Computing with Euclidean lattices

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ÉNS de Lyon

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Goals and plan of the talk

Goals:

- An introduction to the computational aspects of lattices
- An example of how floating-point arithmetic can be used to accelerate an algebraic computation

Plan of the talk:

1. Euclidean lattices
2. Applications of lattices
3. The LLL algorithm
4. Speeding up LLL
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- An introduction to the computational aspects of lattices
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Plan of the talk:

1. **Euclidean lattices**
2. Applications of lattices
3. The LLL algorithm
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Euclidean lattices

Lattice $\equiv$ discrete subgroup of $\mathbb{R}^n$

$\equiv \left\{ \sum_{i \leq n} x_i b_i : x_i \in \mathbb{Z} \right\}$

If the $b_i$'s are linearly independent, they are called a basis.

Bases are not unique, but they can be obtained from each other by integer transforms of determinant $\pm 1$:

$\begin{bmatrix} -2 & 1 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$. 
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**Lattice invariants**

**Minimum:**
\[ \lambda(L) = \min(\|b\| : b \in L \setminus 0) \]

**Determinant:**
\[ \det L = |\det(b_i)|, \text{ for any basis} \]

**Minkowski theorem:**
\[ \lambda(L) \leq \sqrt{n} \cdot (\det L)^{1/n} \]

**Algorithmic approach: lattice reduction**
Start from a basis, and progressively improve its norm/orthogonality
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Why do we care about lattices?

- Computer algebra: factorisation of rational polynomials.
- Cryptanalysis of variants of RSA.
- Lattice-based cryptography.
- Communications theory: MIMO, GPS.
- Combinatorial optimisation, algorithmic group theory, algorithmic number theory, computer arithmetic, etc.

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Lattices tend to pop out when one wants to use linear algebra but is restricted to discrete transformations.
Main computational problem: SVP

- \( SVP_\gamma \): Given a basis of \( L \), find \( b \in L \) with

\[
0 < \|b\| \leq \gamma \cdot \lambda(L).
\]

- \( \text{Dec-SVP}_\gamma \): Given a basis of \( L \) and \( t > 0 \), reply:

**YES** if \( \lambda(L) \leq t \) and **NO** if \( \lambda(L) > \gamma \cdot t \).

---

**Dec-SVP_\gamma on the hardness scale**

- \( \text{NP-hard} \) for any \( \gamma \leq O(1) \), under randomized reductions
- \( \text{In } \text{NP} \cap \text{coNP} \) for \( \gamma \geq \sqrt{n} \)
- \( \text{In } \text{P} \) for \( \gamma \geq 2^{\frac{n \log \log n}{\log n}} \)
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SVP is easy in small dimensions!

That's almost Euclid's algorithm!
- Returns a vector reaching $\lambda(L)$
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1. Euclidean lattices
2. Applications of euclidean lattices
3. The LLL algorithm
4. Speeding up LLL

- Integer relation detection
- Polynomial factorisation
- Cryptanalysis
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1. Euclidean lattices
2. **Applications of euclidean lattices**
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- Integer relation detection
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- Cryptanalysis
Finding small integer relations between real numbers

BBP formula: \[ \pi = \sum_{i \geq 0} \frac{1}{16^i} \left( \frac{4}{8i + 1} - \frac{2}{8i + 4} - \frac{1}{8i + 5} - \frac{1}{8i + 6} \right) \]

\[ \implies \] To compute a base-16 digit of \( \pi \) at any given position.

Assume we search a small \( \mathbb{Z} \)-relation between \( y_1, \ldots, y_d \in \mathbb{R} \)

Take \( L := L[(b_i)] \), with \( B = \begin{pmatrix} y_1 & y_2 & \cdots & y_d \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} \)

A \( \mathbb{Z} \)-relation \( \sum x_i y_i = 0 \) leads to a small vector \( (0, x_1, \ldots, x_d)^T \).
Finding small integer relations between real numbers

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Factoring integer polynomials

The previous idea may be used to factor polynomials in $\mathbb{Z}[x]$

*Factoring polynomials with rational coefficients*,

⇒ Cited ≥ 2500 times!!!

Given $P \in \mathbb{Z}[x]$:

0- If $\deg P \leq 1$, then stop

1- Compute a root $\alpha \in \mathbb{C}$ of $P$

2- Find the minimal polynomial $P_{\alpha}(x)$ of $\alpha$, by searching for $\mathbb{Z}$-combinations between $1, \alpha, \ldots, \alpha^i$ for increasing $i$

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Lattice-based cryptography:
- Secret key: very short basis of a lattice
- Public key: long basis of the same lattice
- Relies on the assumed hardness of SVP

Very popular research topic:
- More secure: post-quantum
- More efficient: no modular exponentiation
- More versatile: fully homomorphic encryption

Lattice reduction algorithms are the best known attack.
Cryptographic design and cryptanalysis

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1. Euclidean lattices
2. Applications of euclidean lattices
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4. Speeding up LLL
Gram-Schmidt orthogonalization (GSO)

$(b_i);$ linearly independent

The GSO $(b_i^*)_i$ is defined by:

$$\forall i : b_i^* = b_i - \sum_{j<i} \mu_{ij} b_j^*$$

$$\forall i > j : \mu_{ij} = \frac{(b_i, b_j^*)}{\|b_j^*\|^2}$$

Triangularisation of $B = (b_i)_i$

For any basis $(b_i)_i$ of $L$, we have $\lambda(L) \geq \min_i \|b_i^*\|$
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The Lenstra-Lenstra-Lovász reduction

Let $\delta \in (1/4, 1)$. A basis $B = (b_i)_{i \leq n} \in \mathbb{R}^{n \times n}$ is said **LLL-reduced** if

- $\forall i, j : |\mu_{ij}| \leq 1/2$  
  [Size-reduction]

- $\forall i : \delta \cdot \|b_i^*\|^2 \leq \|b_{i+1}^*\|^2 + \mu_{i+1,i}^2 \|b_i^*\|^2$  
  [Lovász’ condition]
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The $\|b_i^*\|$’s can’t drop too fast:

$\forall i : \|b_{i+1}^*\|^2 \geq (\delta - \frac{1}{4})\|b_i^*\|^2$

$\Rightarrow \lambda(L) \leq \|b_1\| \leq 2^{O(n)} \cdot \lambda(L)$

$\delta < 1$ is important to get a polynomial complexity
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1. Enforce size-reduction, using a modified Gaussian elimination
2. If there is an $i$ with $\delta \cdot \|b_i^*\|^2 > \|b_{i+1}^*\|^2 + \mu_{i+1,i}^2 \|b_i^*\|^2$, then swap $b_i$ and $b_{i+1}$, and go to Step 1
3. Return the current basis $(b_1, \ldots, b_n)$

$\Rightarrow$ Correctness is trivial

$\Rightarrow$ Termination is much less so:

$O(n^2 \beta)$ loop iterations, with $\beta = \max_i \|b_i^{\text{init}}\|
The Lenstra-Lenstra-Lovász algorithm

Let $\delta \in (1/4, 1)$. A basis $B = (b_i)_{i \leq n} \in \mathbb{R}^{n \times n}$ is said \textbf{LLL-reduced} if

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Bit-complexity of LLL and practical run-time

[LLL82,Kaltofen83]

LLL terminates in $O(n^4\beta^2(n + \beta))$ operations, with $\beta = \log \max_i \|b_i^{\text{init}}\|$

With MAGMA V2.16:

```plaintext
> n := 25; B := RMatrixSpace(Integers(),n,n)!0;
> for i:=1 to 25 do
>   B[i][i]:=1; B[i][1]:=RandomBits(2000);
> end for;
> time C := LLL(B:Method:=''Integral'');
Time: 11.700
> time C := LLL(B);
Time: 0.240
```
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Section based on joint works with X.-W. Chang, I. Morel, P. Q. Nguyen, A. Novocin, X. Pujol and G. Villard
LLL in practice: the numeric-symbolic approach

The Gram-Schmidt computations dominate the cost

Odlyzko’s hybrid approach

Replace the rational computations on the GSO by floating-point approximations, but keep the basis operations exact

Floating-point numbers: $x_1.x_2x_3\ldots x_p \cdot B^e$, where:

- $p$ is the precision
- $B$ is the base, and $x_i \in \{0, \ldots, B - 1\}$
- $e \in \mathbb{Z}$ is the exponent

Floating-point arithmetic:

$fl(a \ op b)$ is a nearest fp number to $a \ op b$, for any $op \in \{+, -, /, \times\}$
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LLL in practice: the numeric-symbolic approach

The Gram-Schmidt computations dominate the cost

**Odlyzko’s hybrid approach**

Replace the rational computations on the GSO by floating-point approximations, but keep the basis operations exact

**Floating-point numbers**: \( x_1.x_2x_3 \ldots x_p \cdot B^e \), where:

- \( p \) is the precision
- \( B \) is the base, and \( x_i \in \{0, \ldots, B - 1\} \)
- \( e \in \mathbb{Z} \) is the exponent

**Floating-point arithmetic**:

fl\((a \ op \ b)\) is a nearest fp number to \(a \ op \ b\), for any \(op \in \{+,-,/,\times\}\)
Odlyzko’s hybrid approach is only heuristic

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**Principle:** For $p$ small, fp arith. may efficiently simulate rational arith.

$\Rightarrow$ In practice: we aim for 53-bit machine precision

But Odlyzko’s approach is heuristic:

- Fp arithmetic is inexact
- Small errors can be amplified

$\Rightarrow$ Infinite loops

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Making the numeric-symbolic approach rigorous

Underlying mathematical phenomenon [CSV12]

Any LLL-reduced basis is well-conditioned with respect to GSO

- Well-conditioned? The GSO computed in small precision is close to the genuine GSO
  ⇒ We’d like to rely on LLL-reduced bases as much as we can

Use a greedy LLL algorithm [NS05, MSV09]:

- Consider the first $i$ s.t. $b_1, \ldots, b_i$ is not LLL-reduced
  ⇒ $b_1, \ldots, b_{i-1}$ is well-conditioned
- Iterate on $b_i$ until nothing happens (iterative refinement)
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Bit complexity of floating-point LLL

Small precision? $O(n)$ bits suffice for correctness.

Bit-complexity:

$$O(n^2\beta) \cdot O(n^2) \cdot \left[ O(n\beta) + O(n^2) \right] = O(n^5\beta(n + \beta)).$$

1. loop iterations
2. size-reduction arithmetic steps
3. integer arithmetic
4. floating-point arithmetic

Asymptotically not much better than LLL’s $O(n^4\beta^2(n + \beta))$, but much better in practice.
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Can we do better? [NSV10,PSV13?]

The totally numeric approach

LLL can be accelerated further by using approximations for the bases too!
\[ \tilde{O}(n^{5\beta^{1.5}}) \] operations

The totally numeric approach, continued

Do the same with several levels of recursion
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The totally numeric approach with blocking

Consider sub-matrices of the GSO
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Plan of the talk:

1. Euclidean lattices
2. Applications of euclidean lattices
3. The LLL algorithm
4. Speeding up LLL
5. Conclusion
Open problems

On LLL:
- Lower the cost further: as fast as matrix multiplication?
- Improve current implementations

In the general area of lattices
- Faster algorithms computing shorter vectors than LLL
- Quantum algorithms
- Hardness proofs for worst-case lattice problems
- Hardness proofs for average-case lattice problems (crucial for lattice-based cryptography)
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