Group embeddings of partial Latin squares

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Latin squares

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A *latin square* of order n is an $n \times n$ matrix in which each of n symbols occurs exactly once in each row and once in each column.

e.g. $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 1 & 4 & 2 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ is a latin square of order 4.

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A partial Latin square (*PLS*) is a matrix, possibly with some empty cells, where no symbol is repeated within a row or column:

e.g. $\begin{pmatrix} 1 & \cdot & \cdot & 4 \\ \cdot & 4 & \cdot & 3 \\ 3 & 1 & \cdot & \cdot \\ \cdot & \cdot & 2 & \cdot \end{pmatrix}$ is a PLS of order 4.

Embedding PLS in groups

The PLS

1	2	3
·	3	1
2	1	·

embeds in $\mathbb{Z}_4$ since...

0	1	3	2
1	3	2	0
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Formally, an *embedding in a group* G is a triple (α, β, γ) of *injective* maps from respectively the rows, columns and symbols, to G, which respects the structure of the group.

[If $(a, b, c) \mapsto \big(\alpha(a), \beta(b), \gamma(c)\big)$ then $\alpha(a)\beta(b) = \gamma(c)$.]

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Injectivity is crucial!

From a PLS to a group

$$P = \begin{array}{c|ccc} & c_1 & c_2 & c_3 \\ \hline r_1 & 1 & 2 & 3 \\ r_2 & \cdot & 3 & 1 \\ r_3 & 2 & 1 & \cdot \end{array}$$

…defines a group

$$\begin{aligned} \langle r_1, r_2, r_3, c_1, c_2, c_3, s_1, s_2, s_3 \mid r_1c_1 = s_1,\ & r_1c_2 = s_2,\ r_1c_3 = s_3, \\ & r_2c_2 = s_3,\ r_2c_3 = s_1, \\ r_3c_1 = s_2,\ & r_3c_2 = s_1 \rangle \end{aligned}$$

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The resulting group/presentation will be denoted $\langle P \rangle$.

Latin trades

A pair of "exchangeable" PLS are known as *Latin trades*

·	2	3	4
·	·	·	·
·	·	4	2
·	3	2	·

·	3	4	2
·	·	·	·
·	·	2	4
·	2	3	·

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There is no finite trade that embeds in $\mathbb{Z}$.

Spherical Latin trades

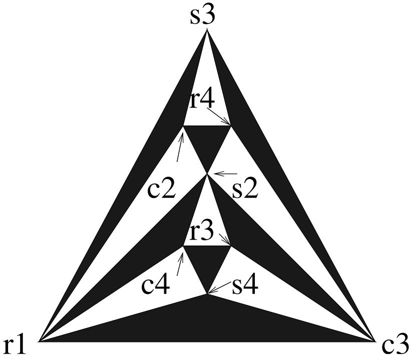
·	2	3	4
·	·	·	·
·	·	4	2
·	3	2	·

·	3	4	2
·	·	·	·
·	·	2	4
·	2	3	·

Spherical Latin trades

·	2	3	4
·	·	·	·
·	·	4	2
·	3	2	·

·	3	4	2
·	·	·	·
·	·	2	4
·	2	3	·



Arguing that black is white!

Cavenagh/W.['09] and Drápal/Hämäläinen/Kala ['10]:

Theorem: Let (W, B) be spherical trades. There is a finite abelian group $A_{W,B}$ such that both W and B embed in $A_{W,B}$.

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0	1	2	3	4	5
1	2	3	4	5	0
2	3	4	5	0	1
3	4	5	0	1	2
4	5	0	1	2	3
5	0	1	2	3	4

W embedded in $\mathbb{Z}_6$

2	3	1	0	·	·
·	·	·	·	·	·
3	1	2	·	·	4
0	·	·	3	·	·
·	·	·	·	·	·
·	·	4	·	·	1

B can't embed in cyclic

Growth rates

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The rank of the canonical group may grow linearly in s.

The minimal group has rank $O(\log s)$.

Smallest PLS not embedding in a group of order n

Open Problem 3.8 in Dénes & Keedwell ['74] asks for the value of $\psi(n)$, the largest number m such that for every PLS P of size m there is some group of order n in which P can be embedded.

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Theorem:

$$\psi(n) = \begin{cases} 1 & \text{when } n = 1, 2, \\ 2 & \text{when } n = 3, \\ 3 & \text{when } n = 4, \text{ or when } n \text{ is odd and } n > 3, \\ 5 & \text{when } n = 6, \text{ or when } n \equiv 2, 4 \mod 6 \text{ and } n > 4, \\ 6 & \text{when } n \equiv 0 \mod 6 \text{ and } n > 6. \end{cases}$$

An abelian variant

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We found that $\psi_+(n)$ is also the largest number m such that every PLS P of size m embeds in the *cyclic* group $\mathbb{Z}_n$.

Upper bounds

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Nevertheless $\psi_+(n) \leqslant \psi(n) < n$ for all n.

Upper bounds

A similar conclusion can be drawn for $n \equiv 2 \mod 4$ because

$$\begin{pmatrix} 1 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & \ddots & \\ & & & & n \end{pmatrix}$$

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We found that $\psi_+(n) = \psi(n) = n - 1$ for $n \in \{2, 3, 4, 6\}$.

Another upper bound

Lemma: For each $\ell \geqslant 2$ there exists a PLS of size 2ℓ that can only be embedded in groups whose order is divisible by ℓ.

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$$C_\ell = \begin{pmatrix} a_1 & a_2 & \cdots & a_{\ell-1} & a_\ell \\ a_2 & a_3 & \cdots & a_\ell & a_1 \end{pmatrix}$$

Suppose that C_ℓ is embedded in rows indexed r_1 and r_2 of the Cayley table of a group G. From the regular representation of G as used in Cayley's theorem, it follows that $r_1^{-1}r_2$ has order ℓ in G. In particular ℓ divides the order of G.

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For odd $n > 5$ it follows that $\psi_+(n) = \psi(n) \leqslant 3$, and
for $n \equiv 2, 4 \mod 6$, $n > 4$ it follows that $\psi_+(n) = \psi(n) \leqslant 5$.

A uniform upper bound

The following pair of PLS of size 7

$$\begin{pmatrix} a & b & \cdot \\ c & a & b \\ \cdot & c & d \end{pmatrix} \qquad \begin{pmatrix} a & b & \cdot \\ c & a & b \\ \cdot & d & a \end{pmatrix}$$

each fail the so-called *quadrangle criterion* and hence neither can be embedded into *any* group.

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Hence $\psi_+(n) = \psi(n) \leqslant 6$ for all n.

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We now have only finitely many PLS to consider.

Reducing the list of candidates

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size	1	2	3	4	5	6	7
#species	1	2	5	18	59	306	1861
reduced#	0	0	0	2	0	11	50

The interesting ones

Of the 11 PLS(6), the two most interesting are

$$\begin{pmatrix} a & \cdot & \cdot & \cdot & \cdot & c \\ \cdot & a & \cdot & \cdot & b & \cdot \\ \cdot & \cdot & b & c & \cdot & \cdot \end{pmatrix} \qquad \begin{pmatrix} a & b & \cdot \\ c & \cdot & b \\ \cdot & c & d \end{pmatrix}$$

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The left one doesn't embed in either group of order 6, and the right one doesn't embed in *any* abelian group.
Of the 50 PLS(7), there are 42 embed in $\mathbb{Z}_6$, 4 others embed in D_6, and 2 don't embed in any group. The other two are

$$\begin{pmatrix} a & b & c \\ b & a & \cdot \\ c & \cdot & a \end{pmatrix} \qquad \begin{pmatrix} a & b & c \\ b & c & \cdot \\ c & \cdot & a \end{pmatrix}$$

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The first embeds in any group that has more than one element of order 2. The second embeds in any group with an element of order 4.

Summary

Smallest PLSs which are obstacles for $\psi(n)$

n	smallest	#	obstacles
2,3,4	n	$\lfloor n/2 \rfloor$	Evans
odd$\geqslant 5$	3	1	C_2
6	6	5	Evans,transversal,sporadic
$2,4 \mod 6$	6	1	C_3
$0 \mod 12$	7	2	Quad.Crit.
$6 \mod 12$	7	3	Quad.Crit., el of order 4

Hirsch and Jackson (2012)

*	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3		10					
3	3							13	
4	4			5		11			
5	5	10							
6	6					7		12	
7	7			11					
8	8	13						9	
9	9					12			

FIGURE 1. A partial group embeddable in a group but not into any finite group.

Hirsch and Jackson (2012)

ROBIN HIRSCH AND MARCEL JACKSON

*	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	3		10					
3	3							13	
4	4			5		11			
5	5	10							
6	6					7		12	
7	7			11					
8	8	13						9	
9	9					12			

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Embeds in Higman's (1951) group

$$\langle a, b, c, d | ab = bba, bc = ccb, cd = ddc, da = aad \rangle$$

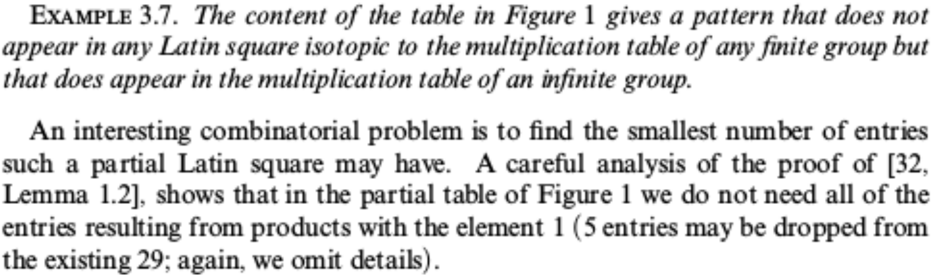
which has no non-trivial *finite* quotients.

Hirsch and Jackson (2012)

Example 3.7. *The content of the table in Figure* 1 *gives a pattern that does not appear in any Latin square isotopic to the multiplication table of any finite group but that does appear in the multiplication table of an infinite group.*

An interesting combinatorial problem is to find the smallest number of entries such a partial Latin square may have. A careful analysis of the proof of [32, Lemma 1.2], shows that in the partial table of Figure 1 we do not need all of the entries resulting from products with the element 1 (5 entries may be dropped from the existing 29; again, we omit details).

Hirsch and Jackson (2012)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After some thought, we found an example of size 14, at which point exhaustive enumeration started to look practical.

Hirsch and Jackson (2012)

Example 3.7. *The content of the table in Figure 1 gives a pattern that does not appear in any Latin square isotopic to the multiplication table of any finite group but that does appear in the multiplication table of an infinite group.*

An interesting combinatorial problem is to find the smallest number of entries such a partial Latin square may have. A careful analysis of the proof of [32, Lemma 1.2], shows that in the partial table of Figure 1 we do not need all of the entries resulting from products with the element 1 (5 entries may be dropped from the existing 29; again, we omit details).

After some thought, we found an example of size 14, at which point exhaustive enumeration started to look practical.

After testing some “likely suspects” we then found one of size 12.

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conn.	1	1	3	11	36	213	1405	12274	125235	1490851
red.	0	0	0	2	0	11	50	489	6057	92533

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But this is only the beginning of the problems. For each PLS, we may have to solve a (potential undecidable!) word problem.

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This last step was only needed for PLS of size 12.

size	None	Fab	Fnonab	nFab	nFnonab	∞
4	0	1	0	1	0	0
6	0	7	1	3	0	0
7	2	37	4	7	0	0
8	16	401	32	34	6	0
9	147	5153	412	294	51	0
10	2402	78343	6784	4212	792	0
11	42884	1272586	120767	66230	14826	0
12	854559	22297343	2365541	1223063	316109	50

None : cannot be embedded in any group

Fab : in free group and in finite abelian group

Fnonab : in free group, not in any abelian group, but in finite non-abelian group

nFab : in finite abelian group but not in free group

nFnonab : in finite non-abelian group, but not in in free group, nor any abelian group

∞ : in an infinite group, but no finite group

The smallest example:

The PLS

$$P = \begin{pmatrix} a & b & c & d & \cdot \\ b & e & f & \cdot & d \\ c & \cdot & \cdot & f & \cdot \\ \cdot & \cdot & \cdot & e & a \end{pmatrix}$$

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Baumslag ['69] considered

$$B = \langle u, v \mid u = [u, u^v] \rangle,$$

where, as usual, $u^v = v^{-1}uv$ and $[u, u^v] = u^{-1}u^{(u^v)}$. He proved that B is infinite, but $u = 1$ in every finite quotient of B.

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If two labels coincided then $\langle P \rangle$ would be cyclic, which it isn't.

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Theorem: Let Δ be the diagonal PLS of size n with $\Delta(i,i) = a$ for $i \leqslant 3$ and $\Delta(i,i) = b$ for $4 \leqslant i \leqslant n$. Then Δ has an embedding into a group G of order n if and only if n is divisible by 3.