Integer-valued polynomials over subsets of matrix rings

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Definition

A non-empty set R together with two binary operations $(+)$ and $(.)$ is called a ring if for every $a, b, c \in R$, the following properties are valid:

(a) $a + b \in R$,
(b) $(a + b) + c = a + (b + c)$,
(c) there exists an element $0 \in R$ such that $a + 0 = a = 0 + a$,
(d) for every $a \in R$, there exists an element $-a \in R$ such that $a + (-a) = 0 = (-a) + a$,
(e) $a + b = b + a$,
(f) $a.b \in R$,
(g) $(a.b).c = a.(b.c)$,
(h) $a.(b + c) = a.b + a.c$ and $(a + b).c = a.c + b.c$,
(i) there exists a element $1 \in R$ such that $1.a = a.1 = a$.

Definition

A polynomial $f(x) \in \mathbb{Q}[x]$ is called integer-valued if $f(a) \in \mathbb{Z}$ for all $a \in \mathbb{Z}$.
The set of all integer-valued polynomials is denoted by Int($\mathbb{Z}$), in fact

$$\text{Int}(\mathbb{Z}) := \{f(x) \in \mathbb{Q}[x] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

Theorem

The set Int($\mathbb{Z}$) is a ring. Also, we have

$$\mathbb{Z}[x] \subsetneq Int(\mathbb{Z}) \subsetneq \mathbb{Q}[x].$$

In fact, the ring Int($\mathbb{Z}$) is an integral domain between $\mathbb{Z}[x]$ and $\mathbb{Q}[X]$.

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Example

Let $f(x) := \frac{x(x-1)}{2}$, then $f(x) \in \text{Int}(\mathbb{Z})$ but $f(x)$ is not an element of $\mathbb{Z}[x]$. Also, if $g(x) := \frac{x}{2}$ then $g(x) \in \mathbb{Q}[x]$ but $g(x)$ is not an element of $\text{Int}(\mathbb{Z})$.

In general, for each $n \in \mathbb{N}$,

$$\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!},$$

is the polynomial of degree n belong to $\text{Int}(\mathbb{Z})$.
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The definition of an integer-valued polynomial is generalized on a subset of $\mathbb{Z}$, as follows:

Definition

Let S be a non-empty subset of $\mathbb{Z}$. Then a polynomial $f(x) \in \mathbb{Q}[x]$ is called integer-valued on S if $f(a) \in \mathbb{Z}$ for each $a \in S$.
The set of all integer-valued polynomials on S is denoted by $\text{Int}(S, \mathbb{Z})$, that is;

$$\text{Int}(S, \mathbb{Z}) := \{f(x) \in \mathbb{Q}[x] \mid f(S) \subseteq \mathbb{Z}\}.$$

For each non-empty subset S of $\mathbb{Z}$, we can easily see that

$$\mathbb{Z}[x] \subsetneq \text{Int}(\mathbb{Z}) \subseteq \text{Int}(S, \mathbb{Z}) \subsetneq \mathbb{Q}[x].$$

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If S be a finite subset of $\mathbb{Z}$, then we have the following theorem.

Theorem

Let $S = \{a_0, a_1, \cdots, a_n\}$ be a finite subset of $\mathbb{Z}$. Then we have

$$\mathrm{Int}(S, \mathbb{Z}) = \sum_{j=0}^{n} \mathbb{Z} \prod_{i \neq j} \frac{x - a_i}{a_j - a_i} + (x - a_0)(x - a_1) \cdots (x - a_n)\mathbb{Q}[x].$$

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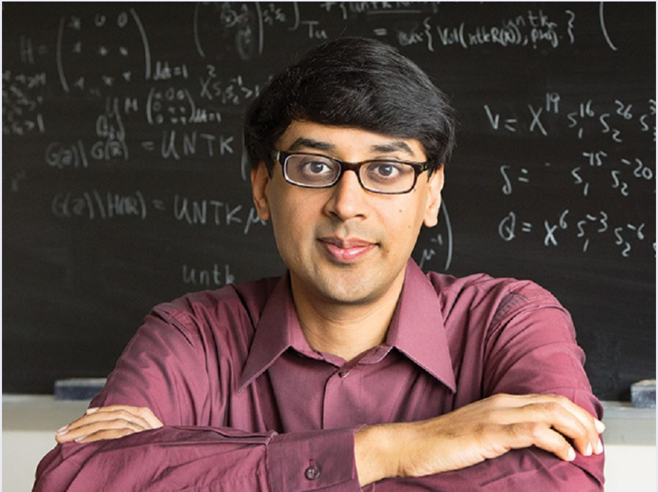
Now, let S be an infinite subset of $\mathbb{Z}$.

Bhargava

Bhargava who won the fields medal in 2014, has several works on integer-valued polynomials.



p-ordering

Let S be an infinite subset of $\mathbb{Z}$ and p be a prime number in $\mathbb{Z}$. A *P-ordering of S* is a sequence $\{a_i\}_{i=1}^{\infty}$ of elements of S that is formed as follows:

- Choose any element $a_0 \in S$,
- Choose an element $a_1 \in S$ that minimizes the highest power of p dividing $(a_1 - a_0)$,
- Choose an element $a_2 \in S$ that minimizes the highest power of p dividing $(a_2 - a_0)(a_2 - a_1)$,

and in general, at the kth step,

- Choose an element $a_k \in S$ that minimizes the highest power of p dividing $(a_k - a_0)(a_k - a_1) \cdots (a_k - a_{k-1})$.

Notice that a p-ordering of S is certainly not unique. In the following definition, we define another sequence which is unique on S.

p-sequence

Let $\{a_i\}_{i=0}^{\infty}$ be an arbitrary p-ordering on S. The associated p-sequence of S corresponding to the p-ordering $\{a_i\}_{i=0}^{\infty}$ is denoted by $\{\nu_k(S,p)\}_{k=0}^{\infty}$ and is defined as follows:

$$\begin{aligned} \nu_0(S,p) &:= 1, \\ \nu_k(S,p) &:= w_p((a_k - a_0)(a_k - a_1)\cdots(a_k - a_{k-1})), \end{aligned} \tag{1}$$

for each $k = 1, 2, \cdots$, where $w_p(a)$ is the highest power of p dividing a, for each a. (for example $w_3(18) = 3^2 = 9$)

Theorem

The associated p-sequence of S is independent of the choice of p-ordering.

Now, we can state the definition of factorial function of S.

factorial function of S.

Let S be a non-empty subset of $\mathbb{Z}$. Then the factorial function of S, denoted $k!_S$, is defined by

$$k!_S := \prod_p \nu_k(S, p).$$

In particular, we have $k!_{\mathbb{Z}} = k!$.

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Proposition

Let S and T be two non-empty subsets of $\mathbb{Z}$ and $S \subseteq T$. Then we have $k!_T$ divides $k!_S$, for each $k \geq 0$. In particular, for each non-empty subset S of $\mathbb{Z}$, $k! \mid k!_S$.

Theorem

Let $\{a_i\}_{i=1}^{\infty}$ be a p-ordering of S for all primes p simultaneously. Then

$$k!_S = \mid (a_k - a_0)(a_k - a_1) \cdots (a_k - a_{k-1}) \mid .$$

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Example

Let S be the set of even integers, that is; $S := 2\mathbb{Z}$. Then by using induction, we can see that the natural ordering $0, 2, 4, \cdots,$ of $2\mathbb{Z}$ forms a p-ordering for all primes p. Hence, by the previous theorem, we have

$$k!_{2\mathbb{Z}} = (2k - 0)(2k - 2)\cdots(2k - (2k - 2)) = 2^k k!.$$

By this factorial function, Bhargava made a basis for the ring $\text{Int}(S, \mathbb{Z})$. He established the following theorem.

Theorem

A polynomial is integer-valued on a subset S of $\mathbb{Z}$ if and only if it can be written as a $\mathbb{Z}$-linear combination of the polynomials

$$\frac{B_{k,S}}{k!_S} := \frac{(x - a_{0,k})(x - a_{1,k}) \cdots (x - a_{k-1,k})}{k!_S}$$

for each $k = 0, 1, 2, \cdots$, where $\{a_{i,k}\}_{i=0}^{\infty}$ is a sequence in $\mathbb{Z}$ that, for each prime p dividing $k!_S$, is term-wise congruent modulo $\nu_k(S, p)$ to some p-ordering of S.

Recently, the set of integer-valued polynomials is considered in some cases for noncommutative rings.
We notice that $R[x]$ is the polynomial ring in one variable x over R, where x commutes with the elements of R. If $f(x), g(x) \in R[x]$, then $(fg)(x)$ denotes the product of $f(x)$ and $g(x)$ in $R[x]$. But, If R is noncommutative and $\alpha \in R$, then $(fg)(\alpha)$ is not necessarily equal to $f(\alpha)g(\alpha)$. In this case, if $f(x) = \sum_i a_i x^i$, then we may express

$$(fg)(x) := \sum_i a_i g(x) x^i. \qquad (*)$$

In this work, we focus on matrix rings.

For any given ring R, let $M_n(R)$ denotes the ring of $n \times n$ matrices with entries from R and $T_n(R)$ denotes the ring of $n \times n$ upper triangular matrices with entries from R. By these notations, we define

$$\mathrm{Int}(M_n(\mathbb{Z})) := \{f \in M_n(\mathbb{Q})[x] \mid f(M_n(\mathbb{Z})) \subseteq M_n(\mathbb{Z})\},$$

and

$$\mathrm{Int}(T_n(\mathbb{Z})) := \{f \in T_n(\mathbb{Q})[x] \mid f(T_n(\mathbb{Z})) \subseteq T_n(\mathbb{Z})\}.$$

In 2012, Werner showed that the set $\text{Int}(M_n(\mathbb{Z}))$ with ordinary addition and multiplication $(*)$ is a noncommutative ring.

In 2017, Frisch proved that $\text{Int}(T_n(\mathbb{Z}))$ is a ring.

We see that if S be a non-empty subset of $\mathbb{Z}$, then $\text{Int}(S, \mathbb{Z})$ is a ring. Therefore, there exist some qouestions here.

Question 1

Let S_1 be an arbitrary subset of $M_n(\mathbb{Z})$ and

$$\text{Int}(S_1, M_n(\mathbb{Z})) := \{f \in M_n(\mathbb{Q})[x] \mid f(S_1) \subseteq M_n(\mathbb{Z})\}.$$

Is $\text{Int}(S_1, M_n(\mathbb{Z}))$ a ring under ordinary addition and multiplication $(*)$?

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Question 2

Let S_2 be an arbitrary subset of $T_n(\mathbb{Z})$ and

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The following example illustrates that, if S_1 is a non-empty subset of $M_n(\mathbb{Z})$ then the set $\text{Int}(S_1, M_n(\mathbb{Z}))$ is not necessary a ring.

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Example

Let $S_1 = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$. Then S_1 is a subset of $M_2(\mathbb{Z})$ and $f(x) := \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} x \in \text{Int}(S_1, M_2(\mathbb{Z}))$. But, by $(*)$ we have

$$f^2(x) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} x \right) x = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} \\ 0 & 0 \end{bmatrix} x^2.$$

Then $f^2\left(\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right) = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 0 \end{bmatrix} \notin M_2(\mathbb{Z})$. This implies that $f^2 \notin \text{Int}(S_1, M_2(\mathbb{Z}))$ and we conclude that $\text{Int}(S_1, M_2(\mathbb{Z}))$ is not closed under multiplication. Therefore, $\text{Int}(S_1, M_2(\mathbb{Z}))$ is not a ring.

Furthermore, The previous example shows that, if S_2 is a non-empty subset of $T_n(\mathbb{Z})$ then the set $\text{Int}(S_2, T_n(\mathbb{Z}))$ is not necessary a ring.

We are going to introduce some subsets S_1 of $M_n(\mathbb{Z})$ such that $\text{Int}(S_1, M_n(\mathbb{Z}))$ be a ring. We need to recall the definition of an ideal.

Ideal

Let R be a commutative ring and I be a non-empty subset of R. The set I is called an ideal of R if the following statements are valid.

- If a and b are elements of I then $a - b \in I$,
- If $a \in I$ and $r \in R$ then $ra \in I$.

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In general, for each $a \in \mathbb{Z}$, the set $a\mathbb{Z} := \{ak \mid k \in \mathbb{Z}\}$ is an ideal of ring $\mathbb{Z}$.

Now, we can state a necessary condition on subset S_1 of $M_n(\mathbb{Z})$ such that $\text{Int}(S_1, M_n(\mathbb{Z}))$ be a ring.

Theorem

Let I be an ideal of $\mathbb{Z}$ and $S_1 := M_n(I) = \{[a_{ij}] \in M_n(R) \mid a_{ij} \in I \; \forall 1 \leq i, j \leq n\}$. Then $\text{Int}(S_1, M_n(\mathbb{Z}))$ is a ring.

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For the upper triangular matrices we use the following notation.

Notation

We write $[a_{ij}]_{j\geq i}$ to denote the following upper triangular matrix,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix}.$$

For the family $\{B_1, B_2, \dots, B_k\}$ of matrices, $b_{ij}^{(r)}$ denotes the (i,j)-th entry of the matrix B_r, where $1 \leq r \leq k$.
Also for each matrix A, we write $a_{ij}^{[r]}$ for the (i,j)-th entry of A^r, that is; $(A^r)_{ij} = a_{ij}^{[r]}$

In the upper triangular matrix ring, we have the following lemma.

Lemma

Let E be a subset of $\mathbb{Z}$ containing zero, $f(x) = B_k x^k + \cdots + B_1 x$ be an element of the set $\text{Int}(T_n(E), T_n(\mathbb{Z}))$ and $A = [a_{ij}]_{j \geq i} \in T_n(E)$. Then we have

$$\sum_{r=1}^{k} b_{il}^{(r)} a_{sj}^{[r]} \in \mathbb{Z}, \qquad (2)$$

where $1 \leq i \leq l \leq s \leq j \leq n$.

Now, we are ready to state the main theorem on the upper triangular matrix ring.

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Theorem

Let E be a subset of $\mathbb{Z}$ containing zero and $S_2 := T_n(E)$. Then the set $\mathrm{Int}(S_2, T_n(\mathbb{Z}))$ is a ring under ordinary addition and multiplication of $(*)$.

Sketch of proof

It is obvious that the set $\mathrm{Int}(S_2, T_n(\mathbb{Z}))$ is non-empty and is closed under addition. Then it is enough to show that $\mathrm{Int}(S_2, T_n(\mathbb{Z}))$ is closed under multiplication. Let $f(x), g(x) \in \mathrm{Int}(S_2, T_n(\mathbb{Z}))$, $A \in S_2$ and $f(x) = B_k x^k + B_{k-1} x^{k-1} + \cdots + B_1 x + B_0$. Suppose that $g(A) := \Gamma = [\gamma_{ij}]_{j \geq i} \in T_n(\mathbb{Z})$, then we obtain

$$(fg)(A) = B_k \Gamma A^k + \cdots + B_1 \Gamma A + B_0 \Gamma.$$

Let $\Omega_r = [\omega_{ij}^{(r)}] := B_r \Gamma$ for $0 \leq r \leq k$, then we have

$$\Omega_r = \left[\sum_{l=i}^{j} b_{il}^{(r)} \gamma_{lj}\right]_{j\geq i}.$$

We can write

$$\begin{aligned}
(fg)(A) &= \Omega_k A^k + \cdots + \Omega_1 A + \Omega_0 \\
&= \left[\left(\sum_{s=i}^{j}\sum_{r=1}^{k} \omega_{is}^{(r)} a_{sj}^{[r]}\right) + \omega_{ij}^{(0)}\right]_{j\geq i} \\
&= \left[\left(\sum_{s=i}^{j}\sum_{r=1}^{k} \left(\sum_{l=i}^{s} b_{il}^{(r)} \gamma_{ls}\right) a_{sj}^{[r]}\right) + \omega_{ij}^{(0)}\right]_{j\geq i} \\
&= \left[\left(\sum_{s=i}^{j}\sum_{l=i}^{s} \gamma_{ls} \sum_{r=1}^{k} b_{il}^{(r)} a_{sj}^{[r]}\right) + \omega_{ij}^{(0)}\right]_{j\geq i},
\end{aligned}$$

where $1 \leq i \leq l \leq s \leq j \leq n$. By using (2), we have $\sum_{r=1}^{k} b_{il}^{(r)} a_{sj}^{[r]} \in \mathbb{Z}$. Also, γ_{ls} and $\omega_{ij}^{(0)}$ are elements of $\mathbb{Z}$, so $(fg)(A) \in T_n(\mathbb{Z})$. Then $fg \in \text{Int}(S_2, T_n(\mathbb{Z}))$ and hence $\text{Int}(S_2, T_n(\mathbb{Z}))$ is a ring.

There are many open problems on the subject of integer-valued polynomials over matrix rings. In the following, we state some open problems on this subject.

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There are many open problems on the subject of integer-valued polynomials over matrix rings. In the following, we state some open problems on this subject.

Open problems

Q_1

Is there a necessary and sufficient condition on the subset S_1 of $M_n(\mathbb{Z})$ such that $\text{Int}(S_1, M_n(\mathbb{Z}))$ be a ring?

Q_2

Is there a necessary and sufficient condition on the subset S_2 of $T_n(\mathbb{Z})$ such that $\text{Int}(S_2, T_n(\mathbb{Z}))$ be a ring?

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Q_1

Is there a necessary and sufficient condition on the subset S_1 of $M_n(\mathbb{Z})$ such that $\mathrm{Int}(S_1, M_n(\mathbb{Z}))$ be a ring?

Q_2

Is there a necessary and sufficient condition on the subset S_2 of $T_n(\mathbb{Z})$ such that $\mathrm{Int}(S_2, T_n(\mathbb{Z}))$ be a ring?

Open problems

Q_3

Is there any regular basis for the ring $\text{Int}(S_1, M_n(\mathbb{Z}))$, where S_1 is a non-empty subset of $M_n(\mathbb{Z})$?

Q_4

Is there any regular basis for the ring $\text{Int}(S_2, T_n(\mathbb{Z}))$, where S_2 is a non-empty subset of $T_n(\mathbb{Z})$?

Open problems

Q_3

Is there any regular basis for the ring $\mathrm{Int}(S_1, M_n(\mathbb{Z}))$, where S_1 is a non-empty subset of $M_n(\mathbb{Z})$?

Q_4

Is there any regular basis for the ring $\mathrm{Int}(S_2, T_n(\mathbb{Z}))$, where S_2 is a non-empty subset of $T_n(\mathbb{Z})$?

Thank you for your attention