Is The Missing Axiom of Matroid Theory Lost Forever?
or
How Hard is Life Over Infinite Fields?

General Theme

- There exist strong theorems for matroids representable over finite fields, but it all turns to custard for infinite fields.

General Theme

- There exist strong theorems for matroids representable over finite fields, but it all turns to custard for infinite fields.
- In this talk "the reals" will be code for any infinite field.

Well-quasi-ordering

Matroids over a finite field are well-quasi-ordered.

Well-quasi-ordering

Matroids over a finite field are well-quasi-ordered.

- Matroids over an infinite field are not.

Serious Custard

Rota's Conjecture

For any fixed finite field $\mathbb{F}$ there are a finite number of forbidden minors for $\mathbb{F}$ representability.

Serious Custard

Rota's Conjecture

For any fixed finite field $\mathbb{F}$ there are a finite number of forbidden minors for $\mathbb{F}$ representability.

Theorem (Mayhew, Newman, W)

For any real-representable matroid M, there is an excluded minor for real representability that contains M as a minor.

Minor-closed properties

Can recognise any minor-closed property in polynomial time for matroids representable over a finite field.

Minor-closed properties

Can recognise any minor-closed property in polynomial time for matroids representable over a finite field.

- Cannot recognise uniform matroids over the reals.

Deciding Representability

(Seymour) Let M be a matroid given by a rank oracle. Then it requires exponentially many calls to the oracle to decide if M is binary.

Deciding Representability

(Seymour) Let M be a matroid given by a rank oracle. Then it requires exponentially many calls to the oracle to decide if M is binary.

- This extends easily to any other field, finite or infinite.

Certifying non-representability

- It requires only a polynomial number of calls to a rank oracle to prove that a matroid is not representable over a prime field.

Certifying non-representability

- It requires only a polynomial number of calls to a rank oracle to prove that a matroid is not representable over a prime field.
- Modulo Rota it requires only a constant number of calls.

Certifying non-representability

- It requires only a polynomial number of calls to a rank oracle to prove that a matroid is not representable over a prime field.
- Modulo Rota it requires only a constant number of calls.
- (ben David and Geelen) It requires exponentially many calls to prove that M is not representable over the reals.

Branch Width

- Bounding branch width gives great control over finite fields.

Branch Width

- Bounding branch width gives great control over finite fields.
- Bounding branch width gives no control over infinite fields.

Whitney's Comment

The fundamental problems of deciding which matroids are matrix is left unsolved.

Whitney's Comment

The fundamental problems of deciding which matroids are matrix is left unsolved.

- Whitney almost certainly had real representable matroids in mind.

Whitney's Comment

The fundamental problems of deciding which matroids are matrix is left unsolved.

- Whitney almost certainly had real representable matroids in mind.
- Search for the missing axiom of matroid theory!

The Rank Axioms

E a finite subset of $\mathbb{R}^n$. For $A \subseteq E$, the *rank* of A, denoted $r(A)$, is the size of a max independent subset of A. We have:

- R1 $r(\emptyset) = 0$.
- R2 If $e \in E$, then $0 \leq r(\{e\}) \leq 1$.
- R3 If $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$.
- R4 If $A, B \subseteq E$, then $r(A) + r(B) \geq r(A \cap B) + r(A \cup B)$.

A *matroid* is a finite set E together with a function $r : 2^E \to \mathbb{Z}$ satisfying **R1**, **R2**, **R3** and **R4**.

Theorem (Tutte)

A matroid is binary if and only if it has no $U_{2,4}$-minor.

Theorem (Tutte)

A matroid is binary if and only if it has no $U_{2,4}$-minor.

No $U_{2,4}$ minor is equivalent to

R5 For all $X \subseteq E$, it is not that case that there exists $Y \subseteq E - X$ with $|Y| = 4$ such that for all $Z \subseteq Y$, $r(X \cup Z) = |X| + |Z|$ if $|Z| \leq 2$, and otherwise $r(X \cup Z) = r(X) + 2$.

Theorem (Tutte)

A matroid is binary if and only if it has no $U_{2,4}$-minor.

No $U_{2,4}$ minor is equivalent to

R5 For all $X \subseteq E$, it is not that case that there exists $Y \subseteq E - X$ with $|Y| = 4$ such that for all $Z \subseteq Y$, $r(X \cup Z) = |X| + |Z|$ if $|Z| \leq 2$, and otherwise $r(X \cup Z) = r(X) + 2$.

We've found the missing axiom of binary matroids!

Theorem

A matroid is binary if and only if it satisfies R1, R2, R3, R4 and R5.

Vamos 1978 paper. "The missing axiom of matroid theory is lost forever."

Theorem (Vamos)

It is not possible to add a finite number of axioms expressed in first order logic *to the matroid axioms to characterise real representability.*

- Vamos' proof uses the fact that reals have an infinite number of excluded minors and the Compactness Theorem from logic.

- Vamos' proof uses the fact that reals have an infinite number of excluded minors and the Compactness Theorem from logic.
- But the proof only *needs* the fact that these are forbidden *submatroids.*

- Vamos' proof uses the fact that reals have an infinite number of excluded minors and the Compactness Theorem from logic.
- But the proof only *needs* the fact that these are forbidden *submatroids.*
- Binary matroids have an infinite number of forbidden submatroids, ie $U_{n,n+2}$ for all $n \geq 2$.

- Vamos' proof uses the fact that reals have an infinite number of excluded minors and the Compactness Theorem from logic.
- But the proof only *needs* the fact that these are forbidden *submatroids*.
- Binary matroids have an infinite number of forbidden submatroids, ie $U_{n,n+2}$ for all $n \geq 2$.
- Therefore Vamos' proof works for binary matroids!

What is going on?

Vamos' First Order Logic

Can quantify over elements. R1 and R2 are first order statement.

What is going on?

Vamos' First Order Logic

Can quantify over elements. R1 and R2 are first order statement.

- R3 and R4 are not first order statements.

What is going on?

Vamos' First Order Logic

Can quantify over elements. R1 and R2 are first order statement.

- R3 and R4 are not first order statements.
- Note that R5 was similar to R3 and R4.

What is going on?

Vamos' First Order Logic

Can quantify over elements. R1 and R2 are first order statement.

- R3 and R4 are not first order statements.
- Note that R5 was similar to R3 and R4.
- In Vamos' logic it's probably not possible to define matroids with a finite number of first order statements.

The Real Question

- Is it possible to add a finite number of axioms in some sort of "natural" logic for matroids that characterises real representability?

Uniform Logic

- Can quantify over elements and *sets*, but we do not allow alternating quantifiers.

Uniform Logic

- Can quantify over elements and *sets*, but we do not allow alternating quantifiers.
- Note that rank axioms are all of this form.

Uniform Logic

- Can quantify over elements and *sets*, but we do not allow alternating quantifiers.
- Note that rank axioms are all of this form.

Theorem (Mayhew, Newman, W.)

Not possible to characterise real representable matroids in this logic.

Uniform Logic

- Can quantify over elements and *sets*, but we do not allow alternating quantifiers.
- Note that rank axioms are all of this form.

Theorem (Mayhew, Newman, W.)

Not possible to characterise real representable matroids in this logic.

Proof uses generalised Ingleton Conditions of Kinser.

Conjecture

It is not possible to characterise real-representable matroids in *monadic second order logic*.

Robertson, Seymour Conjecture

The class of matroids with no $U_{n,2n}$, $M(G_n)$, $B(G_n)$ and $B^*(G_n)$ minor has bounded branch width.

My Favourite Conjecture

Let $\mathcal{R}$ be the set of real representable matroids and $\mathcal{R}^+$ be the set of real representable matroids together with the set of excluded minors for real representability.

Conjecture (Mayhew, Newman, W.)

For all $\epsilon > 0$, there is an N such that if $n > N$, then the proportion of n-element members of $\mathcal{R}^+$ that are in $\mathcal{R}$ is less than ϵ.