The two-modular Fourier transform of binary functions

Yi Hong (Monash University, Melbourne, Australia)
Emanuele Viterbo (Monash University, Melbourne, Australia)
Jean-Claude Belfiore (Telecom ParisTech)

The Discrete Fourier Transform

- N samples of a discrete-time signal (real or complex) form the time-domain vector $\mathbf{x} = (x[n])_{n=0}^{N-1}$
- The discrete Fourier transform (DFT) of $\mathbf{x}$ is the frequency-domain vector $\mathbf{X} = (X[k])_{k=0}^{N-1}$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-\jmath 2\pi \frac{nk}{N}} \qquad k = 0, \dots N-1$$

- The inverse discrete Fourier transform (IDFT) of $\mathbf{X}$ is

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] e^{\jmath 2\pi \frac{nk}{N}} \qquad n = 0, \dots N-1$$

The Discrete Fourier Transform

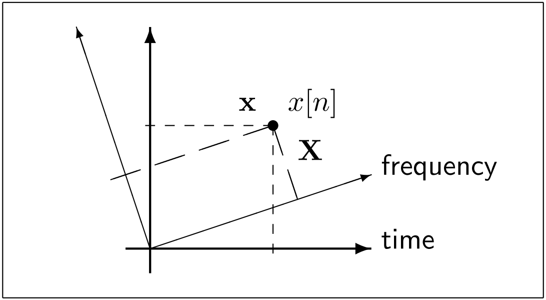
- The vector $\mathbf{x} = (x[n])_{n=0}^{N-1}$ gives the N samples of a time-domain function $f : \mathbb{Z} \to \mathbb{C}$
- If f is periodic by N samples then $f : \mathbb{Z}_N \to \mathbb{C}$ (assumption for DFT to provide the discrete spectrum)
- The time axis $\mathbb{Z}_N = \{0, 1, \ldots N-1\}$ is an additive group $G = (\mathbb{Z}_N, +)$ with addition mod N then $f : G \to \mathbb{C}$
- In the frequency domain $\mathbf{X} = (X[k])_{k=0}^{N-1}$ represents the transform of f as a periodic function $\hat{f} : G \to \mathbb{C}$
- The frequency axis has the same additive group structure $G = (\mathbb{Z}_N, +)$

The Discrete Fourier Transform

- The DFT matrix $\mathbf{F} = \{e^{-\jmath 2\pi \frac{nk}{N}}\}_{n,k=0}^{N-1}$ is a unitary matrix such that

$$\mathbf{X}^T = \mathbf{F}\mathbf{x}^T \qquad \mathbf{x}^T = \frac{1}{N}\mathbf{F}^H\mathbf{X}^T$$

- The vectors $\mathbf{x}$ and $\mathbf{X}$ are a two representations of the signal $x[n]$ in different coordinate systems, defined by the time basis and frequency basis.



One-dimensional group representation

- The Abelian group $G = (\mathbb{Z}_N, +)$ with addition mod N admits the following one-dimensional representations

$$\chi_k : G \to S_k \subset \mathbb{C} \qquad \chi_k(n) = e^{-\jmath 2\pi \frac{nk}{N}}$$

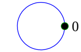
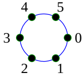
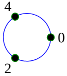
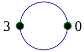
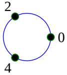
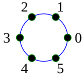
where $k = 0, \ldots, N-1$ and

$$S_k = \left\{1, e^{-\jmath 2\pi \frac{k}{N}}, e^{-\jmath 2\pi \frac{2k}{N}}, \cdots, e^{-\jmath 2\pi \frac{(N-1)k}{N}}\right\}$$

- The representation χ_k is a group homomorphism transforming the addition mod N in G into the multiplication of N-th roots of unity in S_k, i.e., for any $a, b \in G$

$$\chi_k(a+b) = \chi_k(a)\chi_k(b) \text{ since } e^{-\jmath 2\pi \frac{(a+b)k}{N}} = e^{-\jmath 2\pi \frac{ak}{N}} e^{-\jmath 2\pi \frac{bk}{N}}$$

Example $\mathbb{Z}_6$

k		$S_k = \{\chi_k(g), g \in G = \{0,1,2,3,4,5\}\}$	$\mathsf{Ker}(\chi_k)$, $G/\mathsf{Ker}(\chi_k)$
0		$\{1\}$	$\{0,1,2,3,4,5\}, \{0\}$
1		$\{1, e^{-\jmath\frac{2\pi}{6}}, e^{-\jmath\frac{4\pi}{6}}, e^{-\jmath\frac{6\pi}{6}}, e^{-\jmath\frac{8\pi}{6}}, e^{-\jmath\frac{10\pi}{6}}\}$	$\{0\}, \{0,1,2,3,4,5\}$
2		$\{1, e^{-\jmath\frac{4\pi}{6}}, e^{-\jmath\frac{8\pi}{6}}\}$	$\{0,3\}, \{0,2,4\}$
3		$\{1, -1\}$	$\{0,2,4\}, \{0,3\}$
4		$\{1, e^{\jmath\frac{4\pi}{6}}, e^{\jmath\frac{8\pi}{6}}\}$	$\{0,3\}, \{0,2,4\}$
5		$\{1, e^{\jmath\frac{2\pi}{6}}, e^{\jmath\frac{4\pi}{6}}, e^{\jmath\frac{6\pi}{6}}, e^{\jmath\frac{8\pi}{6}}, e^{\jmath\frac{10\pi}{6}}\}$	$\{0\}, \{0,1,2,3,4,5\}$

Example $\mathbb{Z}_6$ (cont.)

$g \in G$	0	1	2	3	4	5	
$\chi_0(g)$	1	1	1	1	1	1	ψ_0
$\chi_1(g)$	1	$e^{-\jmath\frac{2\pi}{6}}$	$e^{-\jmath\frac{4\pi}{6}}$	$e^{-\jmath\frac{6\pi}{6}}$	$e^{-\jmath\frac{8\pi}{6}}$	$e^{-\jmath\frac{10\pi}{6}}$	ψ_1
$\chi_2(g)$	1	$e^{-\jmath\frac{4\pi}{6}}$	$e^{-\jmath\frac{8\pi}{6}}$	1	$e^{-\jmath\frac{4\pi}{6}}$	$e^{-\jmath\frac{8\pi}{6}}$	ψ_2
$\chi_3(g)$	1	-1	1	-1	1	-1	ψ_3
$\chi_4(g)$	1	$e^{\jmath\frac{4\pi}{6}}$	$e^{\jmath\frac{8\pi}{6}}$	1	$e^{\jmath\frac{4\pi}{6}}$	$e^{\jmath\frac{8\pi}{6}}$	ψ_4
$\chi_5(g)$	1	$e^{\jmath\frac{2\pi}{6}}$	$e^{\jmath\frac{4\pi}{6}}$	$e^{\jmath\frac{6\pi}{6}}$	$e^{\jmath\frac{8\pi}{6}}$	$e^{\jmath\frac{10\pi}{6}}$	ψ_5

$$\mathbf{F} = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_5 \end{pmatrix}$$

DFT using representations as Fourier basis

- The representations χ_k for $k = 0, \ldots, N-1$ are all inequivalent.
- Some are one-to-one and some are many-to-one and the images can be associated with subgroups of G
- We can formally rewrite the DFT as

$$X[k] = \langle \mathbf{x}, \boldsymbol{\psi}_k \rangle = \sum_{g \in G} x[g] \chi_k(g) \quad k = 0, \ldots, N-1$$

- The complex vectors $\boldsymbol{\psi}_k = [\chi_k(g)]_{g \in G}$ form the discrete Fourier basis vectors
- Each representation provides a “lens” through which we observe the time-domain signal $x[n]$.

FFT using representations as Fourier basis

- Given a normal subgroup $H \trianglelefteq G$ we define the quotient group G/H consisting of the coset leaders u of the cosets $u+H$
- The direct product of H and G/H is isomorphic to G i.e.,

$$G = \{u+v | u \in H, v \in G/H\} \approx H \times G/H$$

- All $u \in H = \text{Ker}(\chi_k)$ are mapped to the same value $\chi_k(u) = \chi_k(0) = 1 \in S_k$
- Then we can compute the DFT more efficiently as

$$\begin{aligned} X[k] &= \sum_{g \in G} x[g]\chi_k(g) = \sum_{v \in G/H} \sum_{u \in H} x[u+v]\chi_k(u+v) \\ &= \sum_{v \in G/H} \left(\sum_{u \in H} x[u+v] \right) \chi_k(v) \quad k = 0, \ldots, N-1 \end{aligned}$$

Known generalizations of the DFT concept

- $f : G \to \mathbb{C}$, where G can be an arbitrary group (not only Abelian): Fourier coefficients are complex matrices
- These generalizations make use of multi-dimensional representations of the group G with matrices over $\mathbb{C}$
- The inverse Fourier transform uses $\frac{1}{|G|}\text{Tr}(\cdot)$ the Trace operator of a matrix to get back to time domain scalar values of f.
- $f : G \to K$, where K is a field of characteristic p and p does not divide $|G|$: Fourier coefficients are scalars in K since an "exponential" function can be defined using a primitive element $\alpha \in K$.

The missing Fourier Transform for binary functions

- We consider a finite commutative ring $\mathcal{R}$ of characteristic $p = 2$, e.g., $\mathbb{F}_2[X]/\phi(X)$, where $\phi(X)$ is a binary-coefficient polynomial of degree m.
- Elements of $\mathcal{R}$ are represented by m-bit vectors (or polynomials of degree at most $m-1$) and multiplications are computed by polynomial multiplication mod $\phi(X)$.
- Let $G = C_2^n$ be the additive group of $\mathbb{F}_2^n$ (n bit vectors)
- We study binary functions $f : G \to \mathcal{R}$ (n bit to m bit) and their convolutions (group ring $\mathcal{R}[G]$)

What does not work? If $p = 2$ divides $|G| = 2^n = N$, the inverse DFT term $1/N$ is not defined in $\mathcal{R}$ and the Trace fails to work in the generalized inverse DFT.

The two-modular representations of G

- Let $G = C_2 = \{0, 1\}$ then a two-modular representation as 2×2 binary matrices over $\mathcal{R}$, is given by the two matrices

$$E_0 = \pi_1(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad E_1 = \pi_1(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

  The matrix entries are the 'zero' and 'one' element in $\mathcal{R}$.

- The n-fold direct product of C_2, $G = C_2^n = C_2 \times \cdots \times C_2$ can be represented as the Kroneker product of the representations of C_2, i.e.,

$$\pi_n(G) \triangleq \pi_1(C_2) \otimes \cdots \otimes \pi_1(C_2)$$

The two-modular representations of $G = C_2 \times \cdots \times C_2$

- Let the binary vectors $\mathbf{b} = (b_1, \ldots, b_n)$ represent the elements of G with bitwise addition mod 2 (XOR). Then

$$\pi_n(\mathbf{b}) = E_{\mathbf{b}} = \pi_1(b_1) \otimes \cdots \otimes \pi_1(b_n)$$

- Example $G = C_2 \times C_2$

$$E_{00} = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \quad E_{01} = \begin{pmatrix} 1&0&1&0 \\ 0&1&0&1 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix} \quad E_{10} = \begin{pmatrix} 1&1&0&0 \\ 0&1&0&0 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix} \quad E_{11} = \begin{pmatrix} 1&1&1&1 \\ 0&1&0&1 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}$$

The two-modular Fourier basis for $G = C_2 \times \cdots \times C_2$

- Consider the two-modular representations ($2^k \times 2^k$ matrices) of the nested subgroups of $G = C_2^n$,

$$H_0 = \{\mathbf{0}_n\} \lhd H_1 \lhd \cdots \lhd H_k \lhd \cdots \lhd H_{n-1} \lhd H_n = G$$

  where $H_1 \cong C_2$, $H_2 \cong C_2 \times C_2$, $H_3 \cong C_2 \times C_2 \times C_2$, etc.

- The Fourier basis 'vectors' are made up of all the inequivalent two-modular representations π_k

$$\begin{array}{rcl}
H_0 & \cong & \mathrm{Im}(\pi_0) = \{1\} \\
H_1 & \cong & \mathrm{Im}(\pi_1) = \{E_0, E_1\} \\
H_2 & \cong & \mathrm{Im}(\pi_2) = \{E_{00}, E_{01}, E_{10}, E_{11}\} \\
H_3 & \cong & \mathrm{Im}(\pi_3) = \{E_{000}, E_{001}, E_{010}, E_{011}, E_{100}, E_{101}, E_{110}, E_{111}\} \\
 & \vdots &
\end{array}$$

Example $G = C_2^3$

The Fourier basis 'vectors' $\boldsymbol{\psi}_k = [E_{\tau_k(g)} : g \in G]$ are the 2^n-component vectors (indexed by g) of $2^k \times 2^k$ matrices from the set $\mathrm{Im}(\pi_k)$

$g \in G$	000	001	010	011	100	101	110	111	
$\pi_0(g)$	1	1	1	1	1	1	1	1	$\boldsymbol{\psi}_0$
$\pi_1(g)$	E_0	E_1	E_0	E_1	E_0	E_1	E_0	E_1	$\boldsymbol{\psi}_1$
$\pi_2(g)$	E_{00}	E_{01}	E_{11}	E_{10}	E_{00}	E_{01}	E_{11}	E_{10}	$\boldsymbol{\psi}_2$
$\pi_3(g)$	E_{000}	E_{010}	E_{001}	E_{011}	E_{111}	E_{101}	E_{110}	E_{100}	$\boldsymbol{\psi}_3$

The two-modular Fourier Transform of $f : G \to \mathcal{R}$

- We abstractly define the k-th Fourier coefficients as the $2^k \times 2^k$ matrix with elements in $\mathcal{R}$

$$\hat{f}_k = \langle f, \boldsymbol{\psi}_k \rangle \triangleq \sum_{g \in G} f(g) E_{\tau_k(g)} \quad \text{for} \quad k = 0, \ldots, n \qquad (1)$$

where $E_{\tau_k(g)} \in \boldsymbol{\psi}_k$ selected by g according to a (surjective) group homomorphism $\tau_k : G \mapsto C_2^k$ (n-bit to k-bits).

The two-modular Fast Fourier Transform

- We can represent the elements of $H_k \cong C_2^k$ as n-bit vectors with the first $n-k$ bits set to zero i.e.,

$$H_k = \{(0,\dots,0,b_{n-k+1},\dots,b_n)|b_i \in \{0,1\}\} \cong C_2^k \quad k=1,\dots,n$$

- The quotient groups G/H_k are represented as n-bit vectors with the last k bits set to zero i.e.,

$$G/H_k = \begin{cases} \{(b_1,\dots,b_{n-k},0,\dots,0)|b_i \in \{0,1\}\} & k=1,\dots,n-1 \\ \{\mathbf{0}_n\} & k=n\ . \end{cases}$$

- Consider the binary subgroup $\langle d_k \rangle = \{\mathbf{0}, d_k\}$ of H_k where

$$d_k = (0,\dots,0,b_{n-k+1}=1,0\dots,0)$$

- Then

$$\underbrace{G}_{2^n} \approx \underbrace{H_k/\langle d_k\rangle}_{2^{k-1}} \times \underbrace{\langle d_k\rangle}_{2} \times \underbrace{G/H_k}_{2^{n-k}}$$

The two-modular Fast Fourier Transform

The k-th Fourier coefficients $\hat{f}_k$ can be explicitly computed by collecting the terms with the same $E_{\tau_k(g)}$, i.e.,

$$\begin{aligned}\hat{f}_k &= \sum_{u\in H_k/\langle d_k\rangle} \left\{ \left[\sum_{v\in G_n/H_k} f(u+v) \right] E_{\sigma_k(u)} \right. \\ &\qquad \left. + \left[\sum_{v\in G_n/H_k} f(u+d_k+v) \right] E_{\overline{\sigma_k(u)}} \right\} \quad k=0,\ldots,n\end{aligned}$$

where $\sigma_k(u)$ is a map converting the n bit vector $u \in H_k/\langle d_k\rangle$ to a k bit vector in C_2^k, i.e., it removes the first $n-k$ zero bits of the n bit vector u. The corresponding $\overline{\sigma_k(u)}$ is the binary complement of the bits of $\sigma_k(u)$.

The two-modular Inverse Fourier Transform

- Let $\pi_k(g) = E_{\tau_k(g)}$ be the $2^k \times 2^k$ representation of an element $g \in C_2^k$ then we define the character of g as

$$\Phi_k(E_{\tau_k(g)}) \triangleq (E_{\tau_k(g)})_{(1,2^k)} \in \{0,1\}$$

  i.e., Φ extracts the top-right corner element of the matrix E_g.
- The representation of the all ones vector $\mathbf{1} = (1,1,\ldots,1)$ yields $\Phi(E_{\mathbf{1}}) = 1$, while any other binary vector representation is mapped to zero.
- Since Φ_k is an homomorphism, we have

$$\Phi_k(E_{\mathbf{a}}E_{\mathbf{b}}) = \Phi_k(E_{\mathbf{a}\oplus\mathbf{b}}) = \begin{cases} 1 & \text{iff } \mathbf{a}\oplus\mathbf{b} = \mathbf{1} \text{ (or } \mathbf{a} = \bar{\mathbf{b}}) \\ 0 & \text{otherwise} \end{cases}$$

The two-modular Inverse Fourier Transform

The inverse Fourier transform is given by

$$f_j = f_{\mathbf{c}} = \hat{f}_0 + \sum_{k=1}^{n} \Phi_k \left(\hat{f}_k E_{\tau_k(\mathbf{c})} \right) \quad j = 0, \ldots, 2^n - 1 \qquad (2)$$

where $\mathbf{c} = (c_n, \ldots, c_k, \ldots, c_1)$ and $j = D(\mathbf{c})$ is the decimal representation of $\mathbf{c}$.

The convolution theorem

- The two-modular Fourier transform can be used to transform convolution in the 'time-domain' defined as

$$(f_1 * f_2)(a) \triangleq \sum_{b \in G} f_1(ab^{-1}) f_2(b) = \sum_{b \in G} f_1(\mathbf{a} \oplus \mathbf{b}) f_2(\mathbf{b})$$

into the product in the 'frequency domain' i.e.,

$$(\widehat{f_1 * f_2})(\rho) = \hat{f}_1(\rho) \hat{f}_2(\rho)$$

- This is where the multiplicative structure of $\mathcal{R}$ plays a role.

Conclusions

Potential applications of the two-modular Fourier transform

- reliable computation of binary functions
- classification of binary functions (polynomial vs. transcendental over $\mathcal{R}$)
- cryptography
- complexity of binary functions

Thank you!