A moment’s thought: physical derivations of Fibonacci summations

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Two pretty formulas

\[ \sum_{j=1}^{n} j^3 = \left( \sum_{j=1}^{n} j \right)^2 \]

\[ \sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \left( \sum_{j=1}^{n} F_j^2 F_{j+1} \right)^2 \]
Two models of the Fibonacci numbers

1. A combinatorial model
2. A geometric model
A combinatorial model for Fibonacci numbers

**Theorem**

The number of ways to tile a board of length \( j \) with squares and dominoes is \( f_j \) where \( f_0 = f_1 = 1 \) and \( f_j = f_{j-1} + f_{j-2} \).

\[
\begin{array}{c|c|c|c|c}
\hline
& & & & \\
\hline
\text{square} & & & & \\
\hline
\text{domino} & & & & \\
\end{array}
\]
\[ f_4 = 5 \]
Proof.

Consider a $j$-board. Suppose that this can be tiled in $f_j$ ways.

Case 1. If the first tile is a square then there are $f_j - 1$ ways to tile the $(j-1)$-board.

Case 2. If the first tile is a domino then there are $f_j - 2$ ways to tile the $(j-2)$-board.

Therefore $f_j = f_{j-1} + f_{j-2}$. 

\[ \begin{array}{cccc}
  & & & \\
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  & & & \\
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  & & & \\
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  & & & \\
  & & & \\
 \end{array} \]
Proof.

Consider a $j$-board. Suppose that this can be tiled in $f_j$ ways.

Case 1. If the first tile is a square then there are $f_{j-1}$ ways to tile the remaining $(j - 1)$-board.
Proof.

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$\begin{array}{c}
\hline
\vdots \\
\hline
\end{array}$

Case 1. If the first tile is a square then there are $f_{j-1}$ ways to tile the remaining $(j - 1)$-board.

$\begin{array}{c}
\hline
\vdots \\
\hline
\end{array}$

Case 2. If the first tile is a domino then there are $f_{j-2}$ ways to tile the remaining $(j - 2)$-board.

$\begin{array}{c}
\hline
\vdots \\
\hline
\end{array}$
Proof.

Consider a $j$-board. Suppose that this can be tiled in $f_j$ ways.

Case 1. If the first tile is a square then there are $f_{j-1}$ ways to tile the remaining $(j - 1)$-board.

Case 2. If the first tile is a domino then there are $f_{j-2}$ ways to tile the remaining $(j - 2)$-board.

Therefore $f_j = f_{j-1} + f_{j-2}$. 
Sums of squares of Fibonacci numbers

Theorem

\[ f_0^2 + f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1} \]
Proof.

**Question.** How many ways can you tile an $n$-board and an $(n + 1)$-board?
Proof.

**Question.** How many ways can you tile an $n$-board and an $(n + 1)$-board?

**Answer 1.** There are $f_n$ and $f_{n+1}$ tilings of the first and second board, respectively. Therefore there are

$$f_n f_{n+1}$$

tilings of both boards.
**Answer 2.** Condition on the position \( j \) corresponding to the last *common edge* of each tiling.

To avoid future common edges, there is exactly one way to finish the tiling. Prior to this, the first and second board can each be tiled \( f_j \) ways, so both can be tiled \( f_2^j \) ways. Summing over all possible values of \( j \) gives

\[
\sum_{n \geq 0} f_j^2 \text{ tilings.}
\]
Answer 2. Condition on the position $j$ corresponding to the last *common edge* of each tiling.

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Prior to this, the first and second board can each be tiled $f_j$ ways, so both can be tiled $f_j^2$ ways.

Summing over all possible values of $j$ gives $\sum_{j=0}^n f_j^2$ tilings.
A geometric model for Fibonacci numbers

Construct a rectangle comprising two adjacent squares of side $F_1 = 1$ and $F_2 = 1$. 

\[ F_2 \]
\[ F_1 \]
A geometric model

For every $j \geq 2$ we construct a square of side $F_j$ on the larger side of the existing rectangle.
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A geometric model

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A geometric model

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A geometric model

The $j$th square has side $F_j$ where $F_1 = F_2 = 1$ and $F_j = F_{j-1} + F_{j-2}$ for $n \geq 2$. 

\[
\begin{align*}
F_1 & \quad F_2 & \quad F_3 & \quad F_4 & \quad F_5 & \quad F_6 & \quad F_7 \\
F_1F_3 & \quad F_5 & \quad F_7
\end{align*}
\]
Theorem

\[ F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1} \]

Proof.
The total area is equal to the sum of its parts.
Sums of cubes of Fibonacci numbers

**Question.** Is there a closed formula for the sum of cubes of Fibonacci numbers?
Sums of cubes of Fibonacci numbers

**Question.** Is there a closed formula for the sum of cubes of Fibonacci numbers?

**Answer.** Yes, by Binet’s formula for the $n$th Fibonacci number there has to be. However, the answer is not expressible as the product of Fibonacci numbers.
Sums of cubes of Fibonacci numbers

**Question.** Are there combinatorial or geometric methods for determining the sum of cubes of Fibonacci numbers?
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Answer. Yes and yes.
Sums of cubes of Fibonacci numbers

Theorem

$$\sum_{j=1}^{n} F_j^3 = \frac{F_{n+1}F_{n+2}^2 + (-1)^n F_n - 2F_n^3}{2}$$

Proof.
A preliminary result

For the geometric proof we require one preliminary result,

$$\sum_{j=1}^{n} F_j^2 F_{j+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}.$$
The centroid of a composite shape

\[ \bar{x} = \frac{\sum_{j=1}^{n} A_j x_j}{A} \quad \text{and} \quad \bar{y} = \frac{\sum_{j=1}^{n} A_j y_j}{A} \]
A Fibonacci tiling

Example

\[ F_j \quad F_{j+1} \]

\[ F_{j-1} \]
A Fibonacci tiling

Example

1. 

\[ F_{j+2}^2 = 4F_j F_{j+1} + F_{j-1}^2 \]
The centroid of the tiling

Example

\[ F_j - F_{j-1} + 2F_{j+1} - F_{j+2} = 2F_{j+2} \]
The centroid of the tiling

Example

\[ 2F_{j+1}^2 F_j + 4F_j F_{j+1} F_{j+2} = F_{j+2}^3 - F_{j-1}^3 \]
Theorem

$$\sum_{j=1}^{n} F_j F_{j+1} F_{j+2} = \frac{F_j^3 + F_{j+1}^3 + F_{j+2}^3 - F_{j-1} F_j F_{j+1} - 2}{4}.$$
Another Fibonacci tiling

Example

\[
F_j + 1 = F_j F_{j+1} + F_j F_{j+1}
\]
Another Fibonacci tiling

Example

\[ F_{j+2}^2 = 2F_j F_{j+1} + F_j^2 + F_{j+1}^2 \]
The centroid of the tiling

Example

\[ F_j^3 + 3F_j F_{j+1} F_{j+2} = F_{j+2}^3 - F_{j+1}^3 \]
Theorem

\[
\sum_{j=1}^{n} F_j^3 = \frac{3F_{j+1}^2 F_j - F_{j+1}^3 - F_j^3 + 1}{2}.
\]
Sums of cubes

Starting point:

\[ T_n = 1 + 2 + \cdots + n = \frac{1}{2}n(n + 1) \]
Sums of cubes
Sums of cubes

\[ x_j \]

\[ T_{j-1} \quad \frac{1}{x_j} \quad T_{j} \]
Sums of cubes

\[ \frac{1}{2} j^2 \]
Sums of cubes

\[ \sum_{j=1}^{n} j^3 = \left( \sum_{j=1}^{n} j \right)^2 \]
A generalisation

The same trick works more generally. Suppose our starting point is

$$\sum_{j=1}^{n} F_j^2 F_{j+1} = \frac{1}{2} F_n F_{n+1} F_{n+2}.$$
\[ F_1^2 F_2 F_2^2 F_3 F_3^2 F_4 \ldots F_n^2 F_{n+1} \]

\[ \frac{1}{4} F_n F_{n+1} F_{n+2} \]

\[ \frac{1}{2} F_n F_{n+1} F_{n+2} \]
\[ F_j^2 F_{j+1} \]

Diagram showing:
- \( S_{j-1} \)
- A point \( x_j \)
- \( S_j \)
\[ \sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \frac{1}{4} F_n^2 F_{n+1}^2 F_{n+2} \]
\[
\sum_{j=1}^{n} F_j^3 F_{j+1}^3 = \left( \sum_{j=1}^{n} F_j^2 F_{j+1} \right)^2
\]
Apply the method again

\[ \sum_{j=1}^{n} F_j^5 F_{j+1}^5 F_{2j+1} = \frac{1}{8} F_n^4 F_{n+1}^4 F_{n+2}^4 \]