Power Domination and Zero Forcing

Violeta Vasilevska
Utah Valley University
Violeta.Vasilevska@uvu.edu

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AIM, ICERM, NSF, REUF Collaborators (2015)

REUF – **R**esearch **E**xperience for **U**ndergraduate **F**aculty

“a program for undergraduate faculty who are interested in mentoring undergraduate research.”

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Dr. Mary Flagg (University of St. Thomas)
Dr. Veronica Furst (Fort Lewis College)
Dr. Leslie Hogben (Iowa State University)
Dr. Brian Wissman (University of Hawaii at Hilo)

“Zero Forcing and Power Domination for Graph Products.”
Australasian J. Combinatorics 70 (2018), 221-235

Outline

- Power Domination (PD)
- Zero Forcing (ZF)
- Connection between PD and ZF processes
- Computing PD and ZF numbers

P O W E R D O M I N A T I O N

Monitoring Electrical Networks

- Electric power companies need to monitor the state of their networks continuously.

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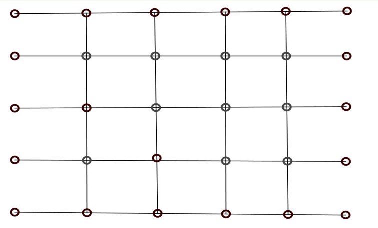
- **Problem:** PMUs are costly, so it is important to minimize the number of PMUs used.
- Where should those PMUs be placed to observe the entire system?

Modeling the Problem

An electric power network	–	modeled by a graph
The electrical nodes	–	graph vertices
Transmission lines joining two electrical nodes	–	graph edges



http://kk.org/thetechnium/Electricity Network.jpg

The Power Domination Problem in Graphs

Find a minimum set of vertices from where the entire graph can be observed according to certain propagation rules.

First studied by

Haynes at al. ("Domination in graphs applied to electric power networks" (2002))

- Start with a graph G whose vertices are colored either white or black.
- Let S be the set of all vertices colored black. Color all neighbors of vertices in S black.
- Apply the following color-change rule as many times as possible.

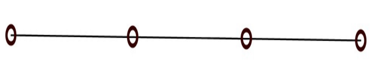
  Color-change Rule:

  - If there is a black vertex that has exactly one white neighbor - color that neighbor black.

- The set S is called a **power dominating set** of a graph G if at the end of applying the propagation rule all vertices in G are colored black.
- A **minimum power dominating set** is a power dominating set with minimum number of vertices.
- **Power domination number** for G, denoted $\gamma_P(G)$, is the number of vertices in a minimum power domination set.

E X A M P L E S

Path P_4



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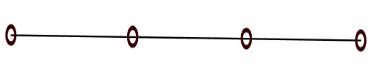

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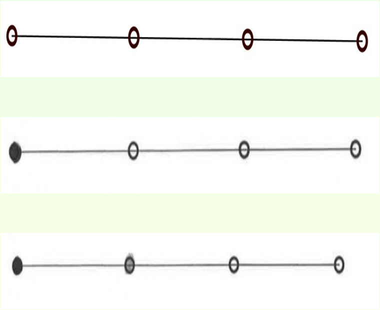

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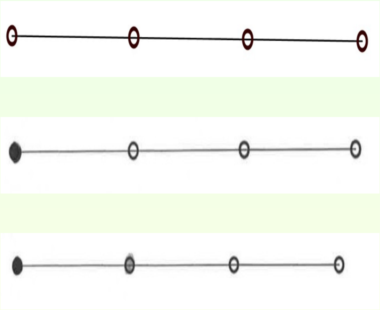
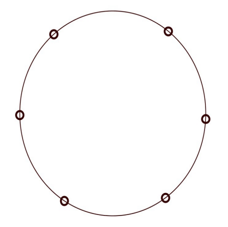


$$\gamma_P(P_4) = 1$$

Circle C_6



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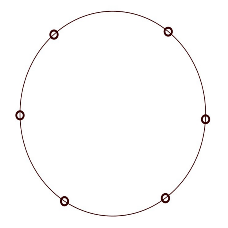

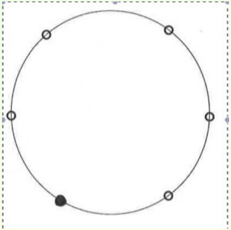

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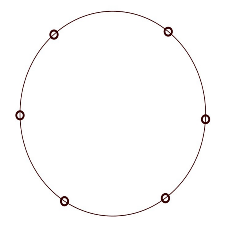

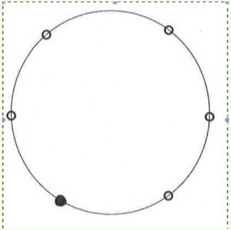

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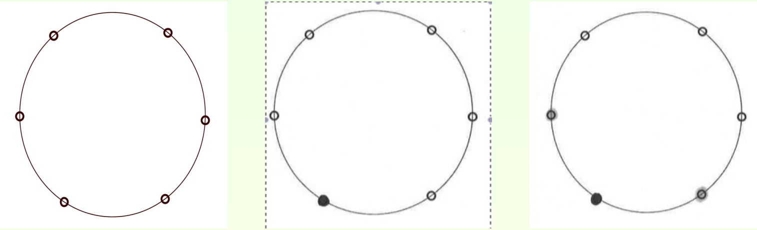
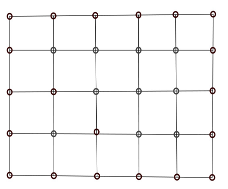


$$\gamma_P(C_6) = 1$$

Grid



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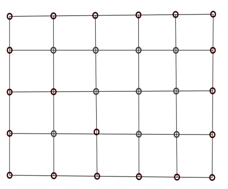

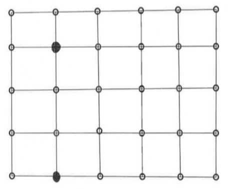

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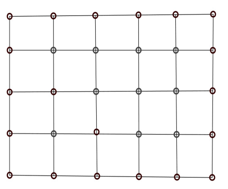

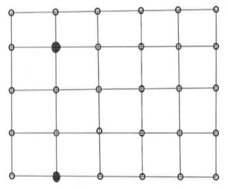

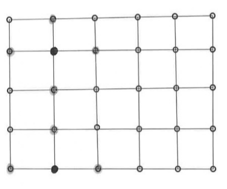

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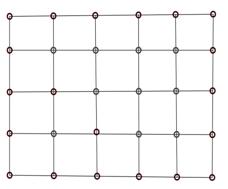
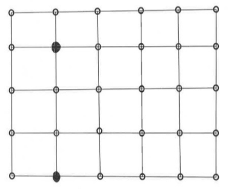
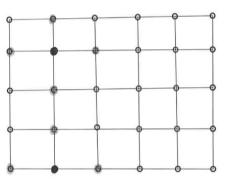
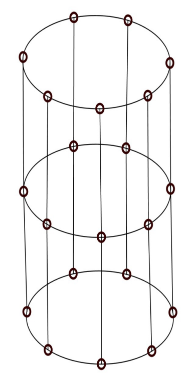






$$\gamma_P(G) = 2$$

Cylinder



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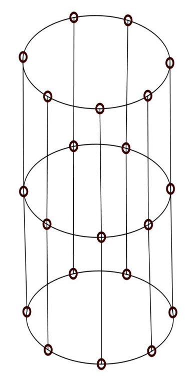

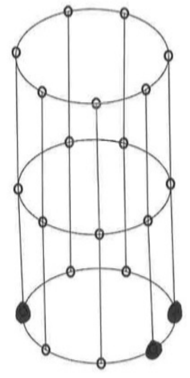

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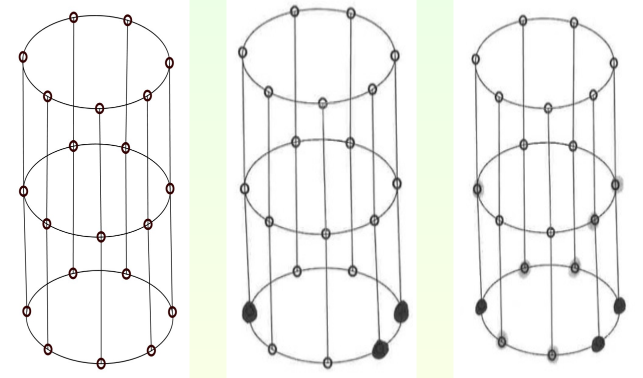
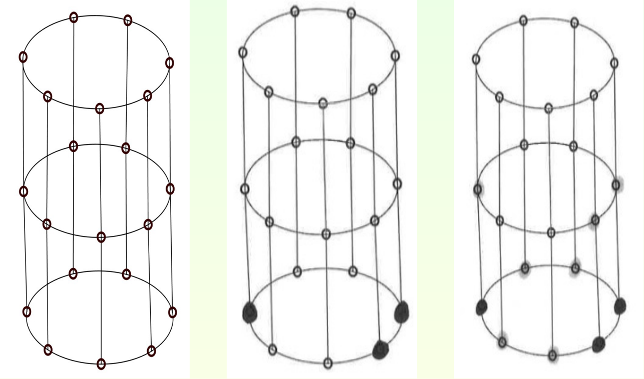


Cylinder



$$\gamma_P(G) = 3$$

ZERO FORCING

A **zero forcing set** for a graph G is a subset of vertices B such that if initially the vertices in B are colored black and the reminding vertices are colored white, repeated application of the color change rule can color all vertices of G black.

A **minimum zero forcing set** is a zero forcing set with minimum number of vertices.

Zero forcing number for G, denoted $Z(G)$, is the number of vertices in a minimum zero forcing set.

The zero forcing number was introduced

- by mathematicians Hogben et al. ("*Zero forcing sets and the minimum rank of graphs*," (2008))
- and independently by mathematical physicists studying control of quantum systems
- and later by computer scientists studying graph search algorithms.

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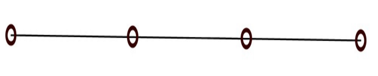

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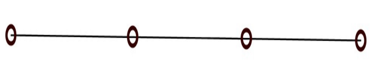

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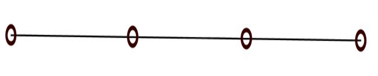

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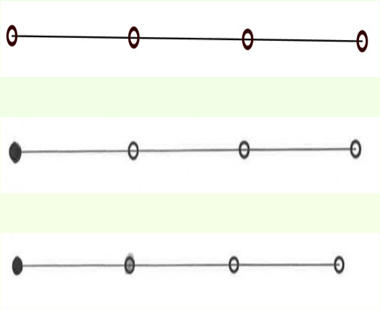
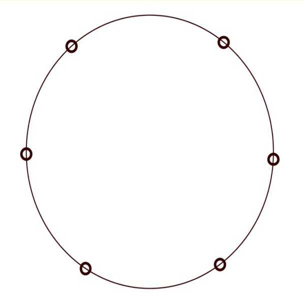


$$Z(P_4) = 1$$

Circle C_6



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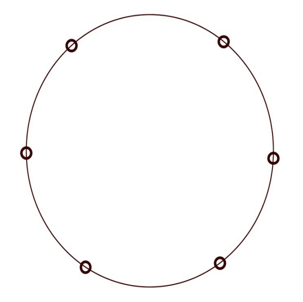

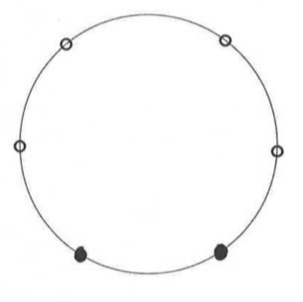

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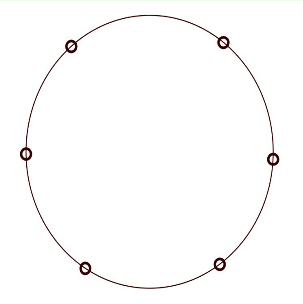

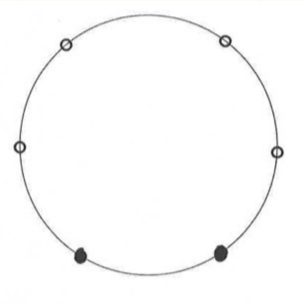

$$Z(C_6) = 2$$

CONNECTION BETWEEN PD AND ZF NUMBERS

Observation

The power domination process on a graph G can be described as

- choosing a set $S \subseteq V(G)$ and
- applying the zero forcing process to the closed neighborhood $N[S]$ of S.

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- choosing a set $S \subseteq V(G)$ and
- applying the zero forcing process to the closed neighborhood $N[S]$ of S.

The set S is a power dominating set of G if and only if $N[S]$ is a zero forcing set for G.

Determining PD

The power domination number of several families of graphs has been determined using the following two-step process:

- find an upper bound:

  The upper bound is usually obtained by providing a pattern to construct a set, together with a proof that constructed set is a power dominating set;

- find a lower bound:

  The lower bound is usually found by exploiting structural properties of the particular family of graphs, and it usually consists of a very technical and lengthy process.

Research Problem

Finding good general lower bounds for the power domination number.

An effort in that direction is the work by Stephen et al. ("*Power domination in certain chemical structures*," (2015))

Theorem (1)

Let G be a graph that has an edge.
Then

$$\left\lceil \frac{Z(G)}{\Delta(G)} \right\rceil \leq \gamma_P(G),$$

where $\Delta(G) = \max\{\deg v : v \in V(G)\}$ *is the maximum degree of G.*

This bound is tight.

Sketch of Proof:

Choose a minimum PD set $\{u_1, u_2, \ldots, u_t\}$. Hence $t = \gamma_P(G)$. Then $\sum_{i=1}^{t} \deg u_i \leq t\Delta(G)$.

- If G has no isolated vertices: Dean et al. ("*On the power dominating sets of hypercubes*," (2011)):

$$Z(G) \leq \sum_{i=1}^{t} \deg u_i.$$

- If G has isolated vertices, they contribute one to each ZF number and PD number.

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Bound is tight:

$$Z(K_n) = \Delta(K_n) = n - 1 \qquad \text{and} \qquad \gamma_P(K_n) = 1. \ \square$$

COMPUTING PD NUMBERS FOR TENSOR PRODUCTS

Tensor Products

Let

$$G = (V(G), E(G)) \text{ and } H = (V(H), E(H))$$

be disjoint graphs.

The *tensor product* (also called the *direct product*) of G and H is denoted by $G \times H$:

- vertex set: $V(G) \times V(H)$
- edge set:

a vertex (g, h) is adjacent to a vertex (g', h') in $G \times H$
if
$\{g, g'\} \in E(G)$ and $\{h, h'\} \in E(H)$.

Computing PD # for Tensor Products

Dorbec et al. ("*Power domination in product graphs*," (2008)) (the power domination problem for the tensor product of two paths)

Computing PD # for Tensor Products

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Question:

What is the power domination number for a tensor product of a path and a complete graph and of a cycle and a complete graph?

Theorem

Let $t \geq 3$ and $G = P_t$ or $G = C_t$.
Suppose t is odd and $n \geq t$, or suppose t is even and either

1. $G = P_t$ *and* $n \geq \frac{t}{2} + 2$, *or*
2. $G = C_t$ *and* $n \geq \frac{t}{2}$.

Then

$$\gamma_P(G \times K_n) = \begin{cases} \left\lceil \frac{t}{2} \right\rceil & \text{if } t \not\equiv 2 \mod 4, \\ \frac{t}{2} \text{ or } \frac{t}{2} + 1 & \text{if } t \equiv 2 \mod 4. \end{cases}$$

Sketch of Proof:

- Upper bound on $\gamma_P(G \times K_n)$:

Theorem

Let $n \geq 3$. If $G = P_t$ with $t \geq 2$ or $G = C_t$ with $t \geq 3$, then

$$\gamma_P(G \times K_n) \leq \begin{cases} \lceil \frac{t}{2} \rceil & \text{if } t \not\equiv 2 \mod 4, \\ \frac{t}{2} + 1 & \text{if } t \equiv 2 \mod 4. \end{cases}$$

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- A lower bound on $\gamma_P(G \times K_n)$:

Sketch of Proof:

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Theorem

Let $n \geq 3$. If $G = P_t$ with $t \geq 2$ or $G = C_t$ with $t \geq 3$, then

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- A lower bound on $\gamma_P(G \times K_n)$:

  Use known ZF #s...

Sketch of Proof - continue

Theorem

1. *(Fernandes, da Fonseca)*
   If $t \geq 1$ *is odd and* $n \geq 2$*, then* $Z(P_t \times K_n) = (n-2)t + 2$.
2. *(V. et al.)*
   If $t \geq 2$ *is even and* $n \geq 3$*, then* $Z(P_t \times K_n) = (n-2)t$.

Theorem

If $n, t \geq 3$*, then*

$$Z(C_t \times K_n) = \begin{cases} (n-2)t + 2 & \text{if } t \text{ is odd,} \\ (n-2)t + 4 & \text{if } t \text{ is even.} \end{cases}$$

Sketch of Proof - continue

Observation:

$\deg(g,h) = \deg_G(g)\deg_H(h) \qquad$ for $(g,h) \in E(G \times H)$.

Hence, $\qquad \Delta(G \times H) = \Delta(G)\Delta(H)$.

Sketch of Proof - continue

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$\deg(g,h) = \deg_G(g)\deg_H(h)$ for $(g,h) \in E(G \times H)$.

Hence, $\Delta(G \times H) = \Delta(G)\Delta(H)$.

$$\Delta(G \times K_n) = \Delta(G)\Delta(K_n) = 2(n-1)$$

Sketch of Proof - continue

Consider two cases depending on the parity of t and use Theorem 1.

- $t = 2k + 1$ for some positive integer k

$$\gamma_P(G \times K_n) \geq \left\lceil \frac{(n-2)(2k+1)+2}{2(n-1)} \right\rceil$$

$$= \left\lceil k + \frac{n-2k}{2(n-1)} \right\rceil \geq k + 1 \text{ if } n - 2k > 0.$$

Hence, $\lceil \frac{t}{2} \rceil \leq \gamma_P(G \times K_n)$ if t is odd and $n \geq t$.

Sketch of Proof - continue

- $t = 2k$ for some positive integer k.
  Take $c = 0$ for $G = P_t$ and $c = 2$ for $G = C_t$.

  $$\gamma_P(G \times K_n) \geq \left\lceil \frac{(n-2)(2k)+2c}{2(n-1)} \right\rceil$$

  $$= \left\lceil k - \frac{k-c}{n-1} \right\rceil = k \text{ if } n-1 > k-c.$$

  Hence,

  $$\tfrac{t}{2} \leq \gamma_P(G \times K_n) \text{ if } G = P_t \text{ and}$$

  $$n \geq \tfrac{t}{2} + 2, \text{ or if } G = C_t \text{ and } n \geq \tfrac{t}{2}. \ \square$$

C O M P U T I N G ZF N U M B E R S F O R G R A P H P R O D U C T S

Lexicographic Product

Let

$$G = (V(G), E(G)) \text{ and } H = (V(H), E(H))$$

be disjoint graphs.

The *lexicographic product* of G and H is denoted by $G * H$:

- vertex set: $V(G) \times V(H)$
- edge set:
  two vertices (g, h) and (g', h') are adjacent in $G * H$ if either
  - $\{g, g'\} \in E(G)$, or
  - $g = g'$ and $\{h, h'\} \in E(H)$.

Domination Number

A vertex v in a graph G is said to **dominate** itself and all of its neighbors in G.

A set of vertices S is a **dominating set** of G if every vertex of G is dominated by a vertex in S.

The minimum cardinality of a dominating set is the **domination number** of G (denoted by $\gamma(G)$).

Theorem

Let G and H be regular graphs with degree d_G and d_H, respectively.

If $\gamma_P(H) = 1$ and $\gamma(G) = 1$, then

$$Z(G * H) = d_G|V(H)| + d_H.$$

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Corollary

For $n \geq 2$ and $m \geq 3$,

$$Z(K_n * C_m) = (n-1)m + 2.$$

Sketch of Proof

- Dorbec et al., ("*Power Domination in Product Graphs*," (2008))

$$\gamma_P(G * H) = \gamma(G) \qquad \text{if } \gamma_P(H) = 1.$$

- For lexicographic product

$$\deg_{G*H}(g,h) = (\deg_G g)|V(H)| + \deg_H h$$

for any vertex $(g,h) \in V(G * H)$, hence

$$\Delta(G * H) = \Delta(G)|V(H)| + \Delta(H).$$

Sketch of Proof - continue

By Theorem 1: $Z(G * H) \leq \gamma_P(G * H)\Delta(G * H)$.
Hence

$$Z(G * H) \leq \gamma(G)\left(\Delta(G)|V(H)| + \Delta(H)\right) \qquad \text{if } \gamma_P(H) = 1.$$

- Since $\gamma(G) = 1$, G is d_G-regular, and H is d_H-regular:

$$Z(G * H) \leq d_G|V(H)| + d_H.$$

- $G * H$ is $(d_G|V(H)| + d_H)$-regular, hence

$$d_G|V(H)| + d_H = \delta(G * H) \leq Z(G * H),$$

where $\delta(G) = \min\{\deg v : v \in V\}$ is the *minimum degree* of G. $\square$

THANK YOU!

Violeta.Vasilevska@uvu.edu