How to determine if a random graph with a fixed degree sequence has a giant component

Felix Joos, Guillem Perarnau, Dieter Rautenbach, Bruce Reed

A walkthrough by Angus Southwell

Monash University

Random graphs with a given degree sequence: $\mathcal{G}_{n,\boldsymbol{d}}$

Definition
Let $\boldsymbol{d} = (d_1, \ldots, d_n)$. Then let $G(\boldsymbol{d})$ be a uniformly chosen simple graph with labelled vertices $\{1, \ldots, n\}$ and degree sequence $\boldsymbol{d}$. The probability space of such graphs is $\mathcal{G}_{n,\boldsymbol{d}}$.

- Pro: these graphs are much more like most real-world graphs than $\mathcal{G}(n, p)$.
- Con: they are much more complicated to analyse.

Example
In $\mathcal{G}(n, p)$, $\mathbb{P}(u \sim v) = p$ trivially. In $\mathcal{G}_{n,\boldsymbol{d}}$, $\mathbb{P}(u \sim v)$ is not known in general.

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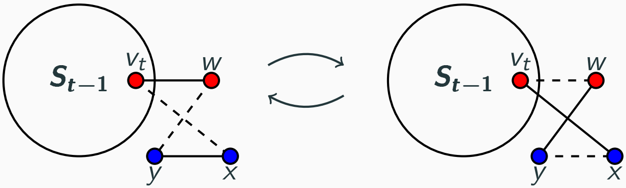

A switching is an operation that takes $G(\boldsymbol{d})$ to $G'(\boldsymbol{d})$.

They are used to find the probability of certain events occurring that we previously got via configuration model, such as:

- probability of specific edges being present,
- probability that a given undiscovered vertex is found at the next step,
- probability of a giant component in the preprocessed vertices.

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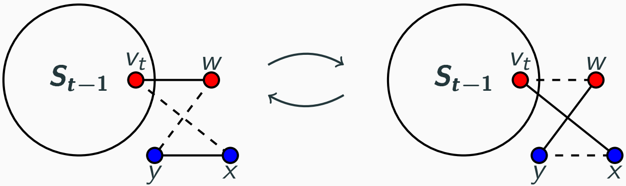

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Giant component problem

Problem

Given a random graph model, what is the distribution of the size of the largest connected component?

Older results

"Double jump" threshold for Erdős–Rényi random graphs at around $\frac{1}{2}n$ edges.

- Below the threshold, all components are order $O(\log n)$.
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Results for fixed degree sequences

Theorem (Molloy and Reed (1995))
Let $\boldsymbol{D}(n)$ be a "well behaved" degree sequence with max. degree at most $n^{\frac{1}{4}-\varepsilon}$. Then define

$$Q(D) := \frac{1}{n}\sum_{j\in[n]} d(j)(d(j)-2).$$

- If $Q(D) < 0$, then all components have size $O(\log n)$.
- If $Q(D) > 0$, then there exists a component with at least αn vertices and βn cycles for $\alpha, \beta > 0$.

Proof sketch

- Breadth first search on the graph.
- Keep track of X_t, the number of "half edges" in your component that can be explored still.

- Show $\mathbb{E}_{t-1}[X_t - X_{t-1}]$ stays positive (or negative) for a sufficiently long time.

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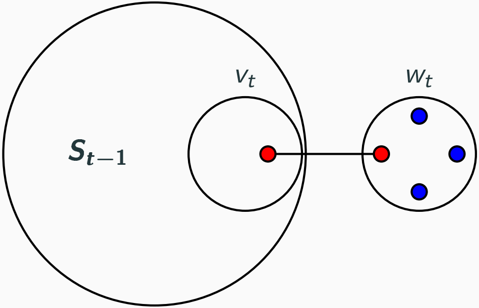

Figure 1: “Superman using his laser vision on the four-vertex empty graph, soon to be the three-vertex empty graph” - Tim

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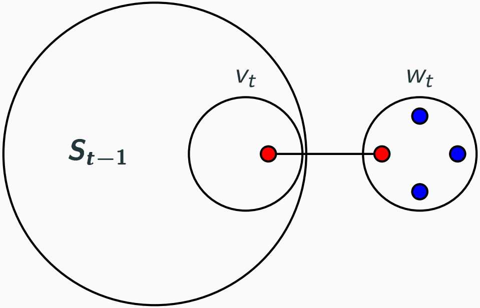

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Limitations of MR result

- Proven in configuration model rather than $\mathcal{G}_{n,\boldsymbol{d}}$.
- Handling of large degree vertices is nonexistent.
- Criterion does not extend to general degree sequences:

Consider $n = k^2$ for large odd k, and $\boldsymbol{d} = (1, \ldots, 1, 2k)$. Then $Q(D) \approx 3$, so we would expect a giant component according to the MR criterion.

Figure 1: " An evil spider that can pick between 1 and n toothpicks as its weapon" - Tim

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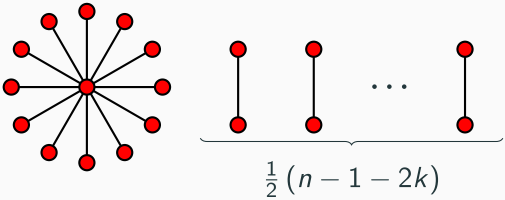

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The more comprehensive result

Theorem (Joos et al. (2018))
For any function $\delta \to 0$ as $n \to \infty$, for every $\gamma > 0$, if $R_D \leq \delta M_D$, the probability that $G(D)$ has a component of order at least γn is $o(1)$.

If there exists an $\varepsilon > 0$ such that $R_D \geq \varepsilon M_D$, then the probability that $G(D)$ contains a component of size γn for some $\gamma > 0$ is $1 - o(1)$.

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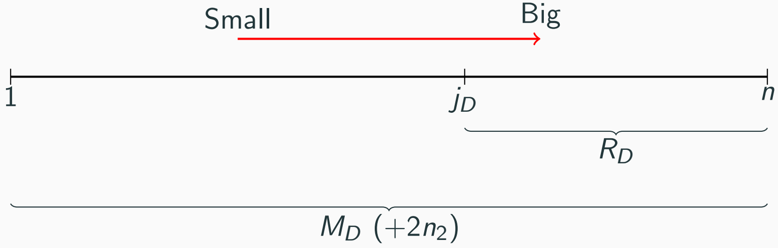

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- Breadth first search again.
- Now with added preprocessing!

- Suppression of degree 2 vertices.
- Switchings used to work in the graph model to get edge probabilities.

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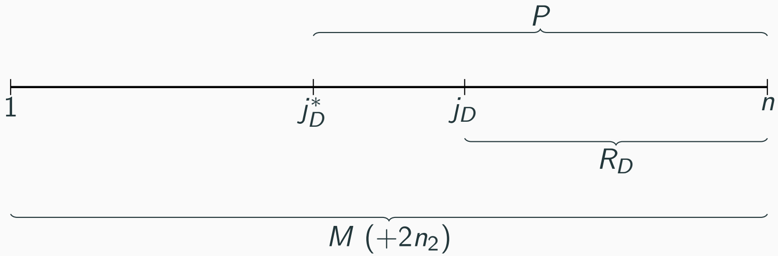

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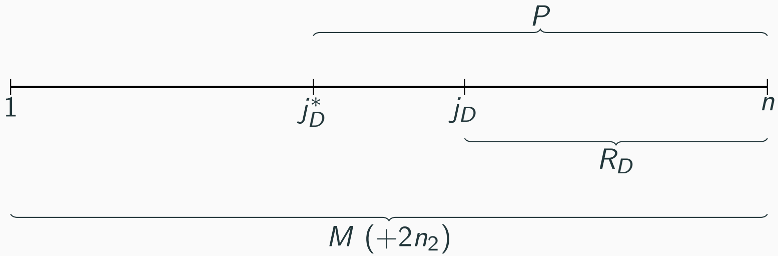

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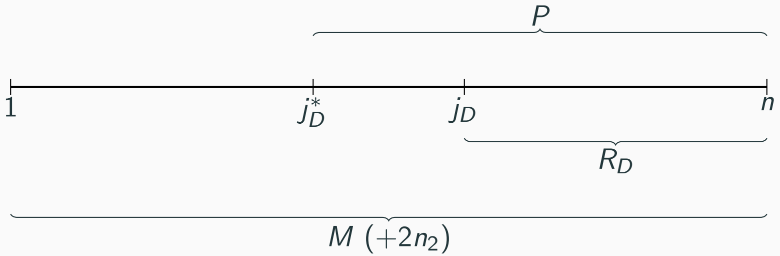

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Subcritical case

Let H be the graph G with all degree 2 vertices contracted, let $\omega > 0$ be small such that $R_D \leq \omega M_D$.

Let S be the smallest set of vertices of H such that $\sum_{i \in S} d_i \geq 5\omega^{1/4} M$ and no vertex outside of S is larger.

Define initial exploration set S_0 to be $S \cup \{v\}$ for any vertex v.

Define X_t' by

$$X_0' = \sum_{u \in S_0} d(u),$$

$$X_t' = X_0' + \sum_{i=1}^{t} (d(w_i) - 2).$$

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We get the following results about the initial stages of the exploration:

Lemma

- $\sum_{w \in V \setminus S} d(w)\,(d(w) - 2) \leq -4\omega^{1/4} M$,
- there is a vertex in S of degree at most $\omega^{-1/4}$.

So the vertices outside the preprocessing set are “small”.

- Process is highly concentrated around its mean.
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The game is now to bound

$$\mathbb{E}_{t-1}\left[d(w_t) - 2\right],$$

the expected increase between X'_{t-1} and X'_t.

$$\mathbb{E}_{t-1}\left[d(w_t) - 2\right] = \sum_{w \notin S_{t-1}} (d(w) - 2)\mathbb{P}_{t-1}(w_t = w).$$

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Lemma
If $t \leq \omega^{1/9} M$ and $X'_{t-1} \leq \omega^{1/5} M$, and $X_{t'} > 0$ for all $t' < t$, then:

- If $w \in V \backslash S_{t-1}$ and $d(w) = 1$, then

$$\mathbb{P}(w_t = w) \geq (1 - 9\omega^{1/5}) \frac{1}{M_{t-1}}.$$

- If $w \in V \backslash S_{t-1}$, then

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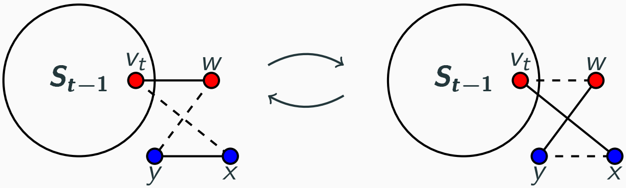

Want to find the number of forward and backward switchings.

Number of forward switchings is at most M_{t-1}. How many of these are 'bad'?

- x or $y \in S_{t-1}$
- $v_t \sim x$
- $w \sim y$
- Vertices overlap

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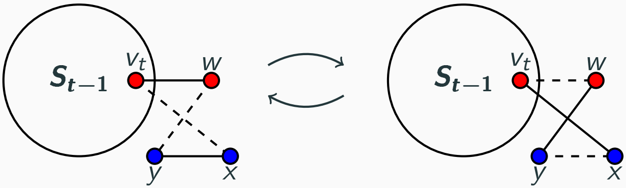

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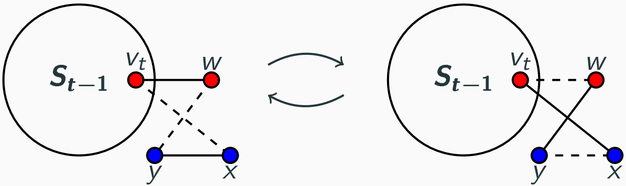

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Subcritical case: finishing touches

Lemma
Define $Y_t = d(w_t) - \mathbb{E}_{t-1}(d(w_t))$. The probability that there exists a t such that $\sum_{t' \le t} Y_{t'} > M^{2/3}$ is less than $e^{-M^{1/4}}$.

Lemma
For $t \le \lfloor \frac{\omega^{1/9} M}{2} \rfloor$, we have that

$$\mathbb{E}_{t-1}\left(d(w_t) - 2\right) \le -\frac{t}{M} + 19\omega^{1/5}.$$

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Supercritical case

Same exploration process, but different preprocessing.

Preprocessing – supercritical case
Expose all components in H containing a vertex of degree larger than $\frac{\sqrt{M}}{\log M}$. Call the set of exposed vertices U.

Analysis splits into two cases:

- $\sum_{u \in U} d(u) \geq \frac{R}{100}$ – then U contains a giant component,
- $\sum_{u \in U} d(u) < \frac{R}{100}$ – same exploration as in the subcritical case.

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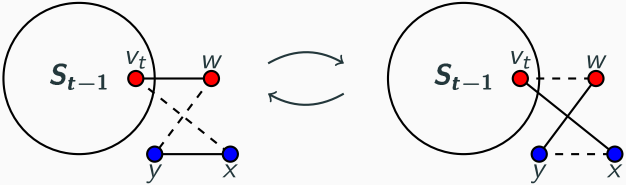
Lemma
Let U be a set of vertices containing all vertices with degree greater than $\frac{\sqrt{M}}{\log M}$ and let $\frac{1}{4} < c < 1$ be such that $\sum_{u \in U} d(u) \leq cR$. Then

$$\sum_{u \in V \setminus U} d(u)\,(d(u) - 2) \geq \frac{(1-c)}{2} R.$$

Supercritical switching analysis



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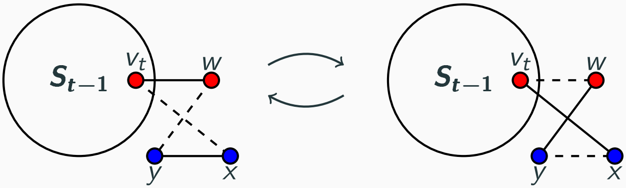

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Let $\beta = 10^{-6}\varepsilon^2$ be a fixed constant. If $M_{t-1} \geq \frac{3M}{4}$ and $X_{t-1} \leq \beta M$, then for every $w \in S_{t-1}$,

$$(1 - 10\sqrt{\beta})\frac{d(w)}{M_{t-1}} \leq \mathbb{P}\left(w = w_t\right) \leq (1 + 10\sqrt{\beta})\frac{d(w)}{M_{t-1}}.$$

Futhermore,

$$\mathbb{P}\left(d_t'(w) \geq \lfloor 2\sqrt{\beta} d(w) \rfloor + i \,\middle|\, w = w_t \right) \leq \beta^{i/2},$$

where $d_t'(w)$ is the number of edges from w to $S_{t-1} \setminus \{v_t\}$ in H and the number of loops at w in H.

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Lemma
For $t \le \tau$, $\mathbb{E}\left[d(w_t) - 2\right] \ge \frac{\varepsilon}{4}$, $\mathbb{E}\left[d'_t(w_t)\right] \le \frac{\mathbb{E}[d(w_t)-2]}{3}$, and thus $\mathbb{E}\left[X_t - X_{t-1}\right] \ge \frac{\varepsilon}{12}$.

Here τ is the smallest t for which either $X_t \ge \beta M$ or $M_t \le \left(1 - \frac{R}{4M}\right) M_0$.

$X_t \ge X_{t-1} + (d(w_t) - 2) - 2d'_t(w_t)$, so we can use this recursively to get...

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$$X_\tau \geq \mathbb{E}[X_\tau] + \sum_{t \leq \tau} A_t + \sum_{t \leq \tau} B_t,$$

where $A_t = d(w_t) - \mathbb{E}[d(w_t)]$ and $B_t = d'_s(w_t) - \mathbb{E}[d'_t(w_t)]$.

Lemma
With probability $1 - o(1)$, there exists no $t \leq \tau$ for which $\sum_{s \leq t} A_s$ or $\sum_{s \leq t} B_s$ are greater than $\frac{M}{\log \log M}$.

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Small caveats

- We found bounds on the number of edges in each component, not vertices!
- What about degree 2 vertices?
  - Degree 2 vertices in components of $H(D)$
  - Degree 2 vertices in cyclic components
  - The case of too many degree 2 vertices

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Thank you!

References

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