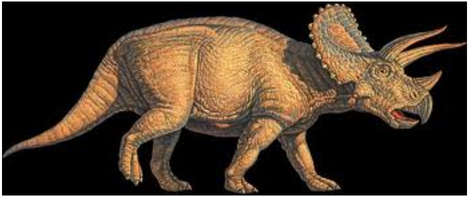
Triceratopisms of Latin Squares

Ian Wanless



Joint work with Brendan McKay and Xiande Zhang

Latin squares

A *Latin square* of order n is an $n \times n$ matrix in which each of n symbols occurs exactly once in each row and once in each column.

e.g.

1	2	3	4
2	4	1	3
3	1	4	2
4	3	2	1

is a Latin square of order 4.

The Cayley table of a finite (quasi-)group is a Latin square.

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- The multiset $\{\alpha, \beta, \gamma\}$.
- The cycle structure of α, β, γ.

Number of possible cycle structures

n	3 diff	2 diff	$\#\text{aut}(n)$	$\#\text{atp}(n)$
1			1	1
2		1	1	2
3		1	3	4
4		5	4	9
5		1	5	6
6	1	11	6	18
7		1	9	10
8		25	12	37
9		10	13	23
10	1	23	14	38
11		1	18	19
12	7	113	26	146
13		1	24	25
14	1	37	24	62
15	1	34	39	74
16		151	50	201
17		1	38	39

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Conjecture: For almost all $\alpha \in \mathcal{S}_n$ there are no $\beta, \gamma \in \mathcal{S}_n$ such that $(\alpha, \beta, \gamma) \in \mathrm{atp}(n)$.

McKay, Meynert and Myrvold 2007

Theorem: Let L be a Latin square of order n and let (α, β, γ) be a nontrivial autotopism of L. Then either

(a) α, β and γ have the same cycle structure with at least 1 and at most $\left\lfloor \frac{1}{2}n \right\rfloor$ fixed points, or

(b) one of α, β or γ has at least 1 fixed point and the other two permutations have the same cycle structure with no fixed points, or

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Corollary: Suppose Q is a quasigroup of order n and that $\alpha \in \mathrm{aut}(Q)$ with $\alpha \neq \varepsilon$.

1. If α has a cycle of length $c > n/2$, then $\mathrm{ord}(\alpha) = c$.
2. If p^a is a prime power divisor of $\mathrm{ord}(\alpha)$ then $\psi(\alpha, p^a) \geqslant \frac{1}{2}n$.

(Here $\psi(\alpha, k)$ is #points that appear in cycles of α for which the cycle length is divisible by k.)

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Corollary: A random permutation is not an automorphism of a quasigroup, Steiner triple system, or 1-factorisation of K_n; nor is it a component of an autotopism, autoparatopism or triceratopism of a latin square.

Prime orders

Theorem: Suppose Q is a quasigroup of order n and that $\theta = (\alpha, \beta, \gamma)$ is an autotopism of Q. If k is a prime power divisor of $\mathrm{ord}(\theta)$ and k does not divide n then

$$\psi(\alpha, k) = \psi(\beta, k) = \psi(\gamma, k) \geqslant \tfrac{1}{2}n.$$

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This is a strong restriction. For prime $n \leqslant 29$ it only leaves

$n = 23$, $(6^2, 3, 2, 1^6)$ and $2 \times (6, 3^3, 2^4)$.
$n = 29$, $(6^2, 3, 2^4, 1^6)$ and $2 \times (6, 3^3, 2^7)$.
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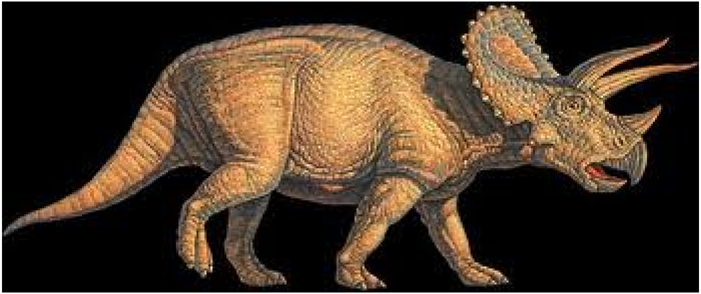
But is it possible for prime order to have three different cycle structures?

Triceratopisms

An autotopism consisting of 3 permutations with different cycle structures is a *triceratopism*.



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Some simple cases

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Corollary: Suppose 2^a is the largest power of 2 dividing n, where $a \geqslant 1$. Suppose each cycle in α, β and γ has length divisible by 2^a. Then $(\alpha, \beta, \gamma) \notin \mathrm{atp}(n)$.

lcm conditions

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Theorem: If at least two of R_Λ, C_Λ and S_Λ are nonempty, then $|R_\Lambda| = |C_\Lambda| = |S_\Lambda|$ and there is a Latin subsquare M on the rows R_Λ, columns C_Λ and symbols S_Λ. Moreover, M admits an autotopism that is a restriction of the original autotopism.