1)}.\end{aligned}$$

$$M(z,q,u) = \sum_n \left(\sum_{k,l} p_{n,k,l} q^k u^l \right) z^n,$$

and

$$\mathbb{P}(A_n = k,\ H_n = l) = \frac{\left[z^n q^k u^l\right] M(z,q,u)}{[z^n]\, M(z,1,1)}.$$

With this representation the moments take a particularly nice form:

$$\mathbb{E}\left(A_n^r H_n^s\right) = \sum_{k,l} k^r l^s \mathbb{P}(A_n = k,\ H_n = l)$$

$$= \frac{[z^n] \left(q \frac{\partial}{\partial q}\right)^r \left(u \frac{\partial}{\partial u}\right)^s M(z,1,1)}{[z^n]\, M(z,1,1)}.$$

So: large n behaviour of the moments by *coefficient asymptotics* of the above series.

Singularity analysis (Flajolet, Odlyzko 1990)

Transfer Theorem: Let $F(z) = \sum f_n z^n$ be analytic in an indented disk and

$$F(z) \sim (1 - \mu z)^{-\alpha} \quad (z \longrightarrow 1/\mu).$$

Then

$$f_n \sim [z^n]\,(1 - \mu z)^{-\alpha} \sim \frac{1}{\Gamma(\alpha)} \times n^{\alpha-1} \times \mu^n \quad (n \longrightarrow \infty).$$

For example, it turns out, that

$$\left(\frac{\partial}{\partial q}\right)^r \left(\frac{\partial}{\partial u}\right)^s M(z,1,1) \sim \frac{b_{r,s}}{(1-2z)^{3r/2+s/2+1/2}} \quad (z \longrightarrow 1/2),$$

Functional equation for $M(z, q, u)$.

The recursive description of the set of meanders

$$\{\text{meanders of length } n\} \simeq$$

$$\{\text{meanders of length } n-1\} \times \{\nearrow, \searrow\}$$

$$\setminus \{\text{excursions of length } n-1\} \times \{\searrow\}$$

translates into

$$M(z,q,u) = 1 + M(z,q,uq)\left(zuq + \frac{z}{uq}\right) - E(z,q)\frac{z}{uq},$$

$E(z, q)$ is the generating function of excursions.

Solution to the equation for $q = 1$ by the *kernel method:*

$$-z(u - u_1(z))(u - v_1(z))M(z, 1, u) = u - zE(z, 1).$$

where $u_1(z) = \frac{1-\sqrt{1-4z^2}}{2z}$ and $v_1(z) = \frac{1+\sqrt{1-4z^2}}{2z}$.
Substitution of $u = u_1(z)$ yields

$$E(z, 1) = \frac{u_1(z)}{z} = \frac{1 - \sqrt{1 - 4z^2}}{2z^2},$$

and finally

$$M(z, 1, u) = \frac{1}{-z(u - v_1(z))}.$$

The partial derivatives $\left(\frac{\partial}{\partial q}\right)^r \left(\frac{\partial}{\partial u}\right)^s M(z,1,u)$ can in principle be obtained inductively by taking derivatives of the functional equation (and setting $q=1$).

- Each derivative w.r.t. u produces one new unknown function $\left(\frac{\partial}{\partial q}\right)^r \left(\frac{\partial}{\partial u}\right)^{s+1} M(z,1,u)$.
- Each derivative w.r.t. q produces two new unknowns, $\left(\frac{\partial}{\partial q}\right)^{r+1} E(z,1)$ and $\left(\frac{\partial}{\partial q}\right)^{r+1} \left(\frac{\partial}{\partial u}\right)^s M(z,1,u)$ and hence requires another application of the kernel method.

The exact expressions for $\left(\frac{\partial}{\partial q}\right)^r \left(\frac{\partial}{\partial u}\right)^s M(z,1,u)$ and for

$$\mathbb{E}\left(A_n^r H_n^s\right) = \frac{[z^n]\left(q\frac{\partial}{\partial q}\right)^r \left(u\frac{\partial}{\partial u}\right)^s M(z,1,1)}{[z^n]\, M(z,1,1)}.$$

are getting intractable.

But we can keep track of the *singular behaviour* of $\left(\frac{\partial}{\partial q}\right)^r \left(\frac{\partial}{\partial u}\right)^s M(z,1,1)$ and $\left(\frac{\partial}{\partial q}\right)^r \left(\frac{\partial}{\partial u}\right)^s M(z,1,u_1(z))$ and via *singularity analysis* large n asymptotics for the moments.

One proceeds in two steps: First show by induction

$$\left(\frac{\partial}{\partial q}\right)^r \left(\frac{\partial}{\partial u}\right)^s M(z,1,u_1(z)) \sim \frac{a_{r,s}}{(1-2z)^{3r/2+s/2+1/2}} \quad (z \longrightarrow 1/2),$$

where

$$a_{r,s} = a_{r,s-1} + (s+2)a_{r-1,s+2},$$

and then by induction

$$\left(\frac{\partial}{\partial q}\right)^r \left(\frac{\partial}{\partial u}\right)^s M(z,1,1) \sim \frac{b_{r,s}}{(1-2z)^{3r/2+s/2+1/2}} \quad (z \longrightarrow 1/2),$$

where

$$b_{r,s} = b_{r,s-2} + (s+1)b_{r-1,s+1},\ (s \geq 1),$$

$$b_{r,0} = b_{r-1,1} + a_{r-1,1}.$$

Application of the transfer theorem finally yields:

$$\mathbb{E}\left(A_n^r H_n^s\right) \sim \frac{b_{r,s}}{b_{0,0}} \frac{\Gamma(1/2)}{\Gamma((3r+s)/2)} n^{(3r+s)/2},$$

and hence (after rescaling $n^{-3/2}A_n$ and $n^{-1/2}H_n$)

- $b_{r,s}$ is essentially $\mathbb{E}\left(\mathcal{M}\mathcal{A}^r\mathcal{M}(1)^s\right)$,
- similarly $a_{r-1,1}$ is essentially $\mathbb{E}\left(\mathcal{E}\mathcal{A}^r\right)$.

Discrete meanders and excursions with *arbitrary finite step sets:*

No result on convergence to $\mathcal{M}$ resp. $\mathcal{E}$! But:

- Generating function satisfies a similar functional equation.
- Area moments for meanders and excursions can be computed in the same fashion,
- and are expressed in terms of the very same $b_{r,s}$ resp. $a_{r-1,1}$!

Result depends on the sign of the *drift* $=$ mean of the step set.

Column convex polyominoes: Area distribution on polyominoes with fixed perimeter n.

- Similar functional equation as above.
- Similar arguments yield an $\mathcal{EA}$ limit law as $n \longrightarrow \infty$.

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Ouch!

Taking derivatives of the fct. eq. w.r.t. q and u allows recursive computation of $F^{n,t}$.

$$
\begin{aligned}
(1 - zS(u))F^{n,0}(u) + z\sum_{i=0}^{c-1} r_i(u)G_i^{(n)} &= zS(u)nF^{n-1,1}(u) \\
&+ zS(u)\sum_{t=2}^{n}\binom{n}{t}F^{n-t,t}(u) \\
&+ z\sum_{l=1}^{n}\sum_{t=0}^{n-l}\binom{n}{l}\binom{n-l}{t}u^{l+t}S^{(l)}(u)F^{n-l-t,t}(u) \\
&- z\sum_{i=0}^{c-1}\sum_{l=1}^{n}\binom{n}{l}u^{l}r^{(l)}(u)G_i^{(n-l)}.
\end{aligned}
$$