# A moment's thought: physical derivations of Fibonacci summations

David Treeby

# Two pretty formulas

$$\sum_{j=1}^{n} j^3 = \left( \sum_{j=1}^{n} j \right)^2$$

$$\sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \left( \sum_{j=1}^{n} F_j^2 F_{j+1} \right)^2$$

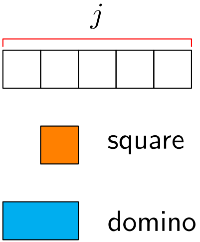
# Two models of the Fibonacci numbers

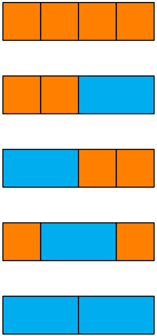
1. A combinatorial model
2. A geometric model

# A combinatorial model for Fibonacci numbers

**Theorem**

*The number of ways to tile a board of length $j$ with squares and dominoes is $f_j$ where $f_0 = f_1 = 1$ and $f_j = f_{j-1} + f_{j-2}$.*

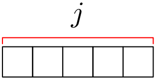




$$f_4 = 5$$

## Proof.

Consider a $j$-board. Suppose that this can be tiled in $f_j$ ways.



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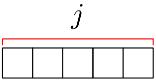
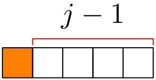



Case 1. If the first tile is a square then there are $f_{j-1}$ ways to tile the remaining $(j-1)$-board.



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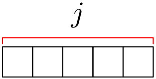


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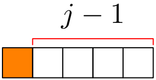
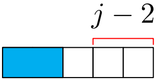



**Case 2.** If the first tile is a domino then there are $f_{j-2}$ ways to tile the remaining $(j-2)$-board.



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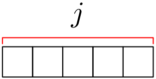


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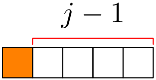


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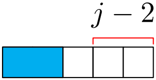



Therefore $f_j = f_{j-1} + f_{j-2}$.

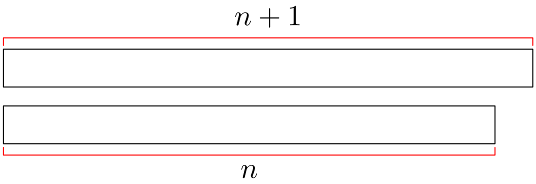
# Sums of squares of Fibonacci numbers

Theorem

$$f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1}$$

Proof.

**Question.** How many ways can you tile an $n$-board and an $(n+1)$-board?



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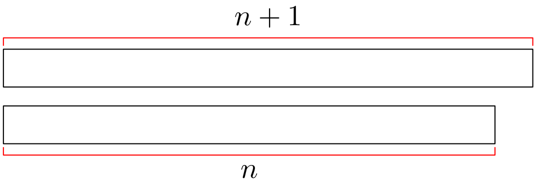


**Answer 1.** There are $f_n$ and $f_{n+1}$ tilings of the first and second board, respectively. Therefore there are

$$f_n f_{n+1}$$

tilings of both boards.

**Answer 2.** Condition on the position $j$ corresponding to the last *common edge* of each tiling.

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To avoid future common edges, there is exactly one way to finish the tiling.

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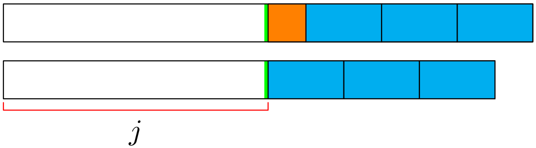


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Prior to this, the first and second board can each be tiled $f_j$ ways, so both can be tiled $f_j^2$ ways.

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To avoid future common edges, there is exactly one way to finish the tiling.

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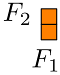
Summing over all possible values of $j$ gives $\sum_{j=0}^{n} {f_j}^2$ tilings.

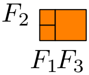
# A geometric model for Fibonacci numbers

Construct a rectangle comprising two adjacent squares of side $F_1 = 1$ and $F_2 = 1$.



# A geometric model

For every $j \geq 2$ we construct a square of side $F_j$ on the larger side of the existing rectangle.



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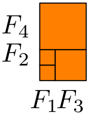

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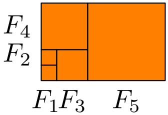

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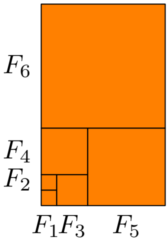

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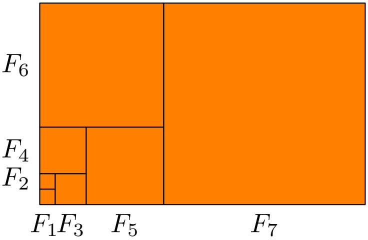
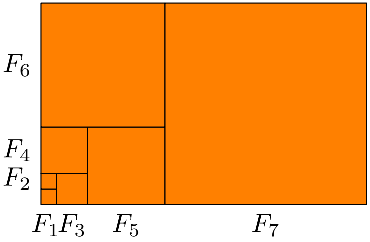


# A geometric model

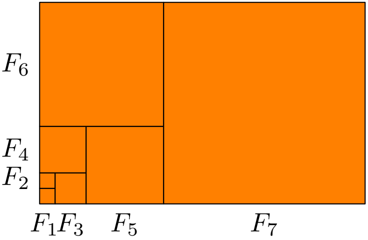
The $j$th square has side $F_j$ where $F_1 = F_2 = 1$ and $F_j = F_{j-1} + F_{j-2}$ for $n \geq 2$.



# Sums of squares of Fibonacci numbers.

## Theorem

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$$




## Proof.

The total area is equal to the sum of its parts. □

# Sums of cubes of Fibonacci numbers

**Question.** Is there a closed formula for the sum of cubes of Fibonacci numbers?

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**Answer.** Yes, by Binet's formula for the $n$th Fibonacci number there has to be. However, the answer is not expressible as the product of Fibonacci numbers.

# Sums of cubes of Fibonacci numbers

**Question.** Are there combinatorial or geometric methods for determining the sum of cubes of Fibonacci numbers?

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**Answer.** Yes and yes.

# Sums of cubes of Fibonacci numbers

Theorem

$$\sum_{j=1}^{n} F_j^3 = \frac{F_{n+1}F_{n+2}^2 + (-1)^n F_n - 2F_n^3}{2}$$

Proof.

A. T. Benjamin, B Cloitre and T. A. Carnes, *Recounting the Sums of Cubes of Fibonacci Numbers*, Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications, (2009), 45-51 □

# A preliminary result

For the geometric proof we require one preliminary result,

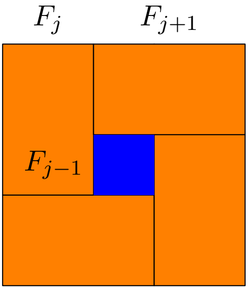
$$\sum_{j=1}^{n} F_j^2 F_{j+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}.$$

# The centroid of a composite shape

$$\overline{x} = \frac{\sum_{j=1}^{n} A_j x_j}{A} \text{ and } \overline{y} = \frac{\sum_{j=1}^{n} A_j y_j}{A}$$

# A Fibonacci tiling

## Example



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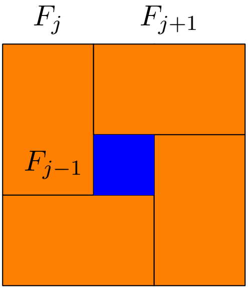
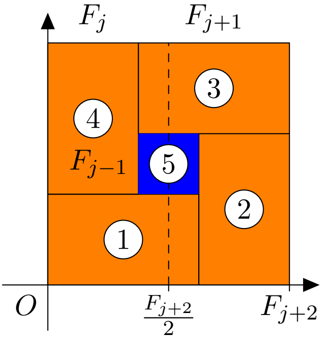



1.

$$F_{j+2}^2 = 4F_jF_{j+1} + F_{j-1}^2$$

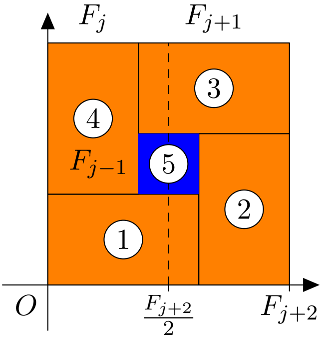
# The centroid of the tiling

Example



# The centroid of the tiling

## Example



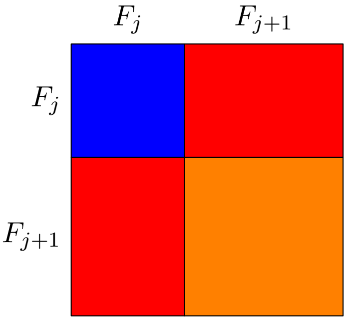

$$2F_{j-1}^2 F_j + 4F_j F_{j+1} F_{j+2} = F_{j+2}^3 - F_{j-1}^3$$

### Theorem

$$\sum_{j=1}^{n} F_j F_{j+1} F_{j+2} = \frac{F_j^3 + F_{j+1}^3 + F_{j+2}^3 - F_{j-1} F_j F_{j+1} - 2}{4}.$$

# Another Fibonacci tiling

## Example



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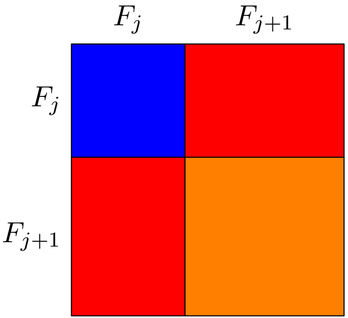
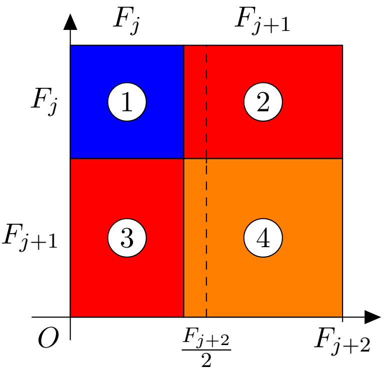
## Example




$$F_{j+2}^2 = 2F_jF_{j+1} + F_j^2 + F_{j+1}^2$$

# The centroid of the tiling

## Example




$$F_j^3 + 3F_jF_{j+1}F_{j+2} = F_{j+2}^3 - F_{j+1}^3$$

## Theorem

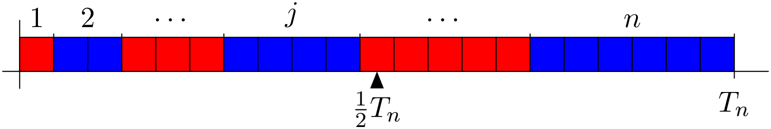
$$\sum_{j=1}^{n} F_j^3 = \frac{3F_{j+1}^2 F_j - F_{j+1}^3 - F_j^3 + 1}{2}.$$

# Sums of cubes

Starting point:

$$T_n = 1 + 2 + \cdots + n = \frac{1}{2}n(n+1)$$

# Sums of cubes



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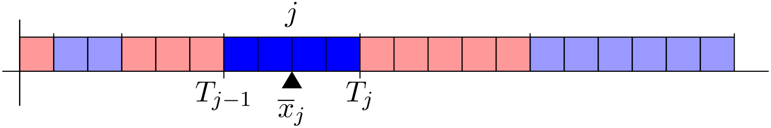
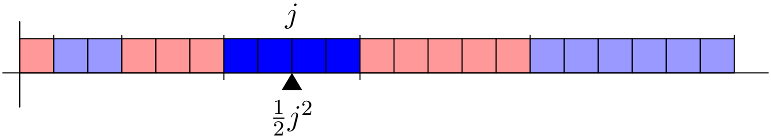


# Sums of cubes



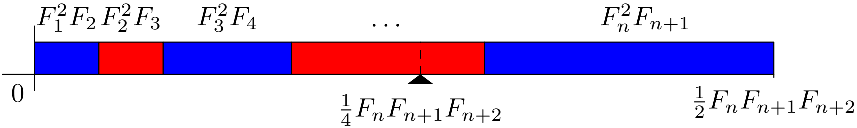
# Sums of cubes

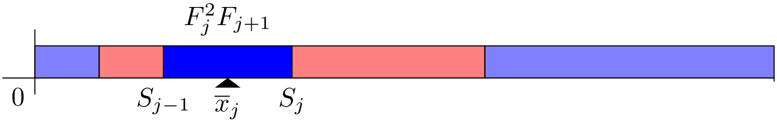
$$\sum_{j=1}^{n} j^3 = \left( \sum_{j=1}^{n} j \right)^2$$

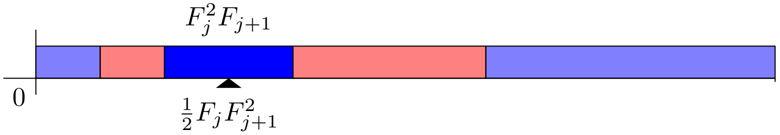
# A generalisation

The same trick works more generally. Suppose our starting point is

$$\sum_{j=1}^{n} F_j^2 F_{j+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}.$$


$F_1^2 F_2$
$F_2^2 F_3$
$F_3^2 F_4$
$\dots$
$F_n^2 F_{n+1}$
0
$\frac{1}{4} F_n F_{n+1} F_{n+2}$
$\frac{1}{2} F_n F_{n+1} F_{n+2}$


$F_j^2 F_{j+1}$
0
$S_{j-1}$
$\overline{x}_j$
$S_j$


$F_j^2 F_{j+1}$
0
$\frac{1}{2} F_j F_{j+1}^2$

$$\sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \frac{1}{4} F_n^2 F_{n+1}^2 F_{n+2}^2$$

$$\sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \left( \sum_{j=1}^{n} F_j^2 F_{j+1} \right)^2$$

# Apply the method again

$$\sum_{j=1}^{n} F_j^5 F_{j+1}^5 F_{2j+1} = \frac{1}{8} F_n^4 F_{n+1}^4 F_{n+2}^4$$