A study of 3-arc graphs

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Outline

- Introduction
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- Domination in 3-arc graphs
- Coloring 3-arc graphs
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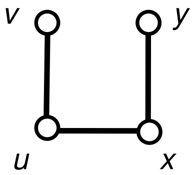
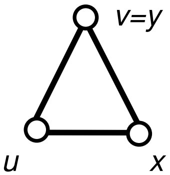
Introduction

definition

An *arc* of a graph G is an ordered pair of adjacent vertices.
A *3-arc* of G is a 4-tuple (v, u, x, y) of vertices such that both (v, u, x) and (u, x, y) are paths of length two G. ($v = y$ is allowed.)

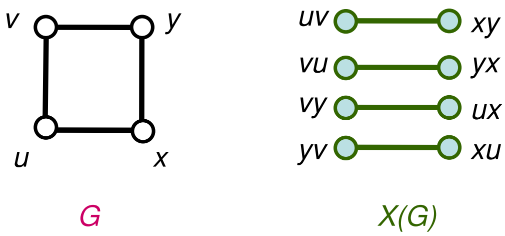




Introduction

definition (Li, Praeger and Zhou'00)

Let G be a graph. The *3-arc graph* of G, denoted by $X(G)$, is defined to have for vertex set the set of arcs of G. Two vertices corresponding to two arcs uv and xy are adjacent in $X(G)$ if and only if (v, u, x, y) is a 3-arc of G.



Introduction

observation

$X(G)$ is an undirected graph with $2|E(G)|$ vertices and $\sum_{\{u,x\}\in E(G)}(\deg_G(u)-1)(\deg_G(x)-1)$ edges.

observation

Relationship between *3-arc graph* and *line graph*:
Split each vertex $\{u, v\}$ of $L(G)$ into two vertices uv and vu. Then, for any two vertices $\{u, v\}$, $\{x, y\}$ of $L(G)$ which are distance two apart in $L(G)$, say, u and x are adjacent in G, join uv and xy by an edge. The graph obtained this way is isomorphic to $X(G)$.

Introduction

theorem (Knor and Zhou'09)

If $\kappa(G) \geq 3$, then

$$\kappa(X(G)) \geq (\kappa(G) - 1)^2$$

and this bound is best possible.

theorem (Knor and Zhou'09)

If G is connected with $\delta(G) \geq 3$, then

$$\mathrm{diam}(G) \leq \mathrm{diam}(X(G)) \leq \mathrm{diam}(G) + 2$$

and both bounds are attainable.

Independence in 3-arc graphs

theorem

Let G be a graph with $\delta(G) \geq 2$. Then

$$\alpha(X(G)) = \max_S \left\{ \alpha(G_S) + \sum_{v \in S} \deg_G(v) \right\},$$

where the maximum is taken over all maximal independent sets S of G, and G_S is the subgraph of G induced by those vertices which have exactly one neighbour in S.

theorem

Let G be a connected d-regular graph with $d \geq 2$. Then

$$d \leq \frac{\alpha(X(G))}{\alpha(G)} \leq d+1,$$

and both bounds are attainable.

Independence in 3-arc graphs

theorem

Let G be a bipartite graph with $\delta(G) \geq 2$. Then

$$\alpha(X(G)) = |E(G)|.$$

Domination in 3-arc graphs

definition

A *k-dominating set* of a graph H is a subset S of $V(H)$ such that $|N(u) \cap S| \geq k$ for every $u \in V(H) - S$. The *k-domination number* of H, denoted by $\gamma_k(H)$, is the minimum cardinality of a k-dominating set of H.

The (normal) *domination number* is $\gamma_1(H) = \gamma(H)$.

Domination in 3-arc graphs

theorem

Let G be a connected graph of order $n \geq 4$ and $\delta(G) \geq 2$. Then

$$3 \leq \gamma(X(G)) \leq n,$$

and both bounds are sharp. Moreover, $\gamma(X(G)) = n$ if and only if G is an n-cycle, and $\gamma(X(G)) = 3$ if and only if G contains a 3-cycle C_3 such that $|N(u) \cap V(C_3)| \geq 2$ for every $u \in V(G)$. Furthermore, for each integer k with $3 \leq k \leq n$, there exists a graph G with $\delta(G) \geq 2$ and order n such that $\gamma(X(G)) = k$.

Domination in 3-arc graphs

theorem

Let G be a graph with $\delta(G) \geq 2$. Then

$$\gamma(X(G)) \leq 2\gamma_2(G),$$

and this bound is sharp.

Domination in 3-arc graphs

theorem

Let G be a graph with n vertices, m edges, maximum degree Δ and minimum degree $\delta \geq 2$. Then

$$\left\lceil \frac{2n}{\Delta} \right\rceil \leq \gamma(X(G)) \leq \frac{2m}{\delta^2 - 2\delta + 2}\left(\ln(\delta^2 - 2\delta + 2) + 1 \right).$$

Moreover, the lower bound is sharp.

corollary

Let G be a d-regular graph of order n, where $delta \geq 3$. Then

$$\left\lceil \frac{2n}{d} \right\rceil \leq \gamma(X(G)) < \frac{n}{d-2}\left(\ln(d^2 - 2d + 2) + 1 \right),$$

and the lower bound is sharp.

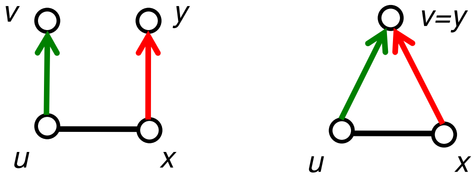
Coloring 3-arc graphs

definition

A coloring of arcs of G is called a *3-arc coloring* if under this coloring any two arcs uv and xy with $v \neq x$, $y \neq u$ and u and x adjacent in G receive different colors.

The *3-arc chromatic index* of G, denoted by $\chi'_3(G)$, is the minimum number of colors needed by a 3-arc coloring of G.

Equivalently, $\chi'_3(G)$ is defined as the chromatic number of $X(G)$.



Coloring 3-arc graphs

theorem

The following hold:

(a) If T_n is a transitive tournament on n vertices, then $\chi'_3(T_n) = n - 1$;

(b) $\chi'_3(K_n) = n - 1$;

(c) for a connected graph G, $\chi'_3(G) = 1$ if and only if G is a star;

(d) for a connected graph G, $\chi'_3(G) = 2$ if and only if G is not a star and the subgraph of G induced by the vertices of degree at least three is bipartite;

(e) if H is a Halin graph, then $\chi'_3(H) = 2$ if H is bipartite and $\chi'_3(H) = 3$ otherwise.

Coloring 3-arc graphs

theorem

Let G be a connected graph. Then

$$\left\lceil \frac{\chi(G)+1}{3} \right\rceil \leq \chi'_3(G) \leq \chi(G),$$

and both bounds are attainable.

theorem

The problem of deciding whether $\chi'_3(G) \leq 3$ for a planar graph G is NP-complete.

Open problems

- Problem 1. Characterize 3-arc graphs of connected graphs.
- Problem 2. Let G be a connected graph with $\delta(G) \geq 3$. Under what conditions is $X(G)$ Hamiltonian?
- Problem 3. Give a sharp upper bound on $\gamma(X(G))$ in terms of $\gamma(G)$ for any connected graph G with $\delta(G) \geq 2$.

Thank you!