Computing with Euclidean lattices

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ÉNS de Lyon

Monash University, November 2012

Goals and plan of the talk

Goals:

- An introduction to the computational aspects of lattices
- An example of how floating-point arithmetic can be used to accelerate an algebraic computation

Plan of the talk:

1. **Euclidean lattices**
2. Applications of lattices
3. The LLL algorithm
4. Speeding up LLL

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Euclidean lattices

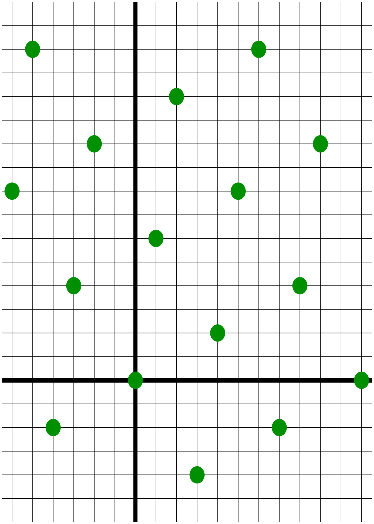
Lattice $\equiv$ discrete subgroup of $\mathbb{R}^n$

$\equiv \quad \{\sum_{i \leq n} x_i \mathbf{b}_i : x_i \in \mathbb{Z}\}$

If the $\mathbf{b}_i$'s are linearly independent, they are called a **basis**.

Bases are not unique, but they can be obtained from each other by integer transforms of determinant ± 1:

$$\begin{bmatrix} -2 & 1 \\ 10 & 6 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 2 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}.$$



Euclidean lattices

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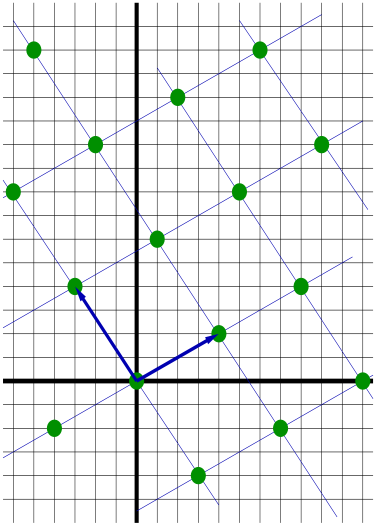

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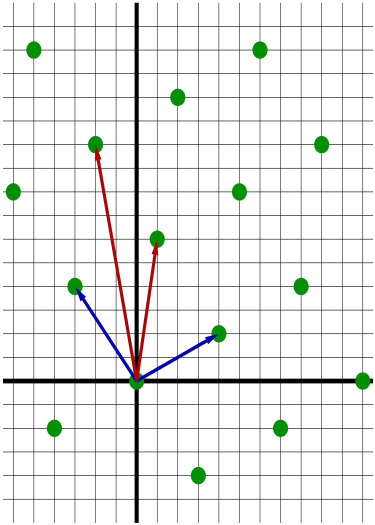


Lattice invariants

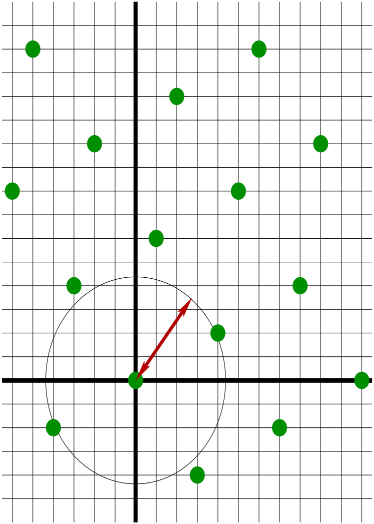
Minimum:
$\lambda(L) = \min(\|\mathbf{b}\| : \mathbf{b} \in L \setminus \mathbf{0})$

Determinant:
$\det L = |\det(\mathbf{b}_i)_i|$, for any basis

Minkowski theorem:
$\lambda(L) \leq \sqrt{n} \cdot (\det L)^{1/n}$

Algorithmic approach: lattice reduction

Start from a basis, and progressively improve its norm/orthogonality



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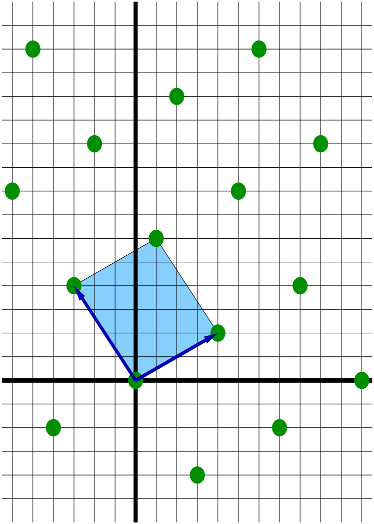

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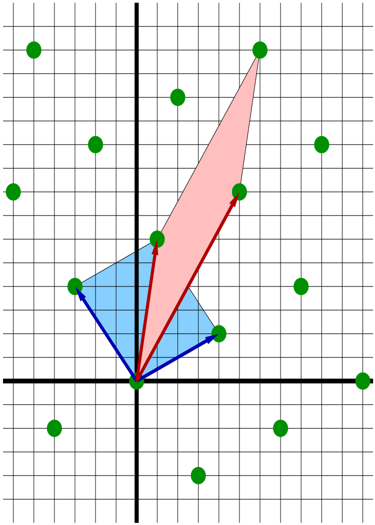

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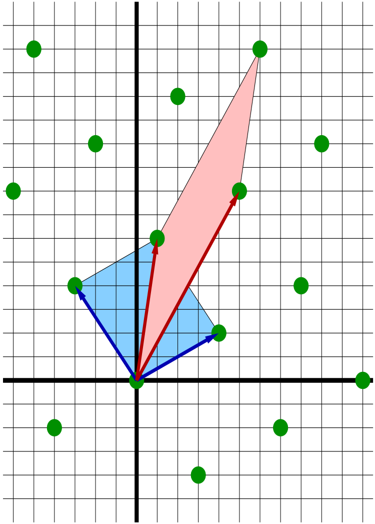

Why do we care about lattices?

- Computer algebra: factorisation of rational polynomials.
- Cryptanalysis of variants of RSA.
- Lattice-based cryptography.
- Communications theory: MIMO, GPS.
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Main computational problem: SVP

- SVP$_\gamma$: Given a basis of L, find $\mathbf{b} \in L$ with

$$0 < \|\mathbf{b}\| \leq \gamma \cdot \lambda(L).$$

- Dec-SVP$_\gamma$: Given a basis of L and $t > 0$, reply:

$$\textbf{YES} \text{ if } \lambda(L) \leq t \quad \text{and} \quad \textbf{NO} \text{ if } \lambda(L) > \gamma \cdot t.$$

Dec-SVP$_\gamma$ on the hardness scale

- NP-hard for any $\gamma \leq \mathcal{O}(1)$, under randomized reductions
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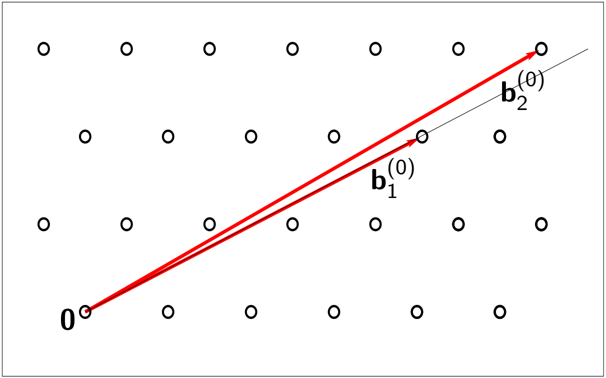

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- Returns a vector reaching $\lambda(L)$
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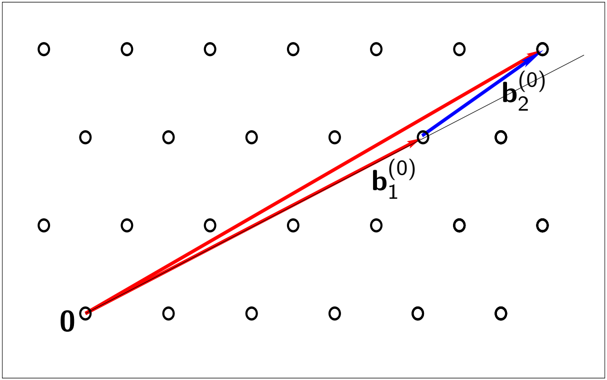

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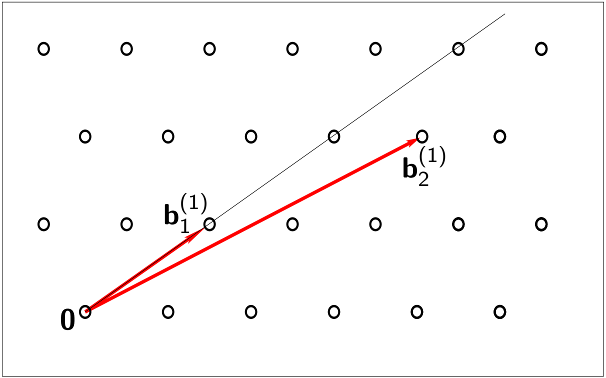

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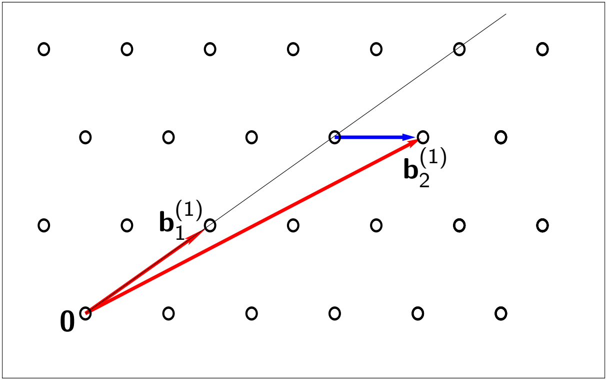

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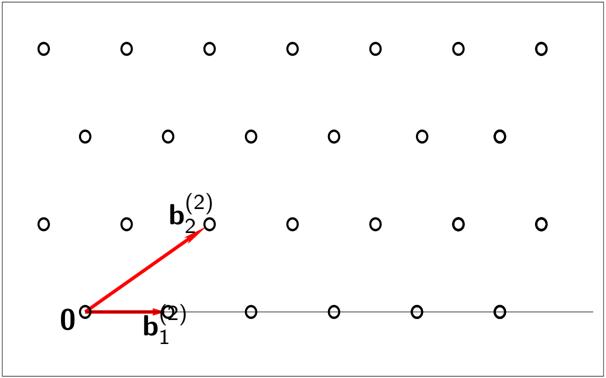

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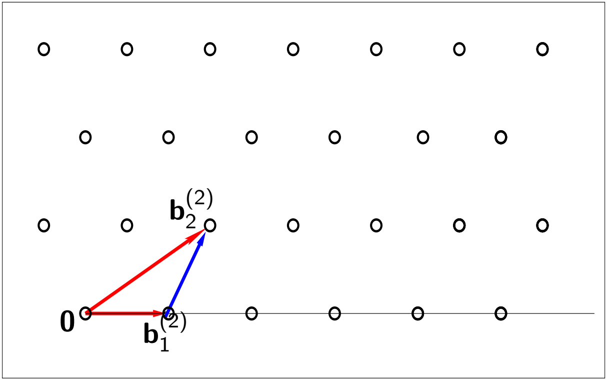

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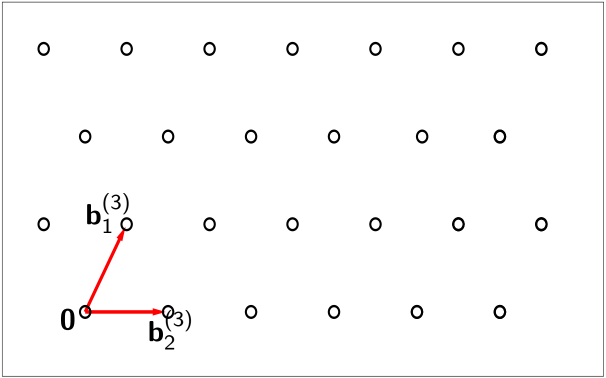

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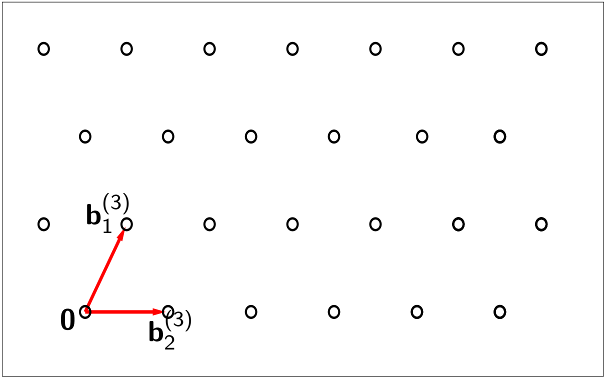

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- Polynomial factorisation
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Finding small integer relations between real numbers

BBP formula: $$\pi = \sum_{i \geq 0} \frac{1}{16^i} \left(\frac{4}{8i+1} - \frac{2}{8i+4} - \frac{1}{8i+5} - \frac{1}{8i+6} \right)$$

$\rightsquigarrow$ To compute a base-16 digit of π at any given position.

Assume we search a small $\mathbb{Z}$-relation between $y_1, \ldots, y_d \in \mathbb{R}$

Take $L := L[(\mathbf{b}_i)_i]$, with $B = \begin{pmatrix} y_1 & y_2 & \ldots & y_d \\ 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix}$

A $\mathbb{Z}$-relation $\sum x_i y_i = 0$ leads to a small vector $(0, x_1, \ldots, x_d)^T$

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Factoring integer polynomials

The previous idea may be used to factor polynomials in $\mathbb{Z}[x]$

Factoring polynomials with rational coefficients,
A. K. Lenstra, H. W. Lenstra Jr. and L. Lovász. Math. Ann., 1982

$\Rightarrow$ Cited $\geq$ 2500 times!!!

Given $P \in \mathbb{Z}[x]$:

0- If $\deg P \leq 1$, then stop

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Cryptographic design and cryptanalysis

Lattice-based cryptography:

- Secret key: very short basis of a lattice
- Public key: long basis of the same lattice
- Relies on the assumed hardness of SVP

Very popular research topic:

- More secure: post-quantum
- More efficient: no modular exponentiation
- More versatile: fully homomorphic encryption

Lattice reduction algorithms are the best known attack.

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$(\mathbf{b}_i)_i$ linearly independent

The GSO $(\mathbf{b}_i^*)_i$ is defined by:

$$\forall i: \quad \mathbf{b}_i^* = \mathbf{b}_i - \sum_{j<i} \mu_{ij} \mathbf{b}_j^*$$

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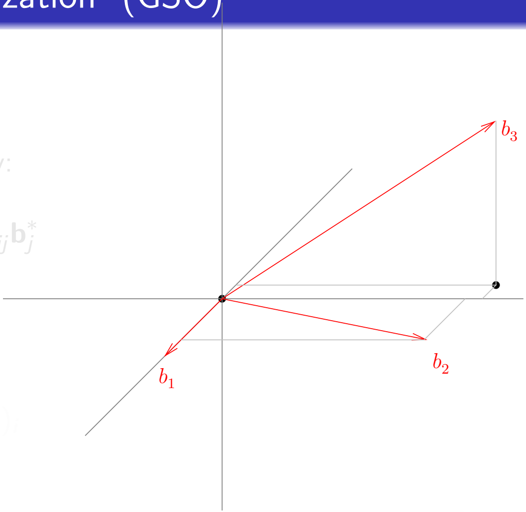
Triangularization of $B = (\mathbf{b}_i)_i$



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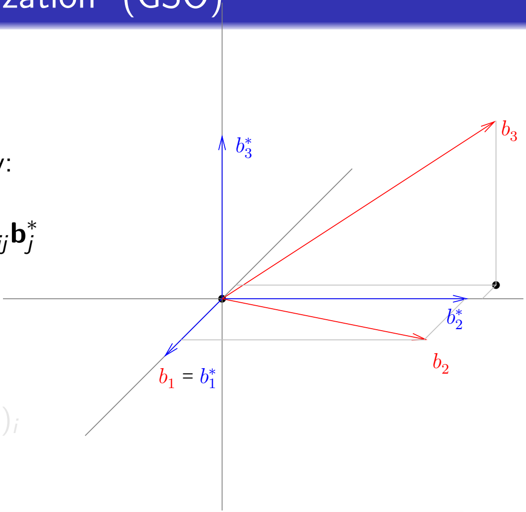

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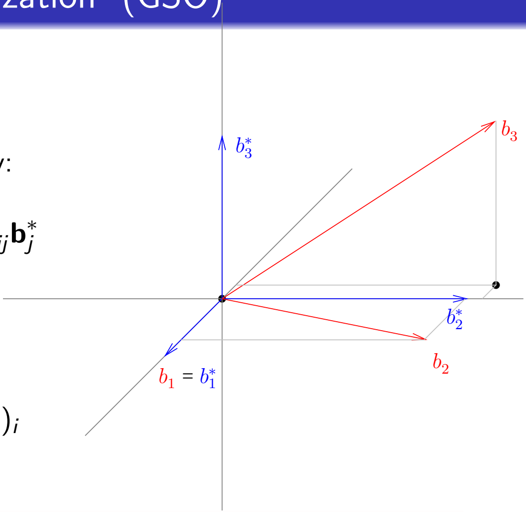

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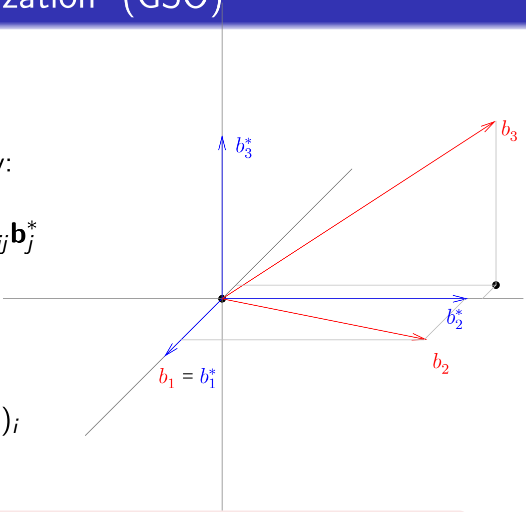

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The Lenstra-Lenstra-Lovász reduction

Let $\delta \in (1/4, 1)$. A basis $B = (\mathbf{b}_i)_{i \leq n} \in \mathbb{R}^{n \times n}$ is said LLL-reduced if

- $\forall i, j: \ |\mu_{ij}| \leq 1/2$ [Size-reduction]
- $\forall i: \ \delta \cdot \|\mathbf{b}_i^*\|^2 \leq \|\mathbf{b}_{i+1}^*\|^2 + \mu_{i+1,i}^2 \|\mathbf{b}_i^*\|^2$ [Lovász' condition]

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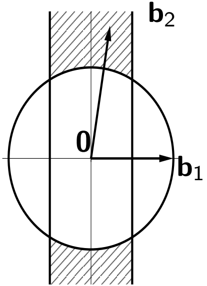

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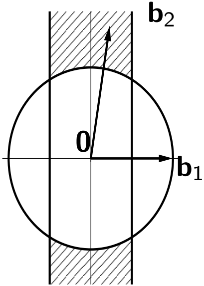

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The Lenstra-Lenstra-Lovász reduction

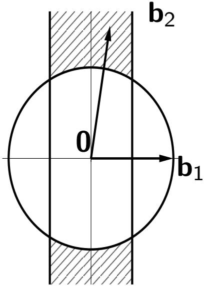
Let $\delta \in (1/4, 1)$. A basis $B = (\mathbf{b}_i)_{i \leq n} \in \mathbb{R}^{n \times n}$ is said LLL-reduced if

- $\forall i, j: \ |\mu_{ij}| \leq 1/2$ [Size-reduction]
- $\forall i: \ \delta \cdot \|\mathbf{b}_i^*\|^2 \leq \|\mathbf{b}_{i+1}^*\|^2 + \mu_{i+1,i}^2 \|\mathbf{b}_i^*\|^2$ [Lovász' condition]

The $\|\mathbf{b}_i^*\|$'s can't drop too fast:

$$\forall i: \ \|\mathbf{b}_{i+1}^*\|^2 \geq (\delta - \tfrac{1}{4}) \|\mathbf{b}_i^*\|^2$$

$$\Rightarrow \lambda(L) \leq \|\mathbf{b}_1\| \leq 2^{\mathcal{O}(n)} \cdot \lambda(L)$$



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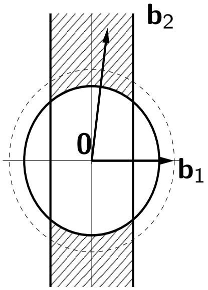

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1. Enforce size-reduction, using a modified Gaussian elimination
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3. Return the current basis $(\mathbf{b}_1, \ldots, \mathbf{b}_n)$

⇒ Correctness is trivial

⇒ Termination is much less so:

$$\mathcal{O}(n^2 \beta) \text{ loop iterations, with } \beta = \max_i \|\mathbf{b}_i^{init}\|$$

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Bit-complexity of LLL and practical run-time

[LLL82,Kaltofen83]

LLL terminates in $\mathcal{O}(n^4\beta^2(n+\beta))$ operations, with $\beta = \log \max_i \|\mathbf{b}_i^{init}\|$

With MAGMA V2.16:

```
> n := 25; B := RMatrixSpace(Integers(),n,n)!0;
> for i:=1 to 25 do
>    B[i][i]:=1; B[i][1]:=RandomBits(2000);
> end for;
> time C := LLL(B:Method:=''Integral'');
```

Time: 11.700

```
> time C := LLL(B);
```

Time: 0.240

Plan of the talk

Plan of the talk:

1. Euclidean lattices
2. Applications of euclidean lattices
3. The LLL algorithm
4. **Speeding up LLL**

Section based on joint works with X.-W. Chang, I. Morel, P. Q. Nguyen, A. Novocin, X. Pujol and G. Villard

LLL in practice: the numeric-symbolic approach

The Gram-Schmidt computations dominate the cost

Odlyzko's hybrid approach

Replace the rational computations on the GSO by floating-point approximations, but keep the basis operations exact

Floating-point numbers: $x_1.x_2x_3\ldots x_p \cdot B^e$, where:

- p is the precision
- B is the base, and $x_i \in \{0, \ldots, B-1\}$
- $e \in \mathbb{Z}$ is the exponent

Floating-point arithmetic:
$fl(a\ op\ b)$ is a nearest fp number to $a\ op\ b$, for any $op \in \{+, -, /, \times\}$

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Principle: For p small, fp arith. may efficiently simulate rational arith.
$\Rightarrow$ In practice: we aim for 53-bit machine precision

But Odlyzko's approach is heuristic:

- Fp arithmetic is inexact
- Small errors can be amplified

$\Rightarrow$ Infinite loops

$\Rightarrow$ Incorrect outputs

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Making the numeric-symbolic approach rigorous

Underlying mathematical phenomenon [CSV12]

Any LLL-reduced basis is well-conditioned with respect to GSO

- Well-conditioned? The GSO computed in small precision is close to the genuine GSO

$\Rightarrow$ We'd like to rely on LLL-reduced bases as much as we can

Use a greedy LLL algorithm [NS05,MSV09]:

- Consider the first i s.t. $\mathbf{b}_1, \ldots, \mathbf{b}_i$ is not LLL-reduced

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Small precision? $\mathcal{O}(n)$ bits suffice for correctness.

Bit-complexity:

$$\underbrace{\mathcal{O}(n^2\beta)}_{1} \cdot \underbrace{\mathcal{O}(n^2)}_{2} \cdot \left[\underbrace{\mathcal{O}(n\beta)}_{3} + \underbrace{\mathcal{O}(n^2)}_{4} \right] = \mathcal{O}\left(n^5\beta(n+\beta)\right).$$

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Can we do better? [NSV10,PSV13?]

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LLL can be accelerated further by using approximations for the bases too!
$\Rightarrow \widetilde{\mathcal{O}}(n^5\beta^{1.5})$ operations

The totally numeric approach, continued

Do the same with several levels of recursion
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5. **Conclusion**

Open problems

On LLL:

- Lower the cost further: as fast as matrix multiplication?
- Improve current implementations

In the general area of lattices

- Faster algorithms computing shorter vectors than LLL
- Quantum algorithms
- Hardness proofs for worst-case lattice problems
- Hardness proofs for average-case lattice problems (crucial for lattice-based cryptography)

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