Two domination parameters in graphs

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Joint work with Liying Kang, Erfang Shan and Min Zhao
Outline

1. Domination in graphs

Xu

Two domination parameters in graphs
Outline

1. Domination in graphs

2. Power domination

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Outline

1. Domination in graphs
2. Power domination
3. Rainbow domination
Definition

- A subset $S \subseteq V$ is a *dominating set* of a graph $G = (V, E)$ if every vertex in $V - S$ has at least one neighbor in $S$.

Other definitions:

- $N[S] = V$;
- For every vertex $v \in V - S$, $d(v, S) \leq 1$;
- For every vertex $v \in V$, $|N[v] \cap S| \geq 1$;

The domination number $\gamma(G)$ of $G$ is the minimum cardinality of a dominating set of $G$. 

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Xu Two domination parameters in graphs
Theorem. **DOMINATING SET** is NP-complete for bipartite graphs, split graphs ($\subseteq$ chordal graph), arbitrary grids.
Known results

- **Theorem.** *DOMINATING SET* is NP-complete for bipartite graphs, split graphs ($\subset$ chordal graph), arbitrary grids.

- **Theorem.** (Ore 1962) If a graph $G$ of order $n$ has no isolated vertices, then $\gamma(G) \leq n/2$. 

Domination variants

- Harary and Haynes defined the *conditional domination number* $\gamma(G : P)$: the smallest cardinality of a dominating set $S \subseteq V$ such that the subgraph $\langle S \rangle$ induced by $S$ satisfies some *graph property* $P$. 
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Background

- An **electrical power system** includes a set of buses and a set of lines connecting the buses. A **bus** is a substation where transmission lines are connected.
- The state of an electrical power system can be represented by a set of state variables, for example, the voltage magnitude at loads and the machine phase angle at generators.
- Monitor the system’s state by putting **Phase Measurement Units (PMUs)** at selected locations in the system.
An electrical power system

A typical electrical power system.

http://www.menard.com/mec_power_system.html
A transmission substation/bus.

http://www.menard.com/mec_power_system.html
PMUs - a key component of electric power grid modernization. The PMUs are the two instruments on top of the cabinet.

http://qdev.boulder.nist.gov/817.03/whatwedo/volt/watt/watt.htm
Observation rules: basic rule

- **Basic rule**: A PMU measures the state variables (voltage, phase angle, etc) for the bus (vertex) at which it is placed and its incident edges and their endvertices.
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Observation rules: rule 1

- **Rule 1**: Any bus (vertex) that is incident to an observed line connected to an observed bus is observed (vertex).

\[ V = IR \]  
Ohm's Law, the known current in the line, the known voltage at the observed bus, and the known resistance of the line determine the voltage at the bus.
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Observation rules: rule 2

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Observation rules: rule 3

- **Rule 3**: If all the lines incident to an observed bus are observed, except one, then all of the lines incident to that bus are observed.

![Diagram](image_url)
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Power dominating set: definition

- An electrical power system $\rightarrow$ a graph where the vertices/edges represent the buses/transmission lines, respectively.
A subset $S$ is a \textit{power dominating set (PDS)} of $G$ if every vertex and every edge in $G$ is observed by $S$ according to the following Observation Rules.

- Basic rule: Every edge incident to some vertex of $S$ and every vertex of $N[S]$ are observed;
- $R_1$: Any vertex that is incident to an observed edge is observed;
- $R_2$: Any edge joining two observed vertices is observed;
- $R_3$: If a vertex is incident to a total of $k > 1$ edges and if $k-1$ of these edges are observed, then all $k$ of these edges are observed.

The \textit{power domination number} $\gamma_p(G)$ of $G$ is the minimum cardinality of a power dominating set of $G$. 

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Known results

- **Observation 1** (Haynes et al., 2002). For any graph $G$, $1 \leq \gamma_p(G) \leq \gamma(G)$. 

- **Theorem 3**. *POWER DOMINATING SET* is NP-complete in bipartite, split ($\subseteq$ chordal), circle, and planar graphs.

- **Theorem 4** (Haynes et al., 2002). *POWER DOMINATING SET* is linear time solvable for trees.

- **Theorem 5** (Haynes et al., 2002). For any tree $T$ of order $n \geq 3$, $\gamma_p(G) \leq n/3$ with equality if and only if $T$ is the corona $T'$ $\circ$ $\overline{K}_2$, where $T'$ is any tree.
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Block graphs

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Block graphs

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- $G$ is called a block graph if every block of $G$ is a complete graph.
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Trees are block graphs (i.e., trees $\subseteq$ block graphs).
A block graph $G$ with five blocks $BK_1 = G[\{a, b, d\}]$, $BK_2 = G[\{c, e\}]$, $BK_3 = G[\{d, e\}]$, $BK_4 = G[\{d, g, h\}]$ and $BK_5 = G[\{e, f, i, j\}]$. 
Let $G$ be a block graph with $h$ blocks $BK_1, ..., BK_h$ and $p$ cut-vertices $v_1, ..., v_p$. 
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cut-vertices $v_1,...,v_p$.

The \textit{refined cut-tree} $T^B(V^B, E^B)$ of $G$ is defined as
$V^B = \{B_1,...,B_h, v_1,...,v_p\}$, where each $B_i := \{v \in BK_i \mid v$
is not a cut-vertex$\}$ is called a \textit{block-vertex} of $T^B$, and
$E^B = \{(B_i, v_j) \mid v_j \in BK_i, 1 \leq i \leq h, 1 \leq j \leq p\}$.
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The refined cut-tree of a block graph can be constructed in linear time.
Refined cut-tree: an example

A block graph $G$ and its one refined cut-tree, where $B_1 = \{a, b\}$, $B_2 = \{c\}$, $B_3 = \emptyset$, $B_4 = \{g, h\}$ and $B_5 = \{f, i, j\}$. 

Two domination parameters in graphs
A block graph $G$ and its one refined cut-tree, where $B_1 = \{a, b\}$, $B_2 = \{c\}$, $B_3 = \emptyset$, $B_4 = \{g, h\}$ and $B_5 = \{f, i, j\}$.
**Lemma.** Let $G$ be a block graph, then there exists a minimum power dominating set in which every vertex is a cut-vertex of $G$. 
Algorithm: part one

Algorithm. Find a minimum PDS of $G$.

**Input:** A connected block graph $G$ of order $n \geq 3$.

**Output:** A minimum PDS of $G$.

Construct a refined cut-tree $T^B(V^B, E^B)$ of $G$ with vertex set \{v_1, v_2, \ldots, v_n\}, and the root is a cut-vertex $v_n$ (in $G$). For every vertex $v_j$ that lies in the odd levels $H_1, H_3, \ldots, H_k$, relabel $v_j$ as $v_j^B$ (the superscript $B$ of $v_j^B$ indicates that it is a block-vertex and $v_i^B$ corresponds to $B_i$ one by one).

Initialization: $S := \emptyset$; for every vertex $v \in V^B$, mark $v$ with white and set $\text{bound}(v) := 0$. 
for \( i := k - 1 \) down to 0 by step-length 2 do
  for every \( v_j \in H_i \) do
    if there exists a gray vertex \( v_a^B \in N(v_j) \cap H_{i+1} \) or a white vertex \( v_b^B \in N(v_j) \cap H_{i+1} \) so that \( B_b = \emptyset \) and all vertices of \( N(v_b^B) \cap H_{i+2} \) are gray and there exists at least one vertex \( v \in N(v_b^B) \cap H_{i+2} \) with \( \text{bound}(v) = 0 \) then
      mark \( v_j \) with gray;
      for every white \( v_z^B \in N(v_j) \cap H_{i+1} \) do
        if \( N(v_z^B) \cap H_{i+2} \) contains at least one gray vertex \( v \) with \( \text{bound}(v) = 0 \) then
          \{if \( |B_z| = 0 \) and \( N(v_z^B) \cap H_{i+2} \) contains at most one white vertex or \( |B_z| = 1 \) and \( N(v_z^B) \cap H_{i+2} \) contains no white vertex then mark \( v_z^B \) with gray\};
        if \( |B_z| = 0 \) and every vertex \( v \in N(v_z^B) \cap H_{i+2} \) is gray and \( \text{bound}(v) = 1 \) then
          mark \( v_z^B \) with gray;
      end for
    end if
if $i \geq 0$ then

\[
\{ W := \{ v^B \mid v^B \in N(v_j) \cap H_{i+1} \text{ and } v^B \text{ is white} \}; \\
C^v_j := \{ u \mid u \in N(W) \cap H_{i+2} \text{ and } u \text{ is white} \}; \\
B^v_{Wj} := \bigcup_{\text{forall } v^B_m \in W} B_{m} \};
\]

if $v_j \neq v_n$ then

\{ if $|B^v_{Wj} \cup C^v_j| \geq 2$ then

\{ mark $v_j$ with black and all white vertices in $N(v_j)$ with gray; \\
$S := S \cup \{ v_j \}$;

if $|B^v_{Wj} \cup C^v_j| = 1$ and $v_j$ is gray then set $\text{bound}(v_j) := 1$ \}

if $v_j = v_n$ then

\{ if $|B^v_{Wj} \cup C^v_j| \geq 2$ or $v_j$ is white then

\{ mark $v_j$ with black and all white vertices in $N(v_j)$ with gray; \\
$S := S \cup \{ v_j \}$\};

end for

end for

output $S$.  

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Algorithm: an example

(a) Domination in graphs
(b) Power domination
(c) Rainbow domination

Algorithm: an example

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Two domination parameters in graphs
Theorem

*PDS can be solved in linear time for block graphs.*
For any block graph $G$ with order $n \geq 3$, $\gamma_p(G) \leq n/3$ with equality if and only if $G$ is obtained from $G'$ by attaching to each vertex of $G'$ a copy of $K_2$ or $\overline{K}_2$, where $G'$ is any block graph.
(Bresăr, Henning and Rall, 2005) Let $C = \{1, 2, \ldots, k\}$ be a set of $k$ colors, and $f$ be a function that assigns to each vertex a set of colors chosen from $C$, that is, $f : V(G) \mapsto \mathcal{P}(C)$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have

$$\bigcup_{u \in N(v)} f(u) = C$$

then $f$ is called a \textit{k-rainbow dominating function} (kRDF) of $G$. The \textit{weight}, $w(f)$, of a function $f$ is defined as $w(f) = \sum_{v \in V(G)} |f(v)|$. 

Rainbow domination: definition

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\[
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then $f$ is called a $k$-rainbow dominating function ($k$RDF) of $G$. The weight, $w(f)$, of a function $f$ is defined as
\[
w(f) = \sum_{v \in V(G)} |f(v)|.
\]
The minimum weight, denote by $\gamma_{rk}(G)$, of a $k$RDF is called the $k$-rainbow domination number of $G$.\[\]
Rainbow domination: definition

(Breslar, Henning and Rall, 2005) Let $C = \{1, 2, ..., k\}$ be a set of $k$ colors, and $f$ be a function that assigns to each vertex a set of colors chosen from $C$, that is, $f : V(G) \mapsto \mathcal{P}(C)$. If for each vertex $v \in V(G)$ such that $f(v) = \emptyset$ we have

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- The minimum weight, denote by $\gamma_{rk}(G)$, of a $k$RDF is called the $k$-rainbow domination number of $G$.
- If $k = 1$, ordinary domination.
Known results

- **Observation 1** (Bresar, Henning and Rall, 2005). For $k \geq 1$ and any graph $G$, $\gamma_{rk}(G) = \gamma(G \Box K_k)$. 
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- **Observation 2** (Bresar, Henning and Rall, 2007). For a path $P_n$ and a cycle $C_n$ with $n \geq 3$, $\gamma_{r2}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor + 1$, $\gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor$. 
Known results

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  \gamma_{r2}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor + \left\lceil \frac{n}{4} \right\rceil - \left\lfloor \frac{n}{4} \right\rfloor.
  \]

- **Theorem 3**. (Bresăr, Henning and Rall, 2007) 2-RAINBOW DOMINATING FUNCTION is NP-complete for chordal graphs and bipartite graphs.

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Two domination parameters in graphs
(Watkins, 1969) For each $n$ and $k$ ($n > 2k$), the *generalized Petersen graph* $P(n, k)$ is a graph with vertex set \[
\{u_i, v_i : i = 0, 1, 2, \ldots, n - 1\}\] and edge set \[
\{u_i u_{i+1}, u_i v_i, v_i v_{i+k} : i = 0, 1, 2, \ldots, n - 1\};\] subscripts are taken modulo $n$. 

Theorem (Bresˇ ar, Henning and Rall, 2007). For the generalized Petersen graph $P(n, k)$, 
$$
\lceil \frac{4}{5} n \rceil \leq \gamma_{r2}(P(n, k)) \leq n.
$$
(Watkins, 1969) For each \( n \) and \( k \) (\( n > 2k \)), the \textit{generalized Petersen graph} \( P(n, k) \) is a graph with vertex set 
\( \{u_i, v_i : i = 0, 1, 2, \ldots, n - 1\} \) and edge set 
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taken modulo \( n \).

\textbf{Theorem} (Bresăr, Henning and Rall, 2007). For the 
generalized Petersen graph \( P(n, k) \), 
\[ \left\lceil \frac{4n}{5} \right\rceil \leq \gamma_{r2}(P(n, k)) \leq n. \]
Two questions

In 2007, Bresăr et al. proposed the following questions:

**Question 1.** Is \( \gamma_{r_2}(P(2k + 1, k)) = 2k + 1 \) for all \( k \geq 2 \)?

**Question 2.** Is \( \gamma_{r_2}(P(n, 3)) = n \) for all \( n \geq 7 \) where \( n \) is not divisible by 3?
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  **Question 1.** Is $\gamma_r(P(2k + 1, k)) = 2k + 1$ for all $k \geq 2$?

  **Question 2.** Is $\gamma_r(P(n, 3)) = n$ for all $n \geq 7$ where $n$ is not divisible by 3?

- **Theorem** (Tong et al. 2008).

  $$\gamma_r(P(2k+1, k)) = \begin{cases} 
  \left\lceil \frac{8k + 4}{5} \right\rceil, & \text{if } n \equiv 1, 4 \pmod{5}; \\
  \left\lceil \frac{8k + 4}{5} \right\rceil + 1, & \text{if } n \equiv 0, 2, 3 \pmod{5}.
  \end{cases}$$
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- In 2007, Bresar et al. proposed the following questions:
  
  **Question 1.** Is $\gamma_r(P(2k+1, k)) = 2k + 1$ for all $k \geq 2$?
  
  **Question 2.** Is $\gamma_r(P(n, 3)) = n$ for all $n \geq 7$ where $n$ is not divisible by 3?

- **Theorem** (Tong et al. 2008).

  \[
  \gamma_r(P(2k+1, k)) = \begin{cases} 
  \left\lceil \frac{8k+4}{5} \right\rceil, & \text{if } n \equiv 1, 4 \pmod{5}; \\
  \left\lceil \frac{8k+4}{5} \right\rceil + 1, & \text{if } n \equiv 0, 2, 3 \pmod{5}. 
  \end{cases}
  \]

- If $k \geq 4$ the answer to Question 1 is negative.
Proposition 1 For $n \geq 13$ and $k$ ($n$ can be divided by 3), we have $\gamma_{r2}(P(n, 3)) \leq n - 1$. 
Our results

- **Proposition 1** For \( n \geq 13 \) and \( k \) (\( n \) can be divided by 3), we have \( \gamma_{r2}(P(n, 3)) \leq n - 1 \).

- **Theorem 2** For \( n \geq 13 \), we have

\[
\gamma_{r2}(P(n, 3)) \leq n - \left\lfloor \frac{n}{8} \right\rfloor + \beta,
\]

where \( \beta = 0 \) for \( n \equiv 0, 2, 4, 5, 6, 7, 13, 14, 15 \pmod{16} \) and \( \beta = 1 \) for \( n \equiv 1, 3, 8, 9, 10, 11, 12 \pmod{16} \).
$\gamma_{r2}(P(13, 3)) \leq 12$

A 2RDF of weight 12 of $P(13, 3)$. 
\[ \gamma_{r2}(P(16, 3)) \leq 14 \]

A 2RDF of weight 14 of \( P(16, 3) \).
Problem

**Conjecture.** For $n \geq 13$,

$$\gamma_{r2}(P(n,3)) = n - \left\lfloor \frac{n}{8} \right\rfloor + \beta,$$

where $\beta = 0$ for $n \equiv 0, 2, 4, 5, 6, 7, 13, 14, 15 \pmod{16}$ and $\beta = 1$ for $n \equiv 1, 3, 8, 9, 10, 11, 12 \pmod{16}$. 
Thank you!