More Efficient Cryptographic Multilinear Maps from Ideal Lattices

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Monash Discrete Math Group, March 2014

Outline of the talk

1- Introduction
- Background: Cryptographic Multilinear Maps and Applications
- Background: Ideal Lattices

2- Review of GGH construction of approx. multilinear maps

3- GGHLite: Our more efficient construction
- Main ingredients
- Construction
- Asymptotic efficiency
- Using GGHLite in applications

4- Concluding Remarks

Background: Cryptographic Multilinear Maps

Non-interactive Key Exchange (NIKE):

- Alice and Bob want to communicate privately over public channel
- Marvin can see everything sent over the public channel
- Non-interactive setup

Solution: Diffie-Hellman Key Exchange (1976)

- Publish a cyclic group G (generator g, order q) where Discrete Log (DL) problem is hard.
- Alice chooses random $x_1 \in \mathbb{Z}_q$, publishes $y_1 = g^{x_1}$.
- Bob chooses random $x_2 \in \mathbb{Z}_q$, publishes $y_2 = g^{x_2}$.
- Correctness: Both Alice and Bob compute agreed secret key $K = g^{x_1 x_2} = y_1^{x_2} = y_2^{x_1}$.
- Security: Eavesdropper Marvin has to solve the **Computational Diffie-Hellman** problem (CDH),

  CDH: Given g, g^{x_1}, g^{x_2}, compute $g^{x_1 x_2}$.

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21st Century variant (privacy for Facebook): Group of $N > 2$ parties want to communicate privately via 'cloud'.

Solution[J00,BS02]: Use a group where DL is hard and there is an efficient $(N-1)$-linear map $e : G^{N-1} \rightarrow G_T$:

$$e(g^{x_1}, g^{x_2}, \ldots, g^{x_{N-1}}) = e(g, \ldots, g)^{x_1 \cdots x_{N-1}} \forall x_1, \ldots, x_{N-1} \in \mathbb{Z}_q.$$

N-party Non-Interactive Key Exchange

- Publish cyclic groups G, G_T (generators g, g_T, order q) where Discrete Log (DL) problem is hard, with an efficient $(N-1)$-linear map e.
- For $i = 1, \ldots, N$, party P_i chooses $x_i \in \mathbb{Z}_q$, publishes $y_i = g^{x_i}$.
- Correctness: All parties can compute agreed secret key $K = e(g, \ldots, g)^{x_1 \cdots x_N} = e(y_2, y_3, \ldots, y_N)^{x_1}$.
- Security: Hardness of **Multilinear CDH** problem (MCDH),

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Background: Cryptographic Multilinear Maps – History

- 2000: Bilinear ($k = 2$) via Weil pairings on algebraic curves, applications:
  - 2000: 3-party non-interactive key agreement [J00]
  - 2000-2001: Identity-Based Encryption (IBE) [SK00,BF01]
  - 2001: Short signatures [BS01]
  - 2000-2013: **lots** of others
- 2002: Applications for k-linear maps [BS02]
  - $(k + 1)$-party non-interactive key agreement
  - Efficient Broadcast Encryption
  - and others...
- 2012: First plausible realization for $k > 2$, via ideal lattices [GGH12], applications:
  - 2012-2013: Functional Encryption for arbitrary functions
  - 2013: Program obfuscation notions for arbitrary functions
- 2014: GGHLite – More efficient variant of GGH construction (this talk)

Approx. Multilin. Maps: GGH 'Graded Encoding Scheme'

GGH realization: not quite a k-linear map, but essentially the same

Technically, a k-graded encoding scheme:

- Replace groups $\mathbb{Z}_q, G$ by
  - Rings R_g, R_q and some public parameters par.
- Replace 'Encode $x \in \mathbb{Z}_q$ as $g^x \in G$' by
  - 'Encode $x \in R_g$ as $\mathsf{Enc}_1(\mathrm{par}, x; \rho) \in R_q$' – randomized 'level 1 encoding' of 'level 0' element x using randomness ρ.
- Replace $e(g_1^{x_1}, \ldots, g_k^{x_k}) = e(g_1, \ldots, g_k)^{x_1 \cdots x_k}$ by
  - Homomorphic up to 'level k':
    $$\mathsf{Enc}_1(\mathrm{par}, x_1; \rho_1) \cdots \mathsf{Enc}_1(\mathrm{par}, x_k; \rho_k) = \mathsf{Enc}_k(\mathrm{par}, x_1 \cdots x_k; \rho)$$
    and
    $$x \cdot \mathsf{Enc}_k(\mathrm{par}, z; \rho) = \mathsf{Enc}_k(\mathrm{par}, x \cdot z; \rho'), \text{ for any } x \in R_g.$$
  - Randomness-independent extraction at level k – $\mathsf{Ext}(\mathrm{par}, \mathsf{Enc}_k(\mathrm{par}, x; \rho)) = r(x) \in \{0,1\}^n$ is independent of randomness ρ, and uniformly random for $x \hookleftarrow U(R_g)$.

Multilinear Maps: GGH 'Graded Encoding Scheme'

N-party NIKE from $N-1$-Graded Encoding Scheme:

- Publish rings R_g, R_q and pub. params. par of $N-1$-Graded Encoding Scheme.
- For $i = 1, \ldots, N$, party P_i chooses $x_i \in R_g$, publishes $y_i = \mathsf{Enc}_1(\mathrm{par}, x_i; \rho_i)$.
- Correctness: All parties can compute agreed secret key

  $$K = \mathsf{Ext}(\mathrm{par}, \mathsf{Enc}_{N-1}(\mathrm{par}, x_1 \cdots x_N; \rho)) = \mathsf{Ext}(\mathrm{par}, x_1 \cdot y_2 \cdot y_3 \cdots y_N)$$

- Security: To compute K, eavesdropper Marvin has to solve the **Extraction Graded Computational Diffie-Hellman** problem – **Ext-GCDH**: Given $\mathrm{par}, y_1 = \mathsf{Enc}_1(\mathrm{par}, x_1; \rho_1), \ldots, y_N = \mathsf{Enc}_1(\mathrm{par}, x_N; \rho_N)$, compute $\mathsf{Ext}(\mathrm{par}, \mathsf{Enc}_{N-1}(\mathrm{par}, x_1 \cdots x_N; \rho))$.

Polynomial Rings

Take $\phi \in \mathbb{Z}[x]$ monic of degree n.

$$R^{\phi} \ := \ \Big[\mathbb{Z}[x]/(\phi), +, \times\Big].$$

Interesting ϕ's:

- $\phi = x^n - 1 \ \rightarrow \ R^{-}, \quad \phi = x^n + 1 \ \rightarrow \ R^{+}.$
- For n a power of 2, the ring R^{+} is isomorphic to the ring of integers of $K = \mathbb{Q}[e^{i\pi/n}]$:

$$K \simeq \mathbb{Q}[x]/(x^n + 1)$$
$$\mathcal{O}_K \simeq \mathbb{Z}[x]/(x^n + 1).$$

$\Rightarrow$ Rich algebraic structure (great for design and proofs).

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- Arithmetic in R_q^{ϕ} costs $\widetilde{O}(n \log q)$.
- R_q^{+} is isomorphic to $\mathcal{O}_K/(q)$.

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Lattices Background: Approx-SVP

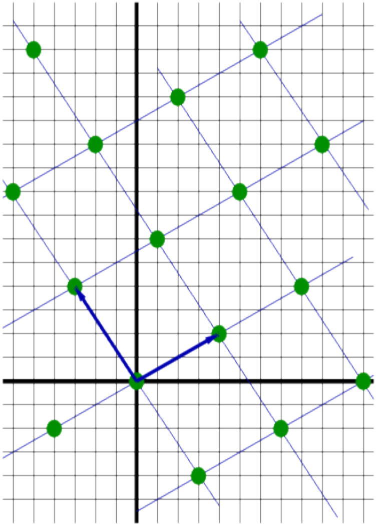
Lattice $\equiv \{\sum_{i \leq n} x_i \mathbf{b}_i : x_i \in \mathbb{Z}\}$,
for some lin. independent $\mathbf{b}_i$'s.

Minimum: $\lambda(L) = \min(\|\mathbf{b}\| : \mathbf{b} \in L \setminus \mathbf{0})$

γ-SVP

Find $\mathbf{b} \in L$ with: $0 < \|\mathbf{b}\| \leq \gamma \cdot \lambda(L)$.

- No known sub-exp. algorithm for $\gamma = Poly(n)$.
- Not even quantumly.
- Seems harder than Int-Fac and DLog.



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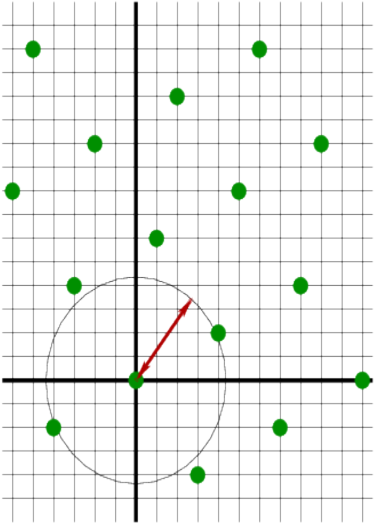

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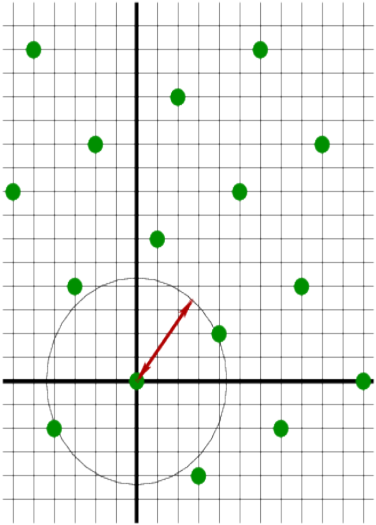

Lattices Background: Approx-Ideal-SVP

- $I \subseteq R^{\phi}$ is an ideal if:

$$\forall a, b \in I, \forall r \in R^{\phi} : \quad a + b \cdot r \in I.$$

- We identify polynomials to vectors via their coefficients:

$$\begin{array}{ccc} R^{\phi} & \rightarrow & \mathbb{Z}^n \\ \sum_{i<n} f_i x^i & \mapsto & (f_0, \ldots, f_{n-1})^t \end{array}$$

- An ideal I can be viewed as a lattice, called an ideal lattice.

$\mathcal{P}oly(n)$-Ideal-SVP: $\mathcal{P}oly(n)$-SVP restricted to ideal lattices.

No significant computational advantage known for this general family of inputs.

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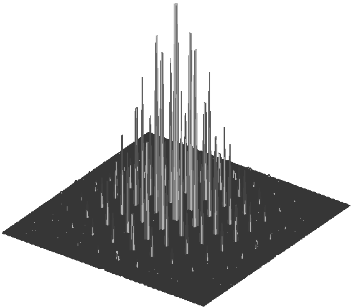
Lattices Background: Discrete Gaussian Distributions

$D_{L,S,c}$ denotes discrete Gaussian distrib. on n-dim. lattice L, full-rank deviation matrix $S \in \mathbb{R}^{n \times n}$, centre c (sample using [GePeVa'08]):

$$\forall x \in L: \quad D_{L,S,c}[x] \sim \exp\left(-\pi(x-c)^T(S^T S)^{-1}(x-c)\right).$$



Approx. Multilin. Maps: GGH k-graded encoded scheme

Public Parameters Generation:

- Sample 'small' $g \hookleftarrow D_{R,\sigma}$ until $\|g^{-1}\| \leq \ell_{g^{-1}}$ and $I = \langle g \rangle$ is a prime ideal. Define encoding domain $R_g = R/\langle g \rangle$.
- Sample $z \hookleftarrow U(R_q)$.
- Sample a level-1 encoding of 1: set $y = [a \cdot z^{-1}]_q$ with $a \hookleftarrow D_{1+I,\sigma'}$.
- Sample m_r level-1 encodings of 0: set $x_j = [b_j \cdot z^{-1}]_q$ with $b_j \hookleftarrow D_{I,\sigma'}$ for all $j \leq m_r$.
- Sample $h \hookleftarrow D_{R,\sqrt{q}}$ and define the zero-testing parameter $p_{zt} = [\frac{h}{g} z^k]_q \in R_q$.
- Return $\mathrm{par} = (n, q, y, \{x_j\}_{j \leq m_r})$ and p_{zt}.

Approx. Multilin. Maps: GGH k-graded encoded scheme

Level-1 encoding $\mathrm{Enc}_1(\mathrm{par}, e)$: Given level-0 $e \in R$:

- Encode e at level 1: $u' = [e \cdot y]_q$ (note $u' = [c'/z]_q$ with $c' \in e + I$).
- Re-randomize: Sample small $\rho_j \hookleftarrow D_{\mathbb{Z},\sigma_1^*}$ for $j \leq m_r$ and return $u = [u' + \sum_{j=1}^{m_r} \rho_j x_j]_q$.
  (Note $u = [c/z]_q$ with $c \in e + I$ and $c = c' + \sum_j \rho_j b_j$.)

Multiplying encodings mult: Given level-k_1 encoding $u_1 = [c_1/z^{k_1}]_q$ and level-k_2 encoding $u_2 = [c_2/z^{k_2}]_q$:

- Return $u = [u_1 \cdot u_2]_q$, a level-$(k_1 + k_2)$ encoding of $[c_1 \cdot c_2]_g$.
  (note $u_1 \cdot u_2 = [c_1 c_2/z^{k_1+k_2}]_q$ and $c_1 \cdot c_2 \in e_1 \cdot e_2 + I$).

Approx. Multilin. Maps: GGH k-graded encoded scheme

Extraction at level k $\mathrm{Ext}(\mathrm{par}, u)$: Given a level-$k$ encoding $u = [c/z^k]_q$, return $v = \mathsf{MSB}_\ell([p_{zt} \cdot u]_q)$ with $\ell < (1/4 - \varepsilon) \log q$.

Correctness of extraction:

- At level 1: if $c = [c]_g + gr$ for some **small** $r \in R$, then $v = \mathsf{MSB}_\ell(\frac{h}{g}([c]_g + gr)) = \mathsf{MSB}_\ell(\frac{h}{g}[c]_g + hr)$, which is equal to $\mathsf{MSB}_\ell(\frac{h}{g}[c]_g)$, with high probability if $q > \|r\|^8$.
- After k multiplications:
  - Let $u_i = [\frac{x_i + g \cdot r_i}{z}]_q$ for $i = 1, \ldots, k$ be encodings of $x_1, \ldots, x_k$.
  - For $u \stackrel{\text{def}}{=} u_1 \cdot u_2 \cdots u_k = [\frac{x + g \cdot r}{z^k}]_q$ to be a valid encoding of $x = x_1 \cdots x_k$, need $\|r\|$ to stay **small** compared to q:

    $$\|r\| = O(2^k \cdot \|(g \cdot r_1) \cdots (g \cdot r_k)\|) = O((\mathcal{P}oly(n) \cdot N)^k) < q^{1/8}.$$

    where $N \stackrel{\text{def}}{=} \max_i \|g \cdot r_i\|$.

Approx. Multilin. Maps: GGH k-graded encoded scheme

Security of GDH for GGH scheme: not well understood.
Known attack needs 'small' multiple d of g ($\|d \cdot g\| < q$).

- **Fact:** Easy [GGH12] to compute basis for $\langle g \rangle$ from par .
- **Conclusion:** Security relies on hardness of q-ideal-SVP.

Attack on 'Graded Discrete Log' prob. given
$u = \mathsf{Enc}_1(\mathrm{par}, x; r) = [\frac{x+r\cdot g}{z}]_q$ (idea):

- Compute $p'_{zt} \stackrel{\mathrm{def}}{=} [d \cdot g \cdot p_{zt}]_q = [(d \cdot g) \cdot (\frac{h}{g} z^k)]_q = [d \cdot h \cdot z^k]_q$.
- Lift: $u' = [u \cdot y^{k-1}]_q = [\frac{x+r' \cdot g}{z^k}]_q$, $y' = [y^k]_q = [\frac{1+r'_y \cdot g}{z^k}]_q$.
- Compute $u'' = [u' \cdot p'_{zt}]_q = d \cdot h \cdot (x + r' \cdot g) \in R$ and $y'' = [y' \cdot p'_{zt}]_q = d \cdot h \cdot (1 + r'_y \cdot g) \in R$.
- Using basis for $\langle g \rangle$, easy to compute a ('large') rep. $x' \in R$ with $x' \equiv u'' \cdot (y'')^{-1} \bmod \langle g \rangle$, so $x' \equiv x \bmod \langle g \rangle$.
- Compute a 'small' rep. $x'' = x' \bmod \langle d \cdot g \rangle$ with $x'' \equiv x \bmod \langle g \rangle$.

GGHLite: Main Ingredients

We improve encoding re-randomization in GGH:

- Pub. Pars. contain level-1 encodings of 0, namely $\{x_j = [b_j/z]_q\}_{j \leq m_r}$ and level-1 encoding of 1, namely y.
- To randomize level-1 encoding $u' = [e \cdot y]_q$, output $u = [u' + \sum_j \rho_j x_j]_q = [c/z]_q$ with $c = c' + \sum_j \rho_j b_j$.
- Randomizers ρ_j's are sampled from a discrete Gaussian distribution over $\mathbb{Z}$ with deviation parameter σ^*.

Re-randomization is essential for security of GDH:

- Without re-randomization, e can be be efficiently recovered from $u' = [e \cdot y]_q$ and y ($u = [u' y^{-1}]_q$).
- Re-randomization can prevent this attack.

GGHLite: First Main Ingredient

But, how to choose the re-randomization parameters for security level 2^λ?

Question: How large should re-randomization deviation σ^* be?

- in GGH, exponential drowning: $\sigma^*/\|c'\| \geq 2^\lambda$
- Makes distribution of u (almost) independent of u'
- But incurs severe efficiency penalty.
  - Need $q \geq 2^\lambda$.
  - Security of q-ideal-SVP deteriorates exponentially with $\log q$.
  - Need quadratic dimension: $n \geq \lambda^2$!

GGHLite First Ingredient: We show that polynomial drowning is sufficient for security: $\sigma^*/\|c'\| \geq \mathcal{P}oly(\lambda)$

But, our analysis only seems to apply to computational GDH problem.

- We use Rényi Divergence in place of Statistical Distance in analysing re-randomized distribution vs. 'canonical' one

GGHLite: Second Main Ingredient

Question: How many encodings of 0 are needed?

GGH construction:

- Needs $m_r = \Omega(n \log n)$ encodings of 0
- Uses rational integer Gaussian randomizers ($\rho_j \in \mathbb{Z}$) as coefficients
- Uses a 'discrete Gaussian Leftover Hash Lemma' to show $\sum_{j \le m_r} \rho_j b_j$ distrib. is close to a discrete Gaussian on I

GGHLite Second Ingredient: $m_r = 2$ encodings of 0 are sufficient

- Uses Gaussian randomizers over full ring ($\rho_j \in R$)
- New algebraic variant of 'discrete Gaussian Leftover Hash Lemma' over R: we show $\sum_{j \le m_r} \rho_j b_j$ distribution is close to a discrete Gaussian on I

GGHLite: Our simplified k-graded encoded scheme

Public Parameters Generation:

- Sample $g \hookleftarrow D_{R,\sigma}$ until $\|g^{-1}\| \leq \ell_{g^{-1}}$ and $I = \langle g \rangle$ is prime.
- Sample $z \hookleftarrow U(R_q)$.
- Sample a level-1 encoding of 1: $y = [a \cdot z^{-1}]_q$ with $a \hookleftarrow D_{1+I,\sigma'}$.
- Sample $B = (b_1, b_2)$ from $(D_{I,\sigma'})^2$. If $\langle b_1, b_2 \rangle \neq I$, or $\sigma_n(\mathrm{rot} B) < \ell_b$, then re-sample.
- Define level-1 encodings of 0: $x_1 = [b_1 \cdot z^{-1}]_q$, $x_2 = [b_2 \cdot z^{-1}]_q$.
- Sample $h \hookleftarrow D_{R,\sqrt{q}}$ and define the zero-testing parameter $p_{zt} = [\frac{h}{g} z^k]_q \in R_q$.
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Level-1 encoding $\mathrm{Enc}_1(\mathrm{par}, e)$: Given level-0 $e \in R$:

- Encode e at level 1: Compute $u' = [e \cdot y]_q$.
- Return $u = [(u' + \rho_1 \cdot x_1 + \rho_2 \cdot x_2)/z]_q$, with $\rho_1, \rho_2 \hookleftarrow D_{R,\sigma_1^*}$.

GGHLite: Formalizing Re-randomization Security

How to formalize re-randomization security requirement?

Informal req.: Prevent correlation of statistical properties of re-randomized encoding with encoded element.

Formal req.: Breaking Ext-GCDH problem is as hard as breaking canonical Ext-GCDH problem

- **Ext-GCDH**: Given $\mathrm{par}, y_1 = [e_1 \cdot y + \rho_{1,1} \cdot x_1 + \rho_{2,1} \cdot x_2]_q, \ldots, y_N = [e_N \cdot y + \rho_{1,N} \cdot x_1 + \rho_{2,N} \cdot x_2]_q$, compute $\mathsf{Ext}(\mathrm{par}, \mathsf{Enc}_{N-1}(\mathrm{par}, x_1 \cdots x_N; \rho)) = MSB_\ell(p_{zt} \cdot e_1 \cdots e_N)$.
- canonical **Ext-GCDH**: Given $\mathrm{par}, y_1 = [c_1 z^{-1}]_q, \ldots, y_N = [c_N z^{-1}]_q$ with $c_i \hookleftarrow D_{I + e_i, \sigma_1^* B^T}$ for $i = 1, \ldots, N$, compute $\mathsf{Ext}(\mathrm{par}, \mathsf{Enc}_{N-1}(\mathrm{par}, x_1 \cdots x_N; \rho)) = MSB_\ell(p_{zt} \cdot e_1 \cdots e_N)$.

Theorem. This requirement is satisfied, i.e. such a reduction exists for GGHLite, under suitable parameter conditions.

GGHLite Re-randomization Security: First Ingredient

D_1: distrib. of $y_i = [v_i/z]_q$ in **Ext-GCDH** problem

- v_i distrib. $\approx D_{I+e_i,\sigma_1^* B^T, c_i'}$ – 'small' centre c_i'.

D_2: distrib. of $y_i = [v_i/z]_q$ in canonical **Ext-GCDH** problem

- v_i distrib. $\approx D_{I+e_i,\sigma_1^* B^T}$ – zero centre.

GGH strong requirement based on statistical distance (SD) Δ:

$$\Delta(D_1, D_2) \stackrel{\text{def}}{=} \sum_x |D_1(x) - D_2(x)| \le 2^{-\lambda},$$

Prob. Preservation Property of SD: Any adversary A with succ. prob. ε against **Ext-GCDH** problem, has succ. prob. ε' against canonical **Ext-GCDH** problem with:

$$\varepsilon' \ge \varepsilon - \Delta(D_1, D_2) \ge \varepsilon - 2^{-\lambda},$$

- To handle $\varepsilon = 2^{-\lambda}$, need $\Delta(D_1, D_2) < 2^{-\lambda}$!
- Consequently, need $\frac{\sigma_1^*}{\|c_i'\|} = 2^{\Omega(\lambda)}$ (exponential drowning).

GGHLite Re-randomization Security: First Ingredient

D_1: distrib. of $y_i = [v_i/z]_q$ in **Ext-GCDH** problem

- v_i distrib. $\approx D_{I+e_i,\sigma_1^* B^T,c_i'}$ – 'small' centre c_i'.

D_2: distrib. of $y_i = [v_i/z]_q$ in canonical **Ext-GCDH** problem

- v_i distrib. $\approx D_{I+e_i,\sigma_1^* B^T}$ – zero centre.

GGHLite weak requirement based on Rényi divergence (RD) R:

$$R(D_1 \| D_2) \stackrel{\text{def}}{=} \sum_x D_1^2(x)/D_2(x) \le \mathcal{P}oly(\lambda),$$

Prob. Preservation Property of RD: Any adversary A with succ. prob. ε against **Ext-GCDH** problem, has succ. prob. ε' against canonical **Ext-GCDH** problem with:

$$\varepsilon' \ge \varepsilon / R(D_1 \| D_2)^2 \ge \varepsilon / \mathcal{P}oly(\lambda),$$

- Useful even if $\varepsilon < R(D_1, D_2)^{-1}$ – use $R(D_1 \| D_2) \le \mathcal{P}oly(\lambda)$.
- We show: $R(D_1 \| D_2) \le \exp\left(2\pi \|c_i'\|^2 / \sigma_n(\sigma_1^* B^T)^2\right)$.
- For $R(D_1 \| D_2) \le \mathcal{P}oly(\lambda)$, can use $\frac{\sigma_1^*}{\|c_i'\|} = O(\frac{1}{\log \lambda})$.

GGHLite Re-randomization Security: Second Ingredient

D_1: distrib. of $y_i = [v_i/z]_q$ in **Ext-GCDH** problem

- v_i distrib. $\approx D_{I+e_i,\sigma_1^* B^T, c_i'}$ – 'small' centre c_i'.

In actual scheme $(e_i \cdot a + \rho_1 \cdot b_1 + \rho_2 \cdot b_2)/z]_q$ with $\rho_i \sim D_{R,\sigma_1^*}$.
How do we show $\rho_1 \cdot b_1 + \rho_2 \cdot b_2 \approx D_{I,\sigma_1^* B^T}$ $(B = g \cdot [t_1, t_2] \in R^2)$?

- **Step 1:** Show $T \cdot R^2 = [t_1, t_2] \cdot R^2 = R$, except for some constant probability < 1.
  - Probability that two 'random' algebraic integers are co-prime ($\approx \zeta_R(2)^{-1}$).
- **Step 2:** Study the 'orthogonal' lattice $A_T = \{v \in R^2 : T \cdot v = 0\}$.
  - Use equality of Minkowski minima of A_T to bound 'smoothing parameter' $\eta_\varepsilon(A_T)$.
  - Apply known results [AGHS12] on 'smoothing of Gaussians modulo a lattice': If $\sigma_1^* > \eta_\varepsilon(A_T)$, then $\rho_1 \cdot t_1 + \rho_2 \cdot t_2$ is within SD 2ε of $D_{R,\sigma_1^* T^T}$.

GGHLite: Asymptotic Parameters

Parameter	GGHLite	GGH
m_r	2	$\Omega(n \log n)$
σ	$O(n \log n)$	$O(n \log n)$
$\ell_{g^{-1}}$	$O(1/\sqrt{n \log n})$	$O(1/\sqrt{n \log n})$
$\varepsilon_d, \varepsilon_e, \varepsilon_\rho$	$O(k^{-1})$	$O(2^{-\lambda} k^{-1})$
σ'	$\widetilde{O}(n^{2.5})$	$\widetilde{O}(n^{1.5}\sqrt{\lambda})$
σ_1^*	$\widetilde{O}(n^{4.5}\sqrt{\log k})$	$\widetilde{O}(2^\lambda n^{4.5}(\lambda + \log k))$
ε_{ext}	$O(\lambda^{-\omega(1)})$	$O(\lambda^{-\omega(1)})$
q	$\widetilde{O}((n^{8.5}\sqrt{\log k})^{8k})$	$\widetilde{O}((2^\lambda n^8 \lambda^{1.5})^{8k})$
n	$O(k\lambda \log \lambda)$	$O(k\lambda^2)$
$\lvert\text{enc}\rvert$	$O(k^2 \lambda \log^2(k\lambda))$	$O(k^2\lambda^3)$
$\lvert\text{par}\rvert$	$O(k^3 \lambda \log^2(k\lambda))$	$O(k^3\lambda^5 \log(k\lambda))$

Adapting Applications of GGH to GGHLite

Applications often need semantic security: no **partial** information on key leaks.

GGH security analysis applies to Graded Decision Diffie-Hellman problem (GDDH): Distinguish between the distributions

$$\mathcal{D}_{DDH} = \{\mathrm{par}, (u_i = \mathsf{Enc}_1(x_i))_{0 \leq i \leq k}, v = \mathsf{Enc}_1(x_0 \cdot x_1 \cdots x_k)\}$$

and

$$\mathcal{D}_R = \{\mathrm{par}, (u_i = \mathsf{Enc}_1(x_i))_{0 \leq i \leq k}, v = \mathsf{Enc}_1(f_0)\}$$ for indep. unif. dist. f_0.

GGHLite security analysis only applies to Extraction Graded Computational Diffie-Hellman problem (Ext-GCDH).

Adapting Applications of GGH to GGHLite

Question: How to adapt GGH app. to rely on Ext-GCDH rather than GDDH?

Answer: Replace agreed key $K = \mathsf{Ext}(\mathrm{par}, v)$ in original protocol by

$$K = H(\mathsf{Ext}(\mathrm{par}, v))$$

in modified protocol, where $H(\cdot)$ is a cryptographic hash function. If $H(\cdot)$ is modelled as a black-box random function ('Random Oracle Model'), then security of modified protocol relies on Ext-GCDH – our GGHLite analysis applies!

Conclusions

Presented GGHLite, a more efficient variant of GGH graded encoding scheme.

Open Problems:

- Can our Rényi divergence analysis be applied to the Decision Graded Diffie Hellman problem?
- Understand the complexity of our canonical Ext-GCDH problem – provable relation to well studied lattice problems?
- Alternative constructions for graded encoding scheme, with provable security from standard lattice problems?
- Understand relation beteen GGH/GGHLite and more recent 'Jigsaw puzzle' variants (obfuscation).
- Concrete computational / space efficiency of GGHLite based on best known attacks?