Embedding partial Latin squares into Latin squares with many orthogonal mates

Emine Şule Yazıcı
Koç University

Joint work with D. Donovan and M. Grannell

TUBITAK 116F166

LATIN SQUARES

- Latin square of order n is an nxn array on the set of symbols {1,2,...,n}, such that each row and column of the array contains each symbol exactly once.

.

1	2	3	4	5	6	7	8	9
2	3	4	5	6	7	8	9	1
3	4	5	6	7	8	9	1	2
4	5	6	7	8	9	1	2	3
5	6	7	8	9	1	2	3	4
6	7	8	9	1	2	3	4	5
7	8	9	1	2	3	4	5	6
8	9	1	2	3	4	5	6	7
9	1	2	3	4	5	6	7	8

Latin Square of order 9

Mutually orthogonal latin squares

- The latin squares $L_1, L_2, \ldots, L_t$ are said to be ***mutually orthogonal*** if for $1 \leq a \neq b \leq t$, L_a and L_b are orthogonal.

- Latin squares L_a and L_b of order n are said to be orthogonal if for each $(x,y) \in \{1,2,\ldots n\} x \{1,2,\ldots,n\}$, there exists one order pair (i,j) such that the cell (i,j) of L_a contains the symbol x and the cell (i,j) of L_b contains the symbol y

A pair of mutually orthogonal latin squares of order 5

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

All ordered pairs $(x,y) \in \{1,2,...,5\} \times \{1,2,...,5\}$ appears once in the superimposed latin square

EMBEDDINGS OF LATIN SQUARES

Embeddings of Latin Squares

- A latin square L of order n is embedded in a latin square K of order m if K contains L as a subsquare

Example

1	2	3	4	5	6	7	8	9
3	1	2	5	6	4	8	9	7
2	3	1	6	4	5	9	7	8
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
9	7	8	3	1	2	6	4	5
4	5	6	7	8	9	1	2	3
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2

A latin square of order 3 embedded in a latin square of order 9

A pair of orthogonal latin squares of order 3 embedded in a pair of orthogonal latin squares of order 9

1	2	3	4	5	6	7	8	9
2	3	1	5	6	4	8	9	7
3	1	2	6	4	5	9	7	8
7	8	9	1	2	3	4	5	6
8	9	7	2	3	1	5	6	4
9	7	8	3	1	2	6	4	5
4	5	6	7	8	9	1	2	3
5	6	4	8	9	7	2	3	1
6	4	5	9	7	8	3	1	2

1	2	3	4	5	6	7	8	9
3	1	2	6	4	5	9	7	8
2	3	1	5	6	4	8	9	7
4	5	6	7	8	9	1	2	3
6	4	5	9	7	8	3	1	2
5	6	4	8	9	7	2	3	1
7	8	9	1	2	3	4	5	6
9	7	8	3	1	2	6	4	5
8	9	7	2	3	1	5	6	4

Embeddings of Mutually Orthogonal Latin Squares

- (1986) A pair of orthogonal latin squares of order n can be embedded in a pair of orthogonal latin squares of all orders $t \geq 3n$.

Partial Latin Squares

- A partial Latin square is an $n \times n$ array with entries chosen from a set of n symbols such that each symbol occurs at most once in each row and at most once in each column.
- A partial Latin square can be thought of as a subset of a Latin square.

Partial Latin square of order 4

1			4
	3	4	
3			2
		2	3

Embedding partial Latin Squares

0	1	2	3	4	5	6
3	4	5	6	0	1	2
6	0	1	2	3	4	5
2	3	4	5	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3

Embeddings of partial Latin Squares

- Evan’ s Theorem (1960) :

  A partial Latin square of order n can always be embedded in some Latin square of order $t \geq 2n$.

Embeddings of Mutually Orthogonal Partial Latin Squares

- When can k mutually orthogonal ***partial*** latin squares embedded in (completed to) a set of mutually orthogonal Latin squares?

Examples

1	2			
	3	4		
	4			

1	2			
	4	3		
	1			

1	2			
	5	1		
	3			

3 mutually orthogonal partial latin squares

Examples

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

1	2	3	4	5
3	4	5	1	2
5	1	2	3	4
2	3	4	5	1
4	5	1	2	3

1	2	3	4	5
4	5	1	2	3
2	3	4	5	1
5	1	2	3	4
3	4	5	1	2

3 partial mutually orthogonal latin squares embedded in 3 mutually orthogonal latin squares of order 5

Situation so far

- Lindner (1976) : A set of k mutually orthogonal partial Latin squares can always be finitely embedded in k mutually orthogonal Latin squares.
- Hilton, Rodger, Wojciechowski (1992): Formulated some necessary conditions for a pair of partial orthogonal Latin squares to be extended to a pair of Latin squares.

Situation so far

- Jenkins (2005): A partial Latin square of order n can be embedded in a Latin square of order $4n^2$ which has an orthogonal mate.

- Donovan, Yazici (2014) A pair of orthogonal partial Latin squares can always be embedded in a pair of orthogonal Latin squares of polynomial order with respect to the order of the partial squares

Embeddings of 2 orthogonal partial Latin squares (2014)

- A pair of ***partial*** orthogonal latin squares of order n can be embedded in a pair of orthogonal latin squares of order m where m is at most $16n^4$
- A pair of orthogonal ***partial*** latin squares of order n can be embedded in a pair of orthogonal latin squares of all orders $m \geq 48n^4$.

Embedding with many mates

0	**1**	2	**3**	4	5	6
3	**4**	**5**	**6**	**0**	1	2
6	**0**	**1**	**2**	**3**	4	5
2	**3**	4	**5**	6	0	1
5	6	0	1	2	3	4
1	2	3	4	5	6	0
4	5	6	0	1	2	3

0	1	2	3	4	5	6
1	2	3	4	5	6	0
2	3	4	5	6	0	1
3	4	5	6	0	1	2
4	5	6	0	1	2	3
5	6	0	1	2	3	4
6	7	1	2	3	4	5

0	1	2	3	4	5	6
2	3	4	5	6	0	1
4	5	6	0	1	2	3
6	0	1	2	3	4	5
1	2	3	4	5	6	0
3	4	5	6	0	1	2
5	6	0	1	2	3	4

Embedding with many mates

- First embed the partial Latin square into a Latin square of order n

0	**1**	3	4	2
2	**0**	1	3	4
3	4	0	2	1
4	3	2	1	0
1	2	4	0	3

B

Embedding with many mates

- Then we take a set of t mutually orthogonal Latin squares of order n

F_1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

F_2

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

F_3

0	1	2	3	4
3	4	0	1	2
1	2	3	4	0
4	0	1	2	3
2	3	4	0	1

Embedding with many mates

- X_k={((p,r),(q,c),[F_k(F_1(**p**,r),q),F_k(F_1(p,**q**),c)]}
- *B**= {((p,r),(q,c),[F_1(p,q),B(F_1(**p**,r),c)]}
- Let pq=F_1(p,q)

B*

	(0,c)				(q,c)
(0,r)	(0,B)	(q,pB)	(q,pB)	(q,pB)	(q,pB)
	(p,pB)	(pq,pB)	(pq,pB)	(pq,pB)	(pq,pB)
(p,r)	(p,pB)	(pq,pB)	(pq,pB)	(pq,pB)	(pq,pB)
	(p,pB)	(pq,pB)	(pq,pB)	(pq,pB)	(pq,pB)
	(p,pB)	(pq,pB)	(pq,pB)	(pq,pB)	(pq,pB)

$$B^{*} = \{((p,r),(q,c),[F_1(p,q),B(F_1(p,r),c)]\}$$

	(0,c)		X_k		(q,c)
(0,r)	(r,c)	(rq, qc)	(rq, qc)	(rq, qc)	(rq, qc)
	(pr,pc)				
(p,r)	(pr,pc)		$(pr*_k q, pq*_k c)$		
	(pr,pc)				
	(pr,pc)				

$$X_k=\{((p,r),(q,c),[F_k(F_1(p,r),q),F_k(F_1(p,q),c)]\}$$

(0,0)	(0,1)	(0,3)	(0,4)	(0,2)	(1,0)	(1,1)	(1,3)	(1,4)	(1,2)	
(0,2)	(0,0)	(0,1)	(0,3)	(0,4)	(1,2)	(1,0)	(1,1)	(1,3)	(1,4)	
(0,3)	(0,4)	(0,0)	(0,2)	(0,1)	(1,3)	(1,4)	(1,0)	(1,2)	(1,1)	
(0,4)	(0,3)	(0,2)	(0,1)	(0,0)	(1,4)	(1,3)	(1,2)	(1,1)	(1,0)	
(0,1)	(0,2)	(0,4)	(0,0)	(0,3)	(1,1)	(1,2)	(1,4)	(1,0)	(1,3)	.....
(1,2)	(1,0)	(1,1)	(1,3)	(1,4)	(2,2)	(2,0)	(2,1)	(2,3)	(2,4)	
(1,3)	(1,4)	(1,0)	(1,2)	(1,1)	(2,3)	(2,4)	(2,0)	(2,2)	(2,1)	
(1,4)	(1,3)	(1,2)	(1,1)	(1,0)	(2,4)	(2,3)	(2,2)	(2,1)	(2,0)	
(1,1)	(1,2)	(1,4)	(1,0)	(1,3)	(2,1)	(2,2)	(2,4)	(2,0)	(2,3)	
(1,0)	(1,1)	(1,3)	(1,4)	(1,2)	(2,0)	(2,1)	(2,3)	(2,4)	(2,2)	
					. . . .					

B*

(0,0)	(0,1)	(0,2)	(0,3)	(0,4)	(1,2)	(1,3)	(1,4)	(1,0)	(1,1)	
(2,0)	(2,1)	(2,2)	(2,3)	(2,4)	(3,2)	(3,3)	(3,4)	(3,0)	(3,1)	
(4,0)	(4,1)	(4,2)	(4,3)	(4,4)	(0,2)	(0,3)	(0,4)	(0,0)	(0,1)	
(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(2,2)	(2,3)	(2,4)	(2,0)	(2,1)	
(3,0)	(3,1)	(3,2)	(3,3)	(3,4)	(4,2)	(4,3)	(4,4)	(4,0)	(4,1)	.....
(2,2)	(2,3)	(2,4)	(2,0)	(2,1)	(3,4)	(3,0)	(3,1)	(3,2)	(3,3)	
(4,2)	(4,3)	(4,4)	(4,0)	(4,1)	(0,4)	(0,0)	(0,1)	(0,2)	(0,3)	
(1,2)	(1,3)	(1,4)	(1,0)	(1,1)	(2,4)	(2,0)	(2,1)	(2,2)	(2,3)	
(3,2)	(3,3)	(3,4)	(3,0)	(3,1)	(4,4)	(4,0)	(4,1)	(4,2)	(4,3)	
(0,2)	(0,3)	(0,4)	(0,0)	(0,1)	(1,4)	(1,0)	(1,1)	(1,2)	(1,3)	
					. . . .					

X_2

F_1

0	1	2	3	4
1	2	3	4	0
2	3	4	0	1
3	4	0	1	2
4	0	1	2	3

F_2

0	1	2	3	4
2	3	4	0	1
4	0	1	2	3
1	2	3	4	0
3	4	0	1	2

0	**1**	3	4	2
2	**0**	1	3	4
3	4	0	2	1
4	3	2	1	0
1	2	4	0	3

B

Consequences

- This results improves Jenkin's result
- First embed in B
- B has order m where

$$2^k = m > 2n \geq 2^{k-1}$$

- so $2^{k-2} \leq n < 2^{k-1}$ this gives us $2^k \leq 4n < 2^{k+1}$ order of the embedding $\mathbf{m^2 \leq 16n^2}$
- There are at least $m-1 \geq 2n$ orthogonal mates.
- **BONUS: EMBEDDING IS IDEMPOTENT**

Corollary

- Let L be a Latin square of order n with $n \geq 3$ and $n \neq 6$. Then L can be embedded in a Latin square B of order n^2 where B has at least two mutually orthogonal mates.

Corollary

- Let L be a Latin square of order n with $n \geq 7$ and $n \neq 10, 18$ or 22. Then L can be embedded in a Latin square B of order n^2 where B has at least 4 mutually orthogonal mates.

New Construction

- Let A_1, A_2 and B_1, B_2 be pairs of orthogonal Latin squares of order n. Let $C_1,\ldots,C_t$ be a set of t mutually orthogonal Latin squares of order n. Then the squares

$$B_1 = \{((p,r), (q,c), (A_1(p,q), B_1(r,c)))\}$$
$$B_2 = \{((p,r), (q,c), (A_2(p,q), B_2(r,c)))\}$$
$$X_i = \{((p,r), (q,c), (C_i(p,B_1(r,c)),C_i(q,B_2(r,c)))\}$$

where $i \in \{1,\ldots,t\}$ form a set of t+2 mutually orthogonal Latin squares of order n^2.

Corollary

- A pair of mutually orthogonal partial Latin squares of order n can be embedded in a set of t > 2 mutually orthogonal Latin squares of polynomial order with respect to n.

Corollary

- There are 9 mutually orthogonal Latin squares of order 576. (Previously only 8 were known)

Proof. There are at least 7 mutually orthogonal Latin squares of order 24. When applied in the construction given, we may obtain 7 + 2 = 9 mutually orthogonal Latin squares of order 24^2 = 576.

Open Problems

- Make the embedding symmetric
- Embed sets of partial orthogonal Latin squares simultaneously
- Make the embedding Linear with respect to the order of the partial Latin Square

THANK YOU