Computing autotopism groups of partial Latin rectangles: a pilot study

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Rebecca J. Stones (Nankai U.)

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1	·	2	·	·	·	3	·	·
2	·	·	4	1	5	6	·	7
·	1	5	3	·	4	·	·	·
·	2	·	5	·	3	·	4	·
4	3	·	·	5	·	1	·	2
·	·	·	·	2	·	·	1	3

Partial Latin rectangles

An $r \times s$ partial Latin rectangle is an $r \times s$ matrix containing symbols from $[n] \cup \{\cdot\}$ such that each row and each column contains at most one copy of any symbol in $[n]$.

$\cdot$	5	4	3	2
5	$\cdot$	$\cdot$	$\cdot$	1
4	$\cdot$	$\cdot$	1	$\cdot$

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Every partial Latin rectangle $L \in \mathrm{PLR}(r, s, n)$ is uniquely determined by its entry set:

$$\mathrm{Ent}(L) := \{\overbrace{(i, j, L[i, j])}^{\text{entry}} : i \in [r], j \in [l], \text{ and } L[i, j] \in [n]\}.$$

$$\mathrm{Ent}(\text{above}) = \left\{ \begin{matrix} (2,5,1), \\ (3,4,1), \end{matrix} \quad (1,5,2), \quad (1,4,3), \quad \begin{matrix} (1,3,4), \\ (3,1,4), \end{matrix} \quad \begin{matrix} (1,2,5), \\ (2,1,5) \end{matrix} \right\}$$

Isotopisms and autotopisms

$$\left\{\begin{array}{l} r \times s \text{ partial Latin rectangles} \\ \text{on symbol set } [n] \end{array}\right\}$$

The isotopism $\theta := (\alpha, \beta, \gamma) \in S_r \times S_s \times S_n$ acts on $\mathrm{PLR}(r, s, n)$.

1	2	3	4	·	·
3	4	2	·	·	·
·	·	·	5	·	·
·	·	·	6	5	·
·	·	·	·	6	5
·	·	·	·	·	6

swap first two rows $\alpha = (12)$

swap last two columns $\beta = (56)$

do nothing to symbols $\gamma = \mathrm{id}$

3	4	2	·	·	·
1	2	3	4	·	·
·	·	·	5	·	·
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·	·	·	·	5	6
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·	·	·	·	6	·

And, in some cases, we can apply an isotopism θ and end up back where we started $\implies$ θ is an autotopism.

The set of autotopisms form a group, named the autotopism group.

1	2
3	4

This member of $\mathrm{PLR}(2, 2, 4)$ has 4 autotopisms.

- $(\mathrm{id}, \mathrm{id}, \mathrm{id})$,
- $((12), \mathrm{id}, (13)(24))$,
- $(\mathrm{id}, (12), (12)(34))$,
- $((12), (12), (14)(23))$,

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Note: The row and column permutations determine the autotopism.

How to *efficiently* compute the autotopism group?

Input: partial Latin rectangle.
Output: its autotopism group.

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We experimentally compare 6 families of methods...

Backtracking methods...

- **Family 1: Alpha-beta backtracking.**

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At each level of the α search tree,
we designate

$$\text{row } i \quad \overset{\alpha}{\longmapsto} \quad \text{row } a$$

provided it doesn't clash.

$$\begin{array}{c|c|c|c|}
 & 3 & 1 & 4 \\
\hline
a & 1 & \cdot & \cdot \\
\hline
i & 2 & \cdot & \cdot \\
\hline
\end{array}$$

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Once α is determined...

At each level of the β search tree, we designate

$$\text{column } j \quad \overset{\beta}{\mapsto} \quad \text{column } b$$

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$$\begin{array}{ccc} & j & b \\ \hline 3 & 1 & 4 \\ 1 & \cdot & \cdot \\ 2 & \cdot & \cdot \end{array}$$

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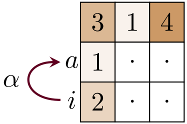

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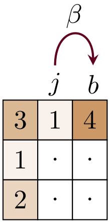

Then we check if $(\alpha, \beta, ??)$ is an autotopism.

Backtracking methods...

- **Family 2: Entrywise backtracking.**

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we designate

$$\text{entry } (i, j, L[i,j]) \quad \overset{\theta}{\mapsto} \quad \text{entry } (a, b, L[a,b])$$

provided it doesn't clash.

$$\alpha(i) = a$$
$$\beta(j) = b$$
$$\gamma(L[i,j]) = L[a,b]$$

$$\begin{array}{c|ccc|}
 & j & & b \\
\hline
a & 3 & 1 & 4 \\
i & 1 & \cdot & \cdot \\
 & 2 & \cdot & \cdot \\
\hline
\end{array}$$

Graph methods...

- **Family 3: McKay, Meynert, and Myrvold method.**

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Vertex set:

$$\begin{aligned} \mathrm{Ent}(L) &\cup \{R_i\colon\ i \in [r] \text{ and row } i \text{ of } L \text{ is non-empty}\} \\ &\cup \{S_j\colon\ j \in [s] \text{ and column } j \text{ of } L \text{ is non-empty}\} \\ &\cup \{N_k\colon\ k \in [n] \text{ and symbol } k \text{ occurs in } L\} \end{aligned}$$

where each of the four subsets, $\mathrm{Ent}(L)$, $\{R_i\}$, $\{S_j\}$, and $\{N_k\}$, are assigned a distinct color.

Edge set:

$$\begin{aligned} &\{R_i L[i,j]\colon\ (i,j,L[i,j]) \in \mathrm{Ent}(L)\} \\ \cup\, &\{S_j L[i,j]\colon\ (i,j,L[i,j]) \in \mathrm{Ent}(L)\} \\ \cup\, &\{N_{L[i,j]} L[i,j]\colon\ (i,j,L[i,j]) \in \mathrm{Ent}(L)\}. \end{aligned}$$

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The <u>automorphism group of this graph</u> is isomorphic to the <u>autotopism group of the partial Latin rectangle</u>.

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The automorphism group of this graph is isomorphic to the autotopism group of the partial Latin rectangle.
We compute this using **Nauty**.

Graph methods...

- **Family 4: Bipartite graph method.**

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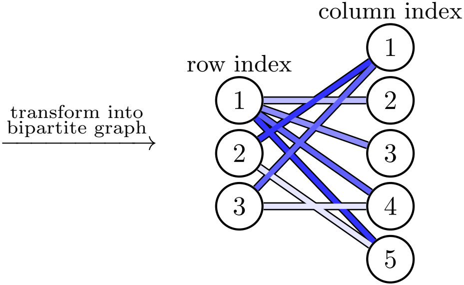
·	5	4	3	2
5	·	·	·	1
4	·	·	1	·



(Edges are colored to illustrate construction.)

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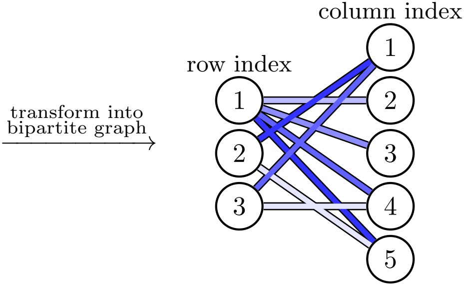

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Then compute the automorphism group of the bipartite graph using **Nauty**.

Graph methods...

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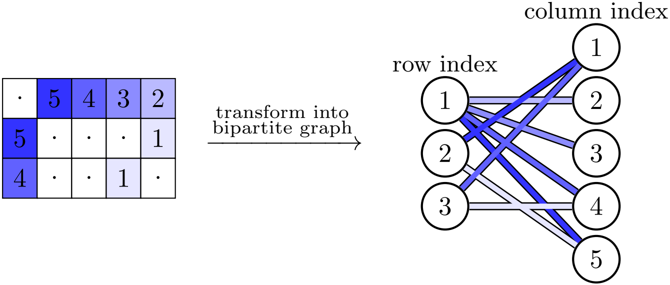

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Then compute the automorphism group of the bipartite graph using **Nauty**. Filter out non-autotopisms.

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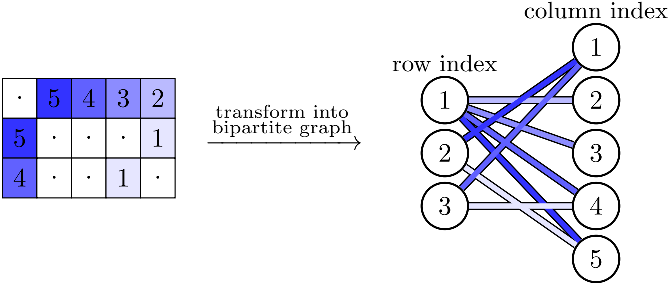

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Then compute the automorphism group of the bipartite graph using **Nauty**. Filter out non-autotopisms.

Nauty can also return (a) the row/column orbits or (b) entry orbits, under the autotopism group. We can alternatively use alpha-beta or entrywise backtracking on these orbits.

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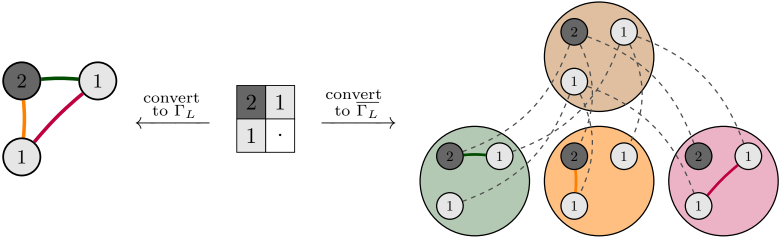
Partial Latin rectangle graph Γ_L (left):



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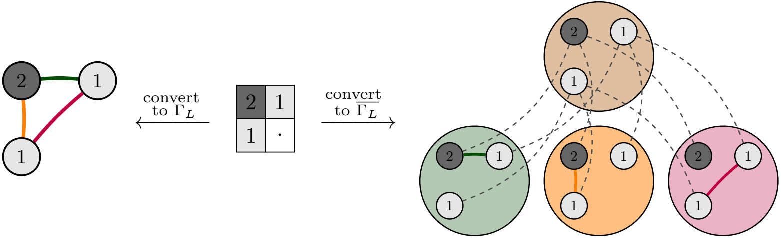

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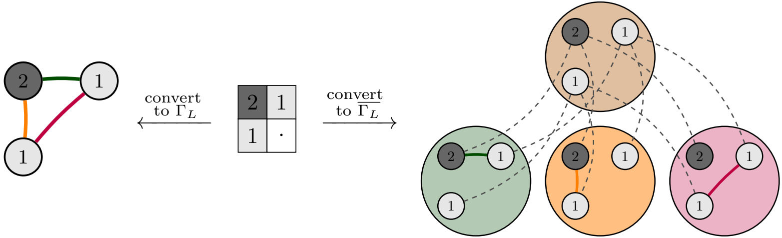

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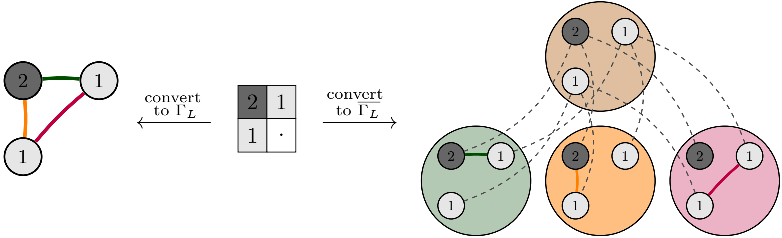

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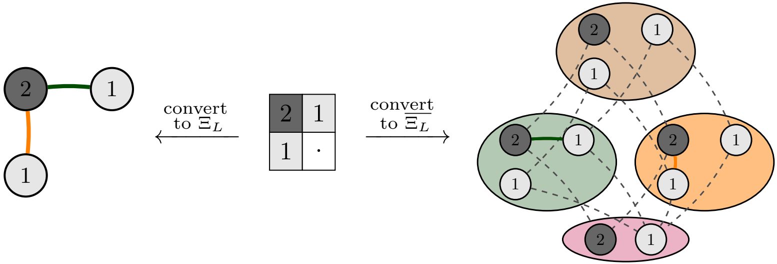

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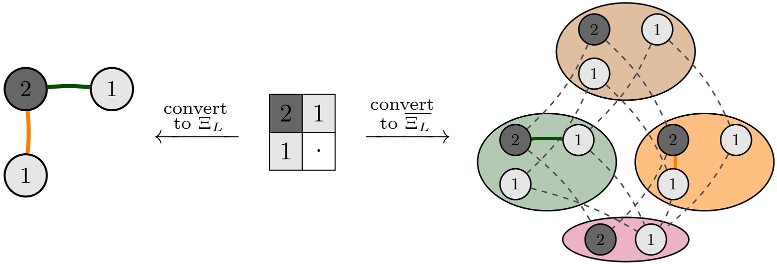

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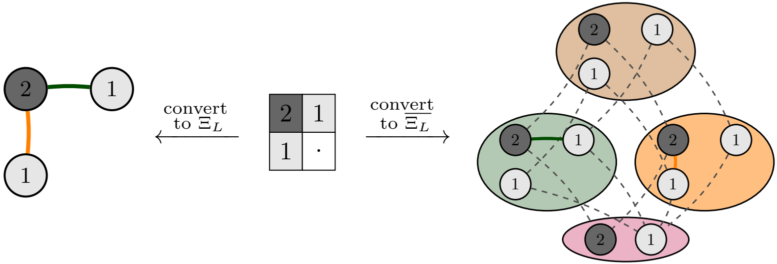

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Summary thus far...

We have six families of methods:

- *Backtracking*: alpha-beta and entrywise.
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We consider two invariants. (More sophisticated invariants may improve run-times, but will improve run-times for every method.)

Strong entry invariants...

For entry (i, j, k) we define the *strong entry invariant* as the vector (a, b, c) where

- a is the number of entries in row i,
- b is the number of entries in column j,
- c is the number of copies of symbol k in the partial Latin rectangle.

1	·	2	·	·	·	3	·	·
2	·	·	4	1	5	6	·	7
·	1	5	3	·	4	·	·	·
·	2	·	5	·	3	·	4	·
4	3	·	·	5	·	1	·	2
·	·	·	·	2	·	·	1	3

$\xrightarrow[\text{relabeled } 1, 2, \ldots]{\text{strong entry invariants}}$

1	·	2	·	·	·	1	·	·
3	·	·	4	3	4	5	·	5
·	6	7	6	·	8	·	·	·
·	6	·	8	·	6	·	7	·
9	10	·	·	9	·	10	·	10
·	·	·	·	1	·	·	2	1

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1	·	2	·	·	·	1	·	·
3	·	·	4	3	4	5	·	5
·	6	7	6	·	8	·	·	·
·	6	·	8	·	6	·	7	·
9	10	·	·	9	·	10	·	10
·	·	·	·	1	·	·	2	1

...useless for Latin rectangles (i.e., no empty cells and number columns = number symbols).

Square invariants...

The entry (i, j, k) belongs to exactly $(r-1)(s-1)$ 2×2 sub-matrices, a typical one looking like:

$$\begin{array}{c|cc} & j & j' \\ \hline i & k & x \\ i' & y & z \end{array}$$

which may have some of the following five properties: (a) x is undefined, (b) y is undefined, (c) z is undefined, (d) $k = z$, and (e) $x = y$.

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2	1	3	4	5
1	4	2	5	3
4	3	5	1	2
5	2	1	3	4
3	5	4	2	1

$\xrightarrow{\text{square invariants relabeled } 1, 2, \ldots}$

3	2	3	3	2
2	3	3	3	2
3	3	3	2	2
3	3	2	3	2
2	2	2	2	1

We call this length-32 vector the square invariant.

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...when we decide that $\beta(j) = b$, the filled/unfilled cells in column j map to filled/unfilled cells in column b.

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In the alpha-beta backtracking method, once α is determined, then...

...when we decide that $\beta(j) = b$, the filled/unfilled cells in column j map to filled/unfilled cells in column b. This can also be used to improve the computation.

Experiments...

So we have six families of methods, and two invariants, etc.

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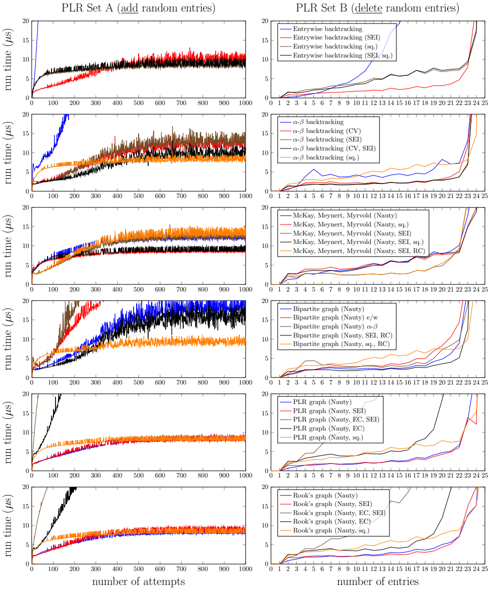
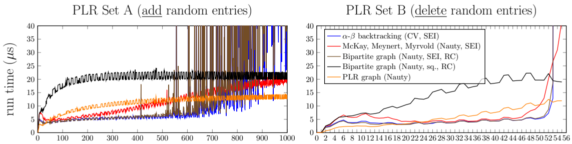
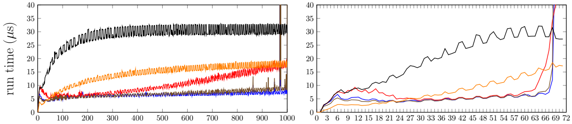
(Each data point is averaged over 10000 samples.)

$(r, s, n) = (5, 5, 5)$; discard bad methods

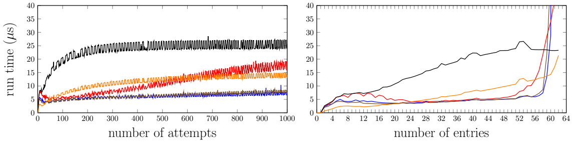




(a) $(r, s, n) = (7, 8, 9)$



(b) $(r, s, n) = (8, 9, 10)$



(c) $(r, s, n) = (8, 8, 8)$

Figure: Average run times of the remaining methods.

Squares vs. rectangles

	run time (μs)					
(r, s, n)	$(17, 18, 19)$			$(7, 7, 7)$		
no. entries	$rs - 0$	$rs - 1$	$rs - 2$	$rs - 0$	$rs - 1$	$rs - 2$
MMM (Nauty)	58.2	30.4	29.0	1203.0	30.8	5.6
MMM (Nauty, SEI)	206.5	42.6	42.7	1200.6	24.2	5.7
MMM (Nauty, sq.)	42.4	33.1	33.0	6.6	2.8	2.2
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PLR graph (Nauty)	**29.7**	**18.5**	**18.5**	840.3	22.1	2.2
PLR graph (Nauty, SEI)	110.7	20.7	20.1	837.7	4.3	2.6
PLR graph (Nauty, sq.)	35.5	33.0	33.1	**5.3**	**2.2**	2.1
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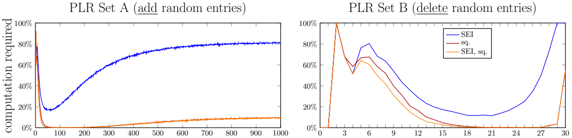
- Square invariants were crucial for Latin squares.

Usefulness of invariants...



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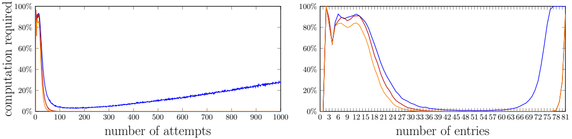

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Figure: Proportion of time computation is required (10000 samples).

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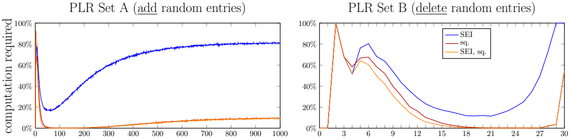

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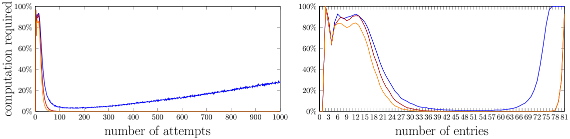

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- Invariants often eliminate the need for computation with an intermediate number of entries.

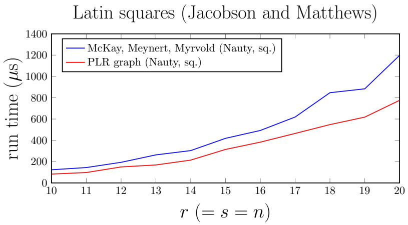

Figure: MMM method vs. PLR graph method, both using square entry invariants, for random Latin squares (10000 samples).

- I'm really surprised by this—the MMM method is the usual method.

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It may be worthwhile re-implementing **Nauty**'s individualization-refinement method for this purpose.

Thank You!